Interface modes in topologically protected edge states using hourglass lattice based metastructures

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INTRODUCTION

- The study of topologically protected phenomena in materials and metamaterials is an active area of research that draws inspiration from quantum systems, and it has been quickly extended to other classical areas of physics, including acoustic, photonic, optomechanical, and elastic media.
- The shape of the hourglass structure in itself is a fascinating design that contains a combination of two oppositely oriented coaxial domes. For more design complexities, this configuration enables us to consider two standard lattices based on regular honeycomb and auxetic (re-entrant structure).[1]
- We have expanded the design space of metastructures by integrating the advantage of lattice geometry with the enhanced tunability of the hourglass shape.
- The unique properties achieved in these media, if the part of a crystalline structure is replaced with an arrangement that is associated with a different value of the invariant, not only will certain frequencies be localized to the interface (as predicted by the classical theory for crystals with defects) but this behaviour will be stable with respect to imperfections.
- These eigenmodes are known as edge modes, and we say they are topologically protected to refer to their robustness. This study introduces hourglass-based invariants to the periodic structures. Such metastructures are based on dome curvatures, suitable for auxetic lattice, and generates different stiffness values.
- The wave propagation at certain frequencies within the isolation (bandgaps) can be precisely achieved by altering the stiffness parameters with the mean values using hourglass invariants.

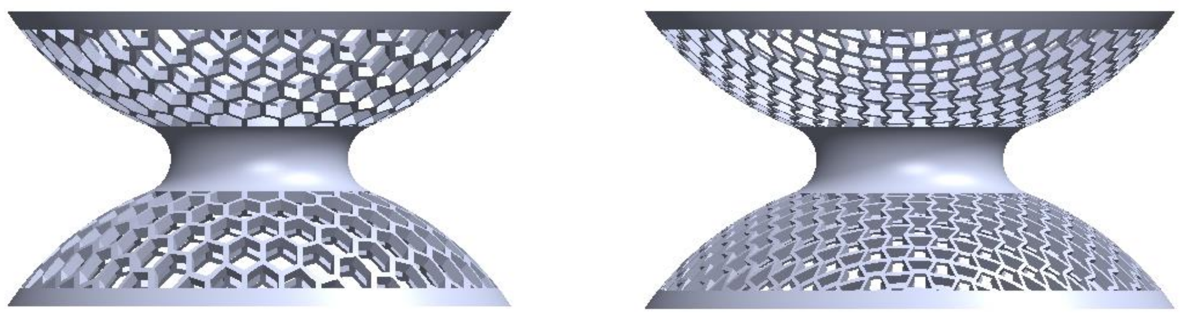

Figure 1. Hourglass shaped lattice metastructures, a combination of two domes d1 and d2 joined by smooth spline surface to avoid stress concentration. The different hourglass structures shown having different lattice namely regular honeycomb and auxetic hourglass respectively.

METHODOLOGY

- **Symmetry Breaking in Topological Metamaterials**
- Considering a finite lattice of one-dimensional spring mass chain, a non-trivial mode is obtained at the interface where left and right sublattice are joined with each other. In the diatomic chain the values of the masses are identical such that $m_1 = m_2 = m$, while the stiffness k_1 and k_2 are alternating in such a way that one is having stiffness higher than the mean value and the other is having stiffness lesser than the mean value.[2]

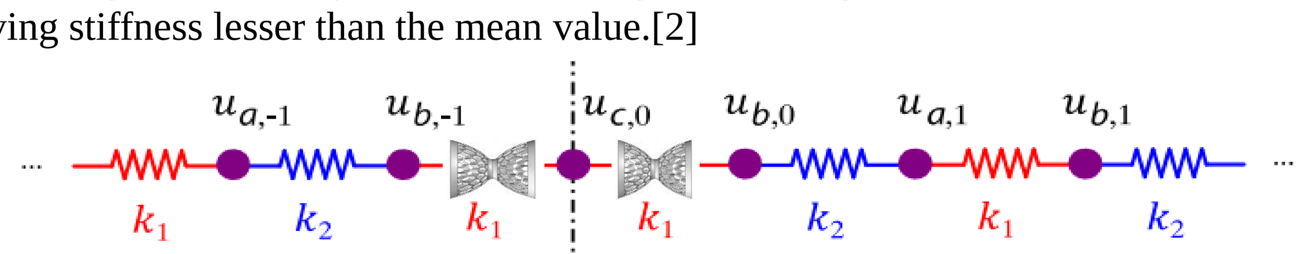

Figure 2. Inversion symmetry breaking at the interface mass which acts as a mirror for the left sublattice and right sublattice. Two sublattices appear as mirror copies of each other about the interface mass. The interface mass would encounter a localized interface mode within the bandgap in the lattice.

- The governing equations for a unit cell p of the sublattice on the **left of interface** is given by

$$m\ddot{u}_{a,p} + k_2(u_{a,p} - u_{b,p}) + k_1(u_{a,p} - u_{b,p-1}) = 0$$
$$m\ddot{u}_{b,p} + k_2(u_{b,p} - u_{a,p}) + k_1(u_{b,p} - u_{a,p+1}) = 0$$

- For a unit cell p on the **right sublattice** the equations of motion are given by

$$m\ddot{u}_{a,p} + k_1(u_{a,p} - u_{b,p}) + k_2(u_{a,p} - u_{b,p-1}) = 0$$
$$m\ddot{u}_{b,p} + k_1(u_{b,p} - u_{a,p}) + k_2(u_{b,p} - u_{a,p+1}) = 0$$

- The governing equation for motion of **interface mass** is given by

$$m\ddot{u}_{c,0} + k_1(u_{c,0} - u_{b,-1}) + k_1(u_{c,0} - u_{b,0}) = 0$$
$$m\ddot{u}_{c,0} + k_1(2u_{c,0} - u_{b,-1} - u_{b,0}) = 0$$
$$m\ddot{u}_{b,0} + k_1(u_{b,0} - u_{c,0}) + k_2(u_{b,0} - u_{a,p+1}) = 0$$

- The equations are normalized by writing the spring constants as k_1 = k(1 + γ) and k_2 = k(1 - γ), with γ = stiffness parameter and k = mean stiffness
- A nondimensional time scale $\tau = \sqrt{k/m}\,t$ is also introduced to express the equations in nondimensional form.

- **Relating different lattice and cell angle of hourglass auxetic and honeycomb hourglass with gamma (γ) parameter**

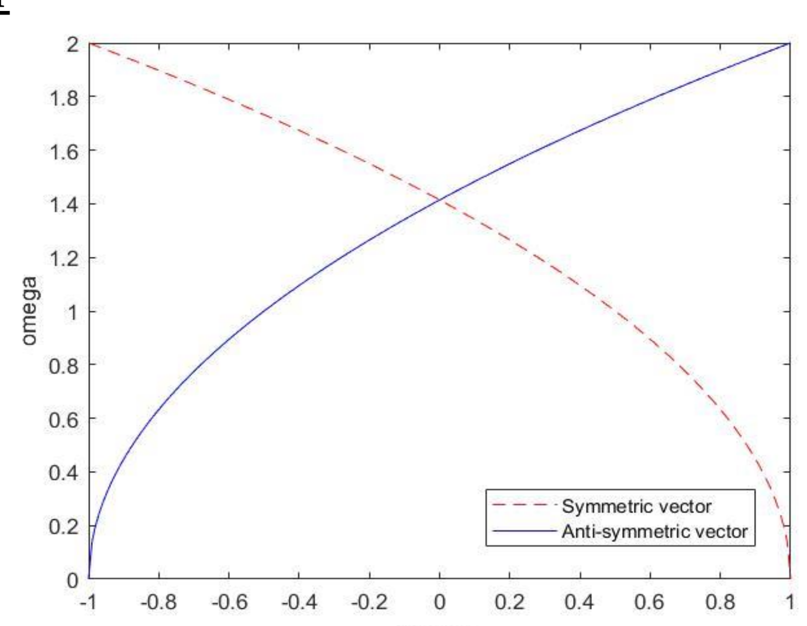

Figure 3. Variation of gamma (γ) parameter within the limits of the bandgap showing the band inversion.

- As gamma varies from -1 to 1 i.e., $\gamma \,\epsilon\, [-1,1]$ then $\Omega \,\epsilon\, \left[\sqrt{2(1-|\gamma|)}, \sqrt{2(1+|\gamma|)}\right]$
- For $\gamma < 0$, the stiffness k_2 will be less than mean stiffness will correspond to auxetic hourglass.
- For $\gamma > 0$, the stiffness k_1 will be less than mean stiffness will correspond to honeycomb hourglass lattice metastructure.
- The dynamic behaviour of the chain obtained using frequency response (FRF) by applying a harmonic displacement at one end of the sublattice (left or right) keeping the other end free and the response is calculated at the interface mass.
- Further their mode shapes can also be obtained to understand interface mass mode shape behaviour.
- The dynamics of the whole lattice can be given by

$$M\ddot{q}(\tau) + Kq(\tau) = f(\tau)$$

- The frequencies at which the diatomic linear chain connected with hourglass attached at both sides of interface can be obtained explicitly by formulating eigenvalue problem.
- Assuming a harmonic solution and formulating the eigen value problem as

$$(K - \Omega^2 M)q = 0$$

RESULTS

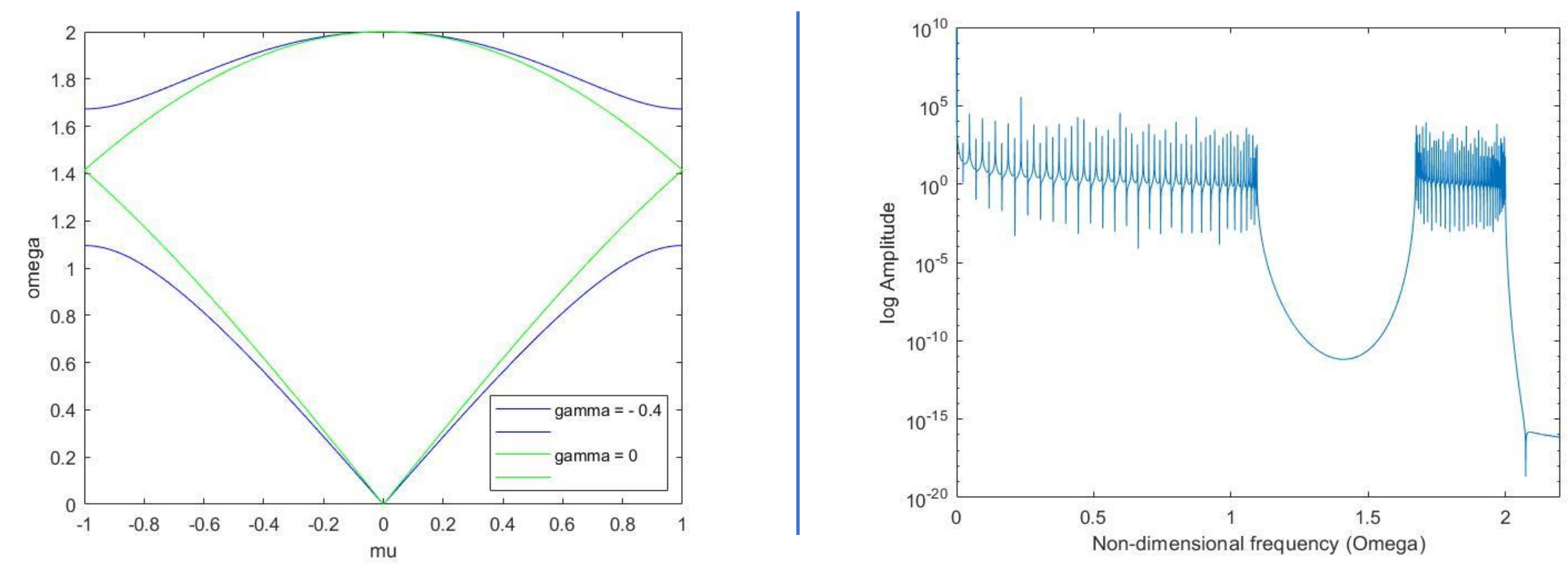

Figure 4. (a) Dispersion relation of the diatomic one-dimensional linear chain with identical masses $m_1 = m_2 =$ m and alternating spring stiffnesses values. **Green color** line corresponds to lattice when spring stiffnesses do not vary i.e. $k_1 = k_2 =$ k representing the absence of the bandgap and acoustic and optical mode meets at $\Omega = \sqrt{2}$ corresponding to **gamma (γ) = 0**. **Blue color** line corresponding to the lattice when spring stiffnesses vary alternatively about the mean stiffness value resulting in presence of bandgap between the acoustic and optical bandgap corresponding to **gamma (γ) = - 0.4**. Assuming the boundary conditions as one end of chain (left) is subjected to harmonic excitation and other end (right) is free.

(b) Frequency response function of one-dimensional diatomic chain having spring mass arrangement with identical masses and spring stiffnesses values alternating about the mean value. The diagram depicts the presence of the bandgap without any localized interface mode present in it.

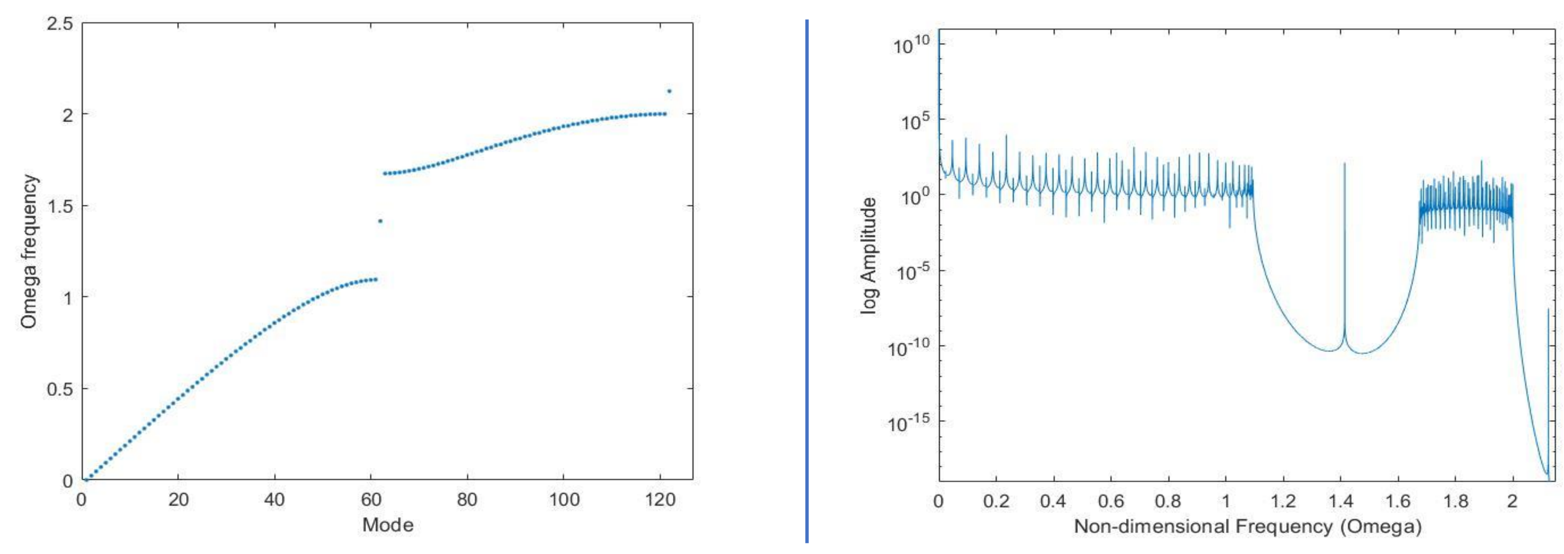

Figure 5. (a) Natural frequency of the mode (mass) number corresponding to the interface mode frequency in bandgap corresponding to **gamma (γ) > 1** topological chain.
(b) Frequency response function of one-dimensional topological chain having **Regular Honeycomb** hourglass attached at the adjacent sides of the interface mass exhibiting a localized interface mode within the bandgap which is shifted towards left side from the center of bandgap. Considering the values as $m_1 = m_2 = m$, γ = - 0.4 , mean stiffness value $k = 1$.

- $k_1 = k(1-\gamma)$ will correspond to regular honeycomb lattice hourglass
- $\gamma > 0 :\ \Omega = \sqrt{3 + \sqrt{1 + 8\gamma^2}}$, Antisymmetric mode
- $\gamma > 0 :\ \Omega = 2$, Symmetric mode

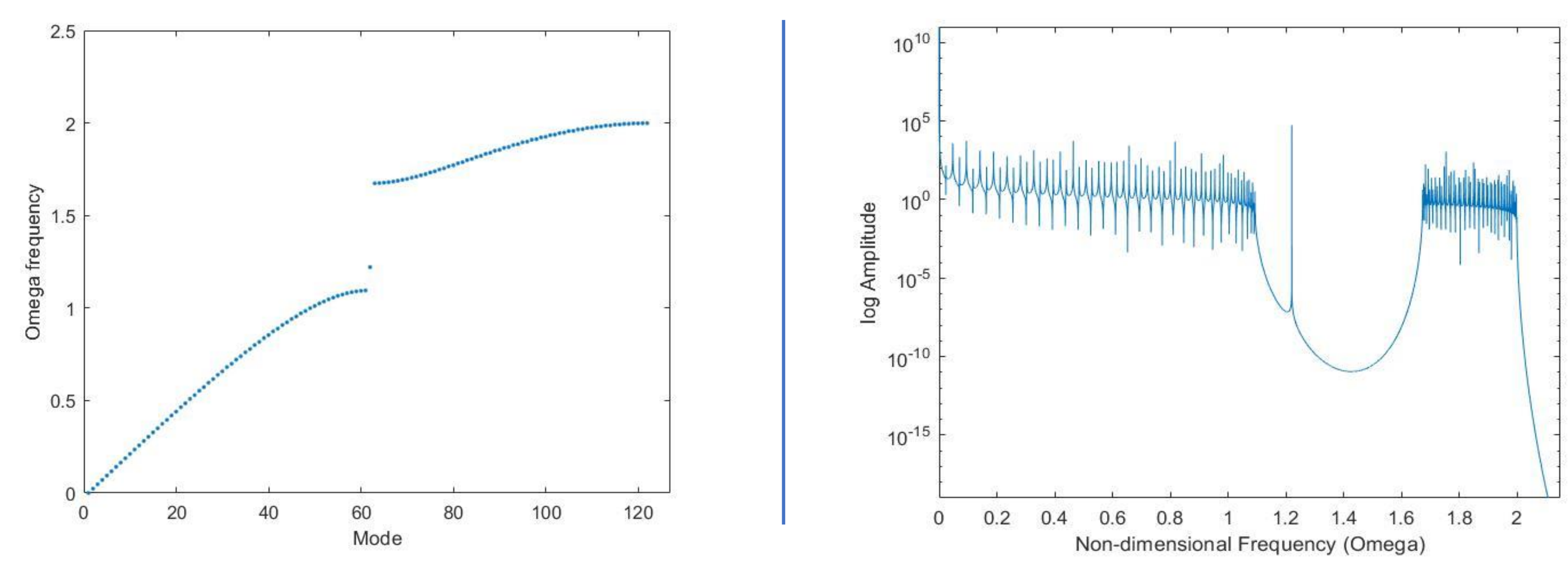

Figure 6. (a) Natural frequency of the mode (mass) number corresponding to the interface mode frequency in bandgap corresponding to **gamma (γ) < 1** topological chain.
(b) Frequency response function of one-dimensional topological chain having **Auxetic** hourglass attached at the adjacent sides of the interface mass exhibiting a localized interface mode within the bandgap which is shifted towards left side from the centre of bandgap. Considering the values as $m_1 = m_2 = m$, γ = - 0.4 , mean stiffness value $k = 1$.

- $k_2 = k(1-\gamma)$ will correspond to auxetic lattice hourglass
- $\gamma < 0 :\ \Omega = \sqrt{3 - \sqrt{1 + 8\gamma^2}}$, Antisymmetric mode

CONCLUSION

- From the analytical results obtained, one can conclude that the localized interface mode obtained within the bandgap is dependent on **non-dimensional *h/t* ratio and lattice cell angle θ_C** of the different classes of hourglass metastructures.
- From this foundation, one can envisage being able to design an hourglass based lattice metastructure as a building block for obtaining the topological interface mode.
- The **interface mode** is achieved successfully by placing the mirror copies of the left sublattice and right sublattice can be strategically placed and used for the potential purpose of **wave guiding at precise frequency** and energy harvesting.
- The possibility of wave propagation at specific frequencies within the bandgaps is strategically achieved by defining lattice-dependent stiffness parameters at the interface modes. The considered configurations define a framework for introducing lattice-based imperfections in the continuous elastic structure that makes it potential engineering relevance.
- The interface mode obtained with the help of **regular honeycomb** lattice is exactly obtained at the center of the bandgap which can be used for guiding the wave at that particular frequency by incorporating desired values of mass and springs involved in the system.
- **The improved functionality of auxetic lattice** has been observed by incorporating them into the dome shape of the hourglass structure, making re-entrant lattice suitable we can get the interface mode easily available at any location in the bandgap from left end of the bandgap to the center of the bandgap.

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