4 Coupled Oscillations

4.1 Two Spring-Coupled Masses

Consider a mechanical system consisting of two identical masses m which are free to slide over a frictionless horizontal surface. Suppose that the masses are attached to one another, and to two immovable walls, by means of three identical light horizontal springs of spring constant k, as shown in Figure 4.1. The instantaneous state of the system is conveniently specified by the displacements of the left and right masses, $x_1(t)$ and $x_2(t)$, respectively. The extensions of the left, middle, and right springs are thus x_1, $x_2 - x_1$, and $-x_2$, respectively, assuming that $x_1 = x_2 = 0$ corresponds to the equilibrium configuration in which the springs are all unextended. The equations of motion of the two masses are thus

$$m\,\ddot{x}_1 = -k\,x_1 + k\,(x_2 - x_1), \tag{4.1}$$

$$m\,\ddot{x}_1 = -k\,(x_2 - x_1) + k\,(-x_2). \tag{4.2}$$

Here, we have made use of the fact that a mass attached to the left end of a spring of extension x and spring constant k experiences a horizontal force $+k\,x$, whereas a mass attached to the right end of the same spring experiences an equal and opposite force $-k\,x$.

Equations (4.1)–(4.2) can be rewritten in the form

$$\ddot{x}_1 = -2\,\omega_0^2\,x_1 + \omega_0^2\,x_2, \tag{4.3}$$

$$\ddot{x}_2 = \omega_0^2\,x_1 - 2\,\omega_0^2\,x_2, \tag{4.4}$$

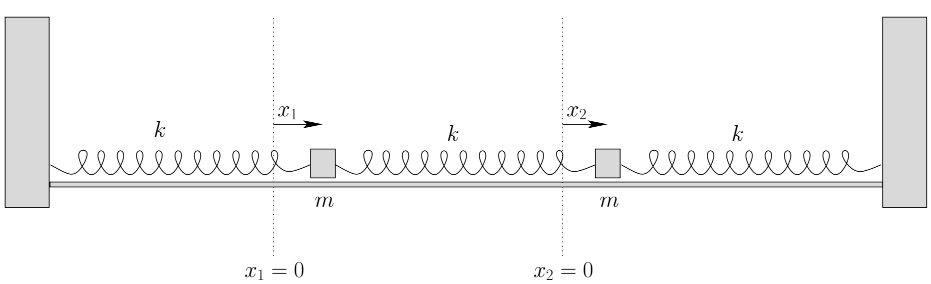

Figure 4.1: *Two degree of freedom mass-spring system.*

where $\omega_0 = \sqrt{k/m}$. Let us search for a solution in which the two masses oscillate *in phase* at the *same angular frequency*, ω. In other words,

$$x_1(t) = \hat{x}_1 \cos(\omega t - \phi), \tag{4.5}$$

$$x_2(t) = \hat{x}_2 \cos(\omega t - \phi), \tag{4.6}$$

where $\hat{x}_1$, $\hat{x}_2$, and ϕ are constants. Equations (4.3) and (4.4) yield

$$-\omega^2 \hat{x}_1 \cos(\omega t - \phi) = \left(-2\,\omega_0^2\,\hat{x}_1 + \omega_0^2\,\hat{x}_2\right) \cos(\omega t - \phi), \tag{4.7}$$

$$-\omega^2 \hat{x}_2 \cos(\omega t - \phi) = \left(\omega_0^2\,\hat{x}_1 - 2\,\omega_0^2\,\hat{x}_2\right) \cos(\omega t - \phi), \tag{4.8}$$

or

$$(\hat{\omega}^2 - 2)\,\hat{x}_1 + \hat{x}_2 = 0, \tag{4.9}$$

$$\hat{x}_1 + (\hat{\omega}^2 - 2)\,\hat{x}_2 = 0, \tag{4.10}$$

where $\hat{\omega} = \omega/\omega_0$. Note that by searching for a solution of the form (4.5)–(4.6) we have effectively converted the system of two coupled *linear differential equations* (4.3)–(4.4) into the much simpler system of two coupled *linear algebraic equations* (4.9)–(4.10). The latter equations have the trivial solutions $\hat{x}_1 = \hat{x}_2 = 0$, but also yield

$$\frac{\hat{x}_1}{\hat{x}_2} = -\frac{1}{(\hat{\omega}^2 - 2)} = -(\hat{\omega}^2 - 2). \tag{4.11}$$

Hence, the condition for a nontrivial solution is clearly

$$(\hat{\omega}^2 - 2)\,(\hat{\omega}^2 - 2) - 1 = 0. \tag{4.12}$$

In fact, if we write Equations (4.9)–(4.10) in the form of a homogenous (*i.e.*, with a null right-hand side) 2×2 matrix equation, so that

$$\begin{pmatrix} \hat{\omega}^2 - 2 & 1 \\ 1 & \hat{\omega}^2 - 2 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.13}$$

then it is clear that the criterion (4.12) can also be obtained by setting the *determinant* of the associated 2×2 matrix to *zero*.

Equation (4.12) can be rewritten

$$\hat{\omega}^4 - 4\,\hat{\omega}^2 + 3 = (\hat{\omega}^2 - 1)\,(\hat{\omega}^2 - 3) = 0. \tag{4.14}$$

It follows that

$$\hat{\omega} = 1 \text{ or } \sqrt{3}. \tag{4.15}$$

Here, we have neglected the two negative frequency roots of (4.14)—*i.e.*, $\hat{\omega} = -1$ and $\hat{\omega} = -\sqrt{3}$—since a negative frequency oscillation is equivalent to an oscillation with an equal and opposite positive frequency, and an equal and opposite phase: *i.e.*, $\cos(\omega\,t - \phi) \equiv \cos(-\omega\,t + \phi)$. It is thus apparent that the dynamical system pictured in Figure 4.1 has *two* unique frequencies of oscillation: *i.e.*, $\omega = \omega_0$ and $\omega = \sqrt{3}\,\omega_0$. These are called the *normal frequencies* of the system. Since the system possesses *two degrees of freedom* (*i.e.*, two independent coordinates are needed to specify its instantaneous configuration) it is not entirely surprising that it possesses *two* normal frequencies. In fact, it is a general rule that a dynamical system with N degrees of freedom possesses N normal frequencies.

The patterns of motion associated with the two normal frequencies can easily be deduced from Equation (4.11). Thus, for $\omega = \omega_0$ (*i.e.*, $\hat{\omega} = 1$), we get $\hat{x}_1 = \hat{x}_2$, so that

$$x_1(t) = \hat{\eta}_1 \cos(\omega_0\,t - \phi_1), \tag{4.16}$$

$$x_2(t) = \hat{\eta}_1 \cos(\omega_0\,t - \phi_1), \tag{4.17}$$

where $\hat{\eta}_1$ and ϕ_1 are constants. This first pattern of motion corresponds to the two masses executing simple harmonic oscillation with the *same amplitude and phase*. Note that such an oscillation does not stretch the middle spring. On the other hand, for $\omega = \sqrt{3}\,\omega_0$ (*i.e.*, $\hat{\omega} = \sqrt{3}$), we get $\hat{x}_1 = -\hat{x}_2$, so that

$$x_1(t) = \hat{\eta}_2 \cos\left(\sqrt{3}\,\omega_0\,t - \phi_2\right), \tag{4.18}$$

$$x_2(t) = -\hat{\eta}_2 \cos\left(\sqrt{3}\,\omega_0\,t - \phi_2\right), \tag{4.19}$$

where $\hat{\eta}_2$ and ϕ_2 are constants. This second pattern of motion corresponds to the two masses executing simple harmonic oscillation with the *same amplitude but in anti-phase*: *i.e.*, with a phase shift of π radians. Such oscillations do stretch the middle spring, implying that the restoring force associated with similar amplitude displacements is greater for the second pattern of motion than for the first. This accounts for the higher oscillation frequency in the second case. (The inertia is the same in both cases, so the oscillation frequency is proportional to the square root of the restoring force associated with similar amplitude displacements.) The two distinctive patterns of motion which we have found are called the *normal modes* *of oscillation* of the system. Incidentally, it is a general rule that a dynamical system possessing N degrees of freedom has N unique normal modes of oscillation.

Now, the most general motion of the system is a *linear combination* of the two normal modes. This immediately follows because Equations (4.1) and (4.2) are *linear equations*. [In other words, if $x_1(t)$ and $x_2(t)$ are solutions then so are $a\,x_1(t)$ and $a\,x_2(t)$, where a is an arbitrary constant.] Thus, we can write

$$x_1(t) = \hat{\eta}_1 \cos(\omega_0\, t - \phi_1) + \hat{\eta}_2 \cos\left(\sqrt{3}\,\omega_0\, t - \phi_2\right), \tag{4.20}$$

$$x_2(t) = \hat{\eta}_1 \cos(\omega_0\, t - \phi_1) - \hat{\eta}_2 \cos\left(\sqrt{3}\,\omega_0\, t - \phi_2\right). \tag{4.21}$$

Note that we can be sure that this represents the most general solution to Equations (4.1) and (4.2) because it contains *four* arbitrary constants: *i.e.*, $\hat{\eta}_1$, ϕ_1, $\hat{\eta}_2$, and ϕ_2. (In general, we expect the solution of a second-order ordinary differential equation to contain two arbitrary constants. It, thus, follows that the solution of a system of two coupled ordinary differential equations should contain four arbitrary constants.) Of course, these constants are determined by the *initial conditions*.

For instance, suppose that $x_1 = a$, $\dot{x}_1 = 0$, $x_2 = 0$, and $\dot{x}_2 = 0$ at $t = 0$. It follows, from (4.20) and (4.21), that

$$a = \hat{\eta}_1 \cos\phi_1 + \hat{\eta}_2 \cos\phi_2, \tag{4.22}$$

$$0 = \hat{\eta}_1 \sin\phi_1 + \sqrt{3}\,\hat{\eta}_2 \sin\phi_2, \tag{4.23}$$

$$0 = \hat{\eta}_1 \cos\phi_1 - \hat{\eta}_2 \cos\phi_2, \tag{4.24}$$

$$0 = \hat{\eta}_1 \sin\phi_1 - \sqrt{3}\,\hat{\eta}_2 \sin\phi_2, \tag{4.25}$$

which implies that $\phi_1 = \phi_2 = 0$ and $\hat{\eta}_1 = \hat{\eta}_2 = a/2$. Thus, the system evolves in time as

$$x_1(t) = a\,\cos(\omega_-\, t)\,\cos(\omega_+\, t), \tag{4.26}$$

$$x_2(t) = a\,\sin(\omega_-\, t)\,\sin(\omega_+\, t), \tag{4.27}$$

where $\omega_\pm = [(\sqrt{3} \pm 1)/2]\,\omega_0$, and use has been made of the trigonometric identities $\cos(a + b) \equiv 2\cos[(a + b)/2]\cos[(a - b)/2]$ and $\cos(a - b) \equiv -2\sin[(a + b)/2]\sin[(a - b)/2]$. This evolution is illustrated in Figure 4.2. [Here, $T_0 = 2\pi/\omega_0$. The solid curve corresponds to x_1, and the dashed curve to x_2.]

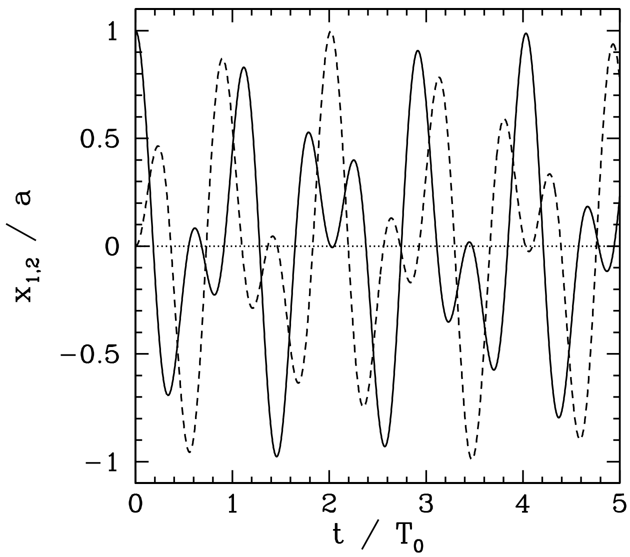

Figure 4.2: *Coupled oscillations in a two degree of freedom mass-spring system.*

Finally, let us define the so-called *normal coordinates*,

$$\eta_1(t) = [x_1(t) + x_2(t)]/2, \tag{4.28}$$

$$\eta_2(t) = [x_1(t) - x_2(t)]/2. \tag{4.29}$$

It follows from (4.20) and (4.21) that, in the presence of both normal modes,

$$\eta_1(t) = \hat{\eta}_1 \cos(\omega_0 t - \phi_1), \tag{4.30}$$

$$\eta_2(t) = \hat{\eta}_2 \cos(\sqrt{3}\,\omega_0 t - \phi_2). \tag{4.31}$$

Thus, in general, the two normal coordinates oscillate sinusoidally with *unique frequencies*, unlike the regular coordinates, $x_1(t)$ and $x_2(t)$—see Figure 4.2. This suggests that the equations of motion of the system should look particularly simple when expressed in terms of the normal coordinates. In fact, it is easily seen that the sum of Equations (4.3) and (4.4) reduces to

$$\ddot{\eta}_1 = -\omega_0^2\,\eta_1, \tag{4.32}$$

whereas the difference gives

$$\ddot{\eta}_2 = -3\,\omega_0^2\,\eta_2. \tag{4.33}$$

Thus, when expressed in terms of the normal coordinates, the equations of motion of the system reduce to two *uncoupled* simple harmonic oscillator equations. Of course, most general solution to Equation (4.32) is (4.30), whereas the most general solution to Equation (4.33) is (4.31). Hence, if we can guess the normal coordinates of a coupled oscillatory system then the determination of the normal modes of oscillation is considerably simplified.

4.2 Two Coupled LC Circuits

Consider the LC circuit pictured in Figure 4.3. Let $I_1(t)$, $I_2(t)$, and $I_3(t)$ be the currents flowing in the three legs of the circuit, which meet at junctions A and B. According to *Kichhoff's first circuital law*, the net current flowing into each junction is zero. It follows that $I_3 = -(I_1 + I_2)$. Hence, this is a *two degree of freedom* system whose instantaneous configuration is specified by the two independent variables $I_1(t)$ and $I_2(t)$. It follows that there are *two* independent normal modes of oscillation. Now, the potential differences across the left, middle, and right legs of the circuit are $Q_1/C + L\,\dot{I}_1$, Q_3/C', and $Q_2/C + L\,\dot{I}_2$, respectively, where $\dot{Q}_1 = I_1$, $\dot{Q}_2 = I_2$, and $Q_3 = -(Q_1 + Q_2)$. However, since the three legs are connected in *parallel*, the potential differences must all be equal, so that

$$Q_1/C + L\,\dot{I}_1 = Q_3/C' = -(Q_1 + Q_2)/C', \tag{4.34}$$

$$Q_2/C + L\,\dot{I}_2 = Q_3/C' = -(Q_1 + Q_2)/C'. \tag{4.35}$$

Differentiating with respect to t, and dividing by L, we obtain the coupled time evolution equations of the system:

$$\ddot{I}_1 + \omega_0^2\,(1+\alpha)\,I_1 + \omega_0^2\,\alpha\,I_2 = 0, \tag{4.36}$$

$$\ddot{I}_2 + \omega_0^2\,(1+\alpha)\,I_2 + \omega_0^2\,\alpha\,I_1 = 0, \tag{4.37}$$

where $\omega_0 = 1/\sqrt{L\,C}$ and $\alpha = C/C'$.

It is fairly easy to guess that the normal coordinates of the system are

$$\eta_1 = (I_1 + I_2)/2, \tag{4.38}$$

$$\eta_2 = (I_1 - I_2)/2. \tag{4.39}$$

Forming the sum and difference of Equations (4.36) and (4.37), we obtain the evolution equations for the two independent normal modes of oscillation:

$$\ddot{\eta}_1 + \omega_0^2\,(1+2\,\alpha)\,\eta_1 = 0, \tag{4.40}$$

$$\ddot{\eta}_2 + \omega_0^2\,\eta_2 = 0. \tag{4.41}$$

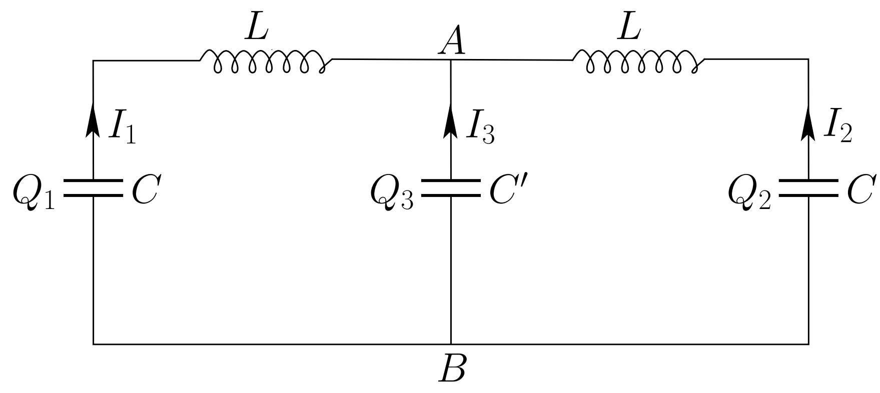

Figure 4.3: *Two degree of freedom* LC *circuit.*

These equations can readily be solved to give

$$\eta_1(t) = \hat{\eta}_1 \cos(\omega_1 t - \phi_1), \tag{4.42}$$

$$\eta_2(t) = \hat{\eta}_2 \cos(\omega_0 t - \phi_2), \tag{4.43}$$

where $\omega_1 = (1 + 2\,\alpha)^{1/2}\,\omega_0$. Here, $\hat{\eta}_1$, ϕ_1, $\hat{\eta}_2$, and ϕ_2 are constants determined by the initial conditions. It follows that

$$I_1(t) = \eta_1(t) + \eta_2(t) = \hat{\eta}_1 \cos(\omega_1 t - \phi_1) + \hat{\eta}_2 \cos(\omega_0 t - \phi_2), \tag{4.44}$$

$$I_2(t) = \eta_1(t) - \eta_2(t) = \hat{\eta}_1 \cos(\omega_1 t - \phi_1) - \hat{\eta}_2 \cos(\omega_0 t - \phi_2). \tag{4.45}$$

As an example, suppose that $\phi_1 = \phi_2 = 0$ and $\hat{\eta}_1 = \hat{\eta}_2 = I_0/2$. We obtain

$$I_1(t) = I_0 \cos(\omega_- t)\cos(\omega_+ t), \tag{4.46}$$

$$I_2(t) = I_0 \sin(\omega_- t)\sin(\omega_+ t), \tag{4.47}$$

where $\omega_\pm = (\omega_0 \pm \omega_1)/2$. This solution is illustrated in Figure 4.4. [Here, $T_0 = 2\pi/\omega_0$ and $\alpha = 0.2$. Thus, the two normal frequencies are ω_0 and $1.18\,\omega_0$.] Note the *beats* generated by the superposition of two normal modes with similar normal frequencies.

We can also solve the problem in a more systematic manner by specifically searching for a normal mode of the form

$$I_1(t) = \hat{I}_1 \cos(\omega t - \phi), \tag{4.48}$$

$$I_2(t) = \hat{I}_2 \cos(\omega t - \phi). \tag{4.49}$$

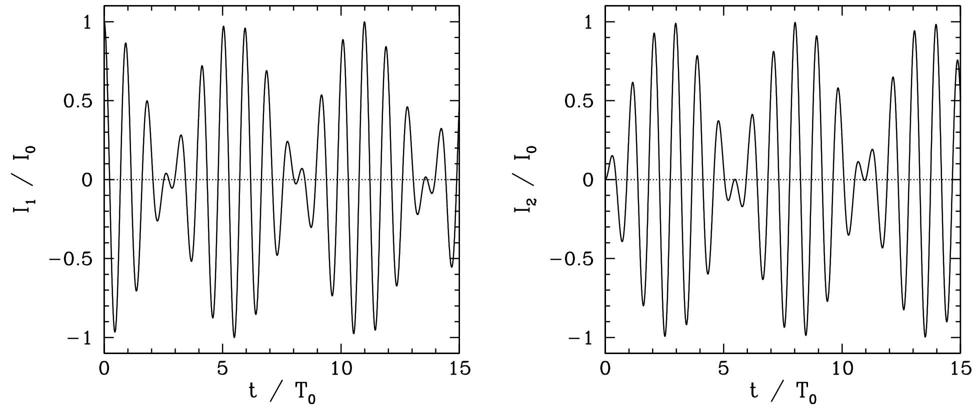

Figure 4.4: *Coupled oscillations in a two degree of freedom* LC *circuit.*

Substitution into the time evolution equations (4.36) and (4.37) yields the matrix equation

$$\begin{pmatrix} \hat{\omega}^2 - (1+\alpha) & -\alpha \\ -\alpha & \hat{\omega}^2 - (1+\alpha) \end{pmatrix} \begin{pmatrix} \hat{I}_1 \\ \hat{I}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.50}$$

where $\hat{\omega} = \omega/\omega_0$. The normal frequencies are determined by setting the determinant of the matrix to zero. This gives

$$\left[\hat{\omega}^2 - (1+\alpha)\right]^2 - \alpha^2 = 0, \tag{4.51}$$

or

$$\hat{\omega}^4 - 2\,(1+\alpha)\,\hat{\omega}^2 + 1 + 2\,\alpha = \left(\hat{\omega}^2 - 1\right)\left(\hat{\omega}^2 - [1 + 2\,\alpha]\right) = 0. \tag{4.52}$$

The roots of the above equation are $\hat{\omega} = 1$ and $\hat{\omega} = (1+2\,\alpha)^{1/2}$. (Again, we neglect the negative frequency roots, since they generate the same patterns of motion as the corresponding positive frequency roots.) Hence, the two normal frequencies are ω_0 and $(1+2\,\alpha)^{1/2}\,\omega_0$. The characteristic patterns of motion associated with the normal modes can be calculated from the first row of the matrix equation (4.50), which can be rearranged to give

$$\frac{\hat{I}_1}{\hat{I}_2} = \frac{\alpha}{\hat{\omega}^2 - (1+\alpha)}. \tag{4.53}$$

It follows that $\hat{I}_1 = -\hat{I}_2$ for the normal mode with $\hat{\omega} = 1$, and $\hat{I}_1 = \hat{I}_2$ for the normal mode with $\hat{\omega} = (1+2\,\alpha)^{1/2}$. We are thus led to Equations (4.44)–(4.45), where $\hat{\eta}_1$ and ϕ_1 are the amplitude and phase of the higher frequency normal mode, whereas $\hat{\eta}_2$ and ϕ_2 are the amplitude and phase of the lower frequency mode.

4.3 Three Spring Coupled Masses

Consider a generalized version of the mechanical system discussed in Section 4.1 that consists of *three* identical masses m which slide over a frictionless horizontal surface, and are connected by identical light horizontal springs of spring constant k. As before, the outermost masses are attached to immovable walls by springs of spring constant k. The instantaneous configuration of the system is specified by the horizontal displacements of the three masses from their equilibrium positions: *i.e.*, $x_1(t)$, $x_2(t)$, and $x_3(t)$. Clearly, this is a *three degree of freedom system*. We, therefore, expect it to possesses *three* independent normal modes of oscillation. Equations (4.1)–(4.2) generalize to

$$m\,\ddot{x}_1 = -k\,x_1 + k\,(x_2 - x_1), \tag{4.54}$$

$$m\,\ddot{x}_2 = -k\,(x_2 - x_1) + k\,(x_3 - x_2), \tag{4.55}$$

$$m\,\ddot{x}_3 = -k\,(x_3 - x_2) + k\,(-x_3). \tag{4.56}$$

These equations can be rewritten

$$\ddot{x}_1 = -2\,\omega_0^2\,x_1 + \omega_0^2\,x_2, \tag{4.57}$$

$$\ddot{x}_2 = \omega_0^2\,x_1 - 2\,\omega_0^2\,x_2 + \omega_0^2\,x_3, \tag{4.58}$$

$$\ddot{x}_3 = \omega_0^2\,x_2 - 2\,\omega_0^2\,x_3, \tag{4.59}$$

where $\omega_0 = \sqrt{k/m}$. Let us search for a normal mode solution of the form

$$x_1(t) = \hat{x}_1\,\cos(\omega\,t - \phi), \tag{4.60}$$

$$x_2(t) = \hat{x}_2\,\cos(\omega\,t - \phi), \tag{4.61}$$

$$x_3(t) = \hat{x}_3\,\cos(\omega\,t - \phi). \tag{4.62}$$

Equations (4.57)–(4.62) can be combined to give the 3×3 homogeneous matrix equation

$$\begin{pmatrix} \hat{\omega}^2 - 2 & 1 & 0 \\ 1 & \hat{\omega}^2 - 2 & 1 \\ 0 & 1 & \hat{\omega}^2 - 2 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.63}$$