ANALYSIS OF DYNAMIC RESPONSE OF MULTI-STORY BUILDINGS UNDER EARTHQUAKE FORCES, FOR DESIGN PURPOSES.

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Synopsis

The 3n natural frequencies of vibration of an n-story building are determined for the general case of three freedoms of movement of each slab. The time functions of the translatory $(u(t), v(t))$ and rotational $(\varphi(t))$ movement of slabs are determined for a general tremor of the ground $u = F(t)$, $v = G(t)$, and $\varphi = H(t)$. For the usual case in practice $H(t) = 0$, $G(t) = \lambda F(t)$, a design procedure is outlined for obtaining the limiting deformations of vertical elements (columns and shear walls) of buildings vibrating under earth tremor effects. Flexibility and stiffness coefficients are assumed easily obtainable by the method of Prof. Roussopoulos developed in 1932 for the static problem.

Notation

Symbols are presented in order of appearance in the text.

O (1,2,3)	Center of cartesian coordinated and principal directions of coordinates.
$i = 1,2...n$	Order of slab (or story) counting fron the ground up.
C_i	Center of gravity of mass enclosed between the center hight of ith and (i-1)th story, assumed lying on ith slab.
W_i'	Total weight of i th story.
g	Gravitational acceleration
W_i	Total mass of i th story.
I_i	Polar moment of inertia of the mass of the i th story about C_i.
d_{ik}^{mn}	Displacement of C_i in the direction m, under the application of a unit load at C_k in the direction n.
$u_i, \upsilon_i, \varphi_i$	Movement of C_i in the directions 1,2,3 respectively.
F(t),G(t),H(t)	Respectively components in the 1,2,3 directions of the function describing earth's tremor.
a_{ik}, b_{ik}, c_{ik}	Symbols respectively equal to d_{ik}^{11}, d_{ik}^{22}, d_{ik}^{33}
e_{ik}, f_{ik}, g_{ik}	Symbols respectively equal to d_{ik}^{12}, d_{ik}^{13}, d_{ik}^{23}
U_i, V_i, Φ_i	Respectively equal to $W_i u_i$, $W_i \upsilon_i$, $I_i \varphi_i$

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σ	order of mode of vibration
$A_{i\sigma}, B_{i\sigma}, C_{i\sigma}$	normalized amplitudes of free vibration of mode σ of C_i
$\mathcal{L}^1_{i\sigma}, \mathcal{L}^2_{i\sigma}, \mathcal{L}^3_{i\sigma}$	normalized characteristic loads
ω_σ	natural frequency of vibration of the order σ
$\mathcal{P}_\sigma$	participation factor for characteristic loads of natural frequency of order σ
$\mathcal{D}_\sigma(t)$	dynamic load factor (of frequency of order σ)
w_i	$= \frac{U_i}{W_i} - F(t) = u_i - F(t)$
v_i	$= \frac{V_i}{W_i} - G(t) = v_i - G(t)$
i	$= \frac{\Phi_i}{I_i} = \varphi_i$

1. The Static Problem.

In 1932, Professor Roussopoulos of the Athens Polytechnic published the first paper on a theory of aseismic analysis which accounted for stresses caused both by translation and rotation of horizontal elements of buildings (slabs). This solution was given for only lateral loads acting on the slabs statically. There were no limitations placed on the symmetry or layout of the buildings nor on the direction of the lateral forces.

Unfortunately this work attracted little attention because it was presented in Greek. Essentially it relied on a methodology for obtaining the stiffness and flexibility coefficients of horizontal slabs of a multi-story building. Since then, and particularly during the last decade, this procedure for structural analysis has become quite popular in view of its advantages for computer application.

In the analysis of the dynamic problem for the general case (including translation as well as rotation of slabs), presented in this paper, it will be considered that flexibility coefficients d_{ik}^{mn} can always be obtained in the easy and straightforward way developed by Roussopoulos. Researchers can obtain the pertinent derivations in the French translation of the basic work of Roussopoulos(2).

2. The Basic Displacement Equation.

Consider an n-story building and designate with i the i th in order slab

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(2) A. Roussopoulos: Calcul Des Construction Hyperstatiques A plusieurs etages solicitees Par Des Effort Obliques et excentrees. Publisher: Association Francaise de Recherches et d'essais Sur Les Materieux et Les Constructions.

counting from the ground up.

The equations of displacements of the points C_i will then be:

$$
\begin{bmatrix} u_n - F(t) \\ \vdots \\ u_i - F(t) \\ \vdots \\ u_1 - F(t) \\ u_n - G(t) \\ \vdots \\ v_i - G(t) \\ \vdots \\ v_1 - G(t) \\ \varphi_n - H(t) \\ \vdots \\ \varphi_i - H(t) \\ \vdots \\ \varphi_1 - H(t) \end{bmatrix}
= -
\begin{bmatrix}
d_{nn}^{11} \cdots d_{ni}^{11} \cdots d_{n1}^{11} & d_{nn}^{12} \cdots d_{ni}^{12} \cdots d_{n1}^{12} & d_{nn}^{13} \cdots d_{ni}^{13} \cdots d_{n1}^{13} \\
\vdots & \vdots & \vdots \\
d_{in}^{11} \cdots d_{ii}^{11} \cdots d_{i1}^{11} & d_{in}^{12} \cdots d_{ii}^{12} \cdots d_{i1}^{12} & d_{in}^{13} \cdots d_{ii}^{13} \cdots d_{i1}^{13} \\
\vdots & \vdots & \vdots \\
d_{1n}^{11} \cdots d_{1i}^{11} \cdots d_{11}^{11} & d_{1n}^{12} \cdots d_{1i}^{12} \cdots d_{11}^{12} & d_{1n}^{13} \cdots d_{1i}^{13} \cdots d_{11}^{13} \\
d_{nn}^{21} \cdots d_{ni}^{21} \cdots d_{n1}^{21} & d_{nn}^{22} \cdots d_{ni}^{22} \cdots d_{n1}^{22} & d_{nn}^{23} \cdots d_{ni}^{23} \cdots d_{n1}^{23} \\
\vdots & \vdots & \vdots \\
d_{in}^{21} \cdots d_{ii}^{21} \cdots d_{i1}^{21} & d_{in}^{22} \cdots d_{ii}^{22} \cdots d_{i1}^{22} & d_{in}^{23} \cdots d_{ii}^{23} \cdots d_{i1}^{23} \\
\vdots & \vdots & \vdots \\
d_{1n}^{21} \cdots d_{1i}^{21} \cdots d_{11}^{21} & d_{1n}^{22} \cdots d_{1i}^{22} \cdots d_{11}^{22} & d_{1n}^{23} \cdots d_{1i}^{23} \cdots d_{11}^{23} \\
d_{nn}^{31} \cdots d_{ni}^{31} \cdots d_{n1}^{31} & d_{nn}^{32} \cdots d_{ni}^{32} \cdots d_{n1}^{32} & d_{nn}^{33} \cdots d_{ni}^{33} \cdots d_{n1}^{33} \\
\vdots & \vdots & \vdots \\
d_{in}^{31} \cdots d_{ii}^{31} \cdots d_{i1}^{31} & d_{in}^{32} \cdots d_{ii}^{32} \cdots d_{i1}^{32} & d_{in}^{33} \cdots d_{ii}^{33} \cdots d_{i1}^{33} \\
\vdots & \vdots & \vdots \\
d_{1n}^{31} \cdots d_{1i}^{31} \cdots d_{11}^{31} & d_{1n}^{32} \cdots d_{1i}^{32} \cdots d_{11}^{32} & d_{1n}^{33} \cdots d_{1i}^{33} \cdots d_{11}^{33}
\end{bmatrix}
\begin{bmatrix} W_n \ddot{u}_n \\ \vdots \\ W_i \ddot{u}_i \\ \vdots \\ W_1 \ddot{u}_1 \\ W_n \ddot{v}_n \\ \vdots \\ W_i \ddot{v}_i \\ \vdots \\ W_1 \ddot{v}_1 \\ I_n \ddot{\varphi}_n \\ \vdots \\ I_i \ddot{\varphi}_i \\ \vdots \\ I_1 \ddot{\varphi}_1 \end{bmatrix}
\qquad (1)
$$

The analysis of dynamic response necessitates the determination of the frequencies and shapes of the various modes of free vibrations of the structure (the homogenuous solution), and the determination of the dynamic load factors and equivalent static loads which will correspond to the forced vibrations caused by the earth's tremor. This is achieved along established procedures of analysis separately for each component of earth tremor. The complete effect of the tremor can then be obtained by supperposition.

3. Free Vibrations.

Considering that by Maxwell's theorem $d_{ik}^{mn} = d_{ki}^{nm}$, and introducing the following symbols:

$$
\left.
\begin{aligned}
&a_{ik}^2 = d_{ik}^{11}, \quad b_{ik}^2 = d_{ik}^{22}, \quad c_{ik}^2 = d_{ik}^{33}, \\
&e_{ik}^2 = d_{ik}^{12}, \quad f_{ik}^2 = d_{ik}^{13}, \quad g_{ik}^2 = d_{ik}^{23}, \\
\text{and} \quad &U_i = W_i u_i, \quad V_i = W_i v_i, \quad \Phi_i = I_i \varphi_i
\end{aligned}
\right\} \qquad (2)
$$

where $i, k = 1, 2, 3, \ldots n$,

and that for the case of free vibrations: $F(t) = G(t) = H(t) = 0$,

the displacement equations (1) become for the case of natural vibration:

$$
\begin{bmatrix} \frac{U_n}{W_n} \\ \vdots \\ \frac{U_i}{W_i} \\ \vdots \\ \frac{U_1}{W_1} \\ \frac{V_n}{W_n} \\ \vdots \\ \frac{V_i}{W_i} \\ \vdots \\ \frac{V_1}{W_1} \\ \frac{\Phi_n}{I_n} \\ \vdots \\ \frac{\Phi_i}{I_i} \\ \vdots \\ \frac{\Phi_1}{I_1} \end{bmatrix}
= -
\begin{bmatrix}
a^2_{nn} & a^2_{ni} & a^2_{n1} & e^2_{nn} & e^2_{ni} & e^2_{n1} & f^2_{nn} & f^2_{ni} & f^2_{n1} \\
a^2_{in} & a^2_{ii} & a^2_{i1} & e^2_{in} & e^2_{ii} & e^2_{i1} & f^2_{in} & f^2_{ii} & f^2_{i1} \\
a^2_{1n} & a^2_{1i} & a^2_{11} & e^2_{1n} & e^2_{1i} & e^2_{11} & f^2_{1n} & f^2_{1i} & f^2_{11} \\
e^2_{nn} & e^2_{ni} & e^2_{n1} & b^2_{nn} & b^2_{ni} & b^2_{n1} & g^2_{nn} & g^2_{ni} & g^2_{n1} \\
e^2_{in} & e^2_{ii} & e^2_{i1} & b^2_{in} & b^2_{ii} & b^2_{i1} & g^2_{in} & g^2_{ii} & g^2_{i1} \\
e^2_{1n} & e^2_{1i} & e^2_{11} & b^2_{1n} & b^2_{1i} & b^2_{11} & g^2_{1n} & g^2_{1i} & g^2_{11} \\
f^2_{nn} & f^2_{ni} & f^2_{n1} & g^2_{nn} & g^2_{ni} & g^2_{n1} & c^2_{nn} & c^2_{ni} & c^2_{n1} \\
f^2_{in} & f^2_{ii} & f^2_{i1} & g^2_{in} & g^2_{ii} & g^2_{i1} & c^2_{in} & c^2_{ii} & c^2_{i1} \\
f^2_{1n} & f^2_{1i} & f^2_{11} & g^2_{1n} & g^2_{1i} & g^2_{11} & c^2_{1n} & c^2_{1i} & c^2_{11}
\end{bmatrix}
\begin{bmatrix} \ddot{U}_n \\ \vdots \\ \ddot{U}_i \\ \vdots \\ \ddot{U}_1 \\ \ddot{V}_n \\ \vdots \\ \ddot{V}_i \\ \vdots \\ \ddot{V}_1 \\ \ddot{\Phi}_n \\ \vdots \\ \ddot{\Phi}_i \\ \vdots \\ \ddot{\Phi}_1 \end{bmatrix}
\qquad (3)
$$

Obviously the matrix of the coefficients of this system of equations is symmetric. General solutions of the following type are attempted:

$$
\left.
\begin{aligned}
U(t) &= A \cos(\omega t + \alpha) \\
V(t) &= B \cos(\omega t + \alpha) \\
\Phi(t) &= C \cos(\omega t + \alpha)
\end{aligned}
\right\} \qquad (4)
$$

For such solutions: $\ddot{Z} = -\omega^2 Z \qquad (Z = U, V, \Phi)$.

The introduction of functions (4) in (3) gives:

$$
\begin{bmatrix}
\left(\frac{1}{W_n} - a^2_{nn}\omega^2\right) \cdots -a^2_{n1}\omega^2 & -e^2_{nn}\omega^2 \cdots -e^2_{n1}\omega^2 & -f^2_{nn}\omega^2 \cdots -f^2_{n1}\omega^2 \\
\vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\
-a^2_{n1}\omega^2 \cdots \left(\frac{1}{W_1} - a^2_{11}\omega^2\right) & -e^2_{n1}\omega^2 \cdots -e^2_{11}\omega^2 & -f^2_{n1}\omega^2 \cdots -f^2_{11}\omega^2 \\
-e^2_{nn}\omega^2 \cdots -e^2_{n1}\omega^2 & \left(\frac{1}{W_n} - b^2_{nn}\omega^2\right) \cdots -b^2_{n1}\omega^2 & -g^2_{nn}\omega^2 \cdots -g^2_{n1}\omega^2 \\
\vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\
-e^2_{n1}\omega^2 \cdots -e^2_{11}\omega^2 & -b^2_{n1}\omega^2 \cdots \left(\frac{1}{W_1} - b^2_{11}\omega^2\right) & -g^2_{n1}\omega^2 \cdots -g^2_{11}\omega^2 \\
-f^2_{nn}\omega^2 \cdots -f^2_{n1}\omega^2 & -g^2_{nn}\omega^2 \cdots -g^2_{n1}\omega^2 & \left(\frac{1}{I_n} - c^2_{nn}\omega^2\right) \cdots -c^2_{n1}\omega^2 \\
\vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\
-f^2_{n1}\omega^2 \cdots -f^2_{11}\omega^2 & -g^2_{n1}\omega^2 \cdots -g^2_{11}\omega^2 & -c^2_{n1}\omega^2 \cdots \left(\frac{1}{I_1} - c^2_{11}\omega^2\right)
\end{bmatrix}
\begin{bmatrix} U_n \\ \vdots \\ U_1 \\ V_n \\ \vdots \\ V_1 \\ \Phi_n \\ \vdots \\ \Phi_1 \end{bmatrix}
= 0 \qquad (5)
$$

To obtain the non-trivial solution of the system of equations (5), the determinant of the coefficients matrix is equated to zero. From this condition a 3n-degree polynomial of (ω^2) is obtained, the 3n solutions of which give the 3n natural frequencies of vibration. For each mode of free vibration (σ) there is a solution (4), and hence the complete solution of the free vibration problem is given by:

$$\left.\begin{aligned} U_i(t) &= \sum_{\sigma=1}^{3n} A_{i\sigma}\cos(\omega_\sigma t+\alpha_\sigma) \\ V_i(t) &= \sum_{\sigma=1}^{3n} B_{i\sigma}\cos(\omega_\sigma t+\alpha_\sigma) \\ \phi_i(t) &= \sum_{\sigma=1}^{3n} C_{i\sigma}\cos(\omega_\sigma t+\alpha_\sigma) \end{aligned}\right\} \qquad (6)$$

The amplitudes of the vibrations A_i, B_i, C_i can be obtained for each mode in terms of A_1 by solving the system of any $3(n-1)$ set of equations (5) for the unknowns A_i/A_1, B_i/A_1, C_i/A_1 and the value of ω_i obtained as above.

In practice the amplitudes as well as the frequencies of natural vibrations can also be obtained, in a more convenient probably manner, by the Vianello-Stodola method. Either way, it is considered that the amplitudes of the free vibrations (6) are obtainable and known. It is further assumed that A_i, B_i, C_i are the normalized amplitudes of free vibrations and that therefore they obey the normalizing condition:

$$\sum_{i=1}^{n}\left[\frac{1}{W_i}\left(A_{i\sigma}^2+B_{i\sigma}^2\right)+\frac{1}{I_i}C_{i\sigma}^2\right]=1 \qquad (7)$$

The normalized characteristic loads are introduced:

$$\begin{aligned} \mathcal{L}_{i\sigma}^1 &= \omega_\sigma^2 A_{i\sigma} \qquad (i=1,2,3,\dots n \\ \mathcal{L}_{i\sigma}^2 &= \omega_\sigma^2 B_{i\sigma} \qquad \sigma=1,2,3\dots 3n) \\ \mathcal{L}_{i\sigma}^3 &= \omega_\sigma^2 C_{i\sigma} \end{aligned} \qquad (8)$$

where $\mathcal{L}_{i\sigma}^1$, $\mathcal{L}_{i\sigma}^2$, $\mathcal{L}_{i\sigma}^3$ are considered acting respectively in the directions 1, 2, and 3.

Consider now the identity:

$$\sum_{k=1}^{n}\left(\mathcal{L}_{k\sigma}^1 d_{ki}^{11}+\mathcal{L}_{k\sigma}^2 d_{ki}^{12}+\mathcal{L}_{k\sigma}^3 d_{ki}^{13}\right)=\omega_\sigma^2\sum_{k=1}^{n}\left(A_{k\sigma}a_{ki}^2+B_{k\sigma}e_{ki}^2+C_{k\sigma}f_{ki}^2\right)=\frac{A_{i\sigma}}{W_i}=max\ u_{i\sigma} \qquad (9)$$

Similar expressions to (9) can be obtained for all $A_{i\sigma}$, $B_{i\sigma}$, $C_{i\sigma}$ from the system of equations (5). They prove that the shape of the displacements corresponding to the mode of vibration σ can be obtained by applying statically on the structure the characteristic loads $\mathcal{L}_{i\sigma}^r$ $(r=1,2,3)$. Indeed:

$\frac{A_{i\sigma}}{W_i}, \frac{B_{i\sigma}}{W_i}, \frac{C_{i\sigma}}{I_i} = \max(u_{i\sigma}, v_{i\sigma}, \varphi_{i\sigma})$ = max. displacement components of point C_i for the σ mode of vibration.

According to this observation, the characteristic loads $\mathcal{L}_{im}^k$ $(k=1,2,3)$ produce $\max(u_{im}, v_{im}, \varphi_{im})$, and the characteristic loads $\mathcal{L}_{in}^k$ $(k=1,2,3)$ produce $\max(u_{in}, v_{in}, \varphi_{in})$.

Therefore, applying Betti's Law for the energies produced by the $\mathcal{L}_{im}^k$

loads during the deformation caused by the $\mathcal{L}_{is}^k$ loads and vice versa:

$$\sum_{i=1}^{n}\left(\mathcal{L}_{im}^1 \frac{A_{is}}{W_i}+\mathcal{L}_{im}^2 \frac{B_{is}}{W_i}+\mathcal{L}_{im}^3 \frac{C_{is}}{I_i}\right)=\sum_{i=1}^{n}\left(\mathcal{L}_{is}^1 \frac{A_{im}}{W_i}+\mathcal{L}_{is}^2 \frac{B_{im}}{W_i}+\mathcal{L}_{is}^3 \frac{C_{im}}{I_i}\right)$$

or introducing the values of $\mathcal{L}_{ik}^k$ given by (8):

$$\sum_{i=1}^{n}\omega_s^2\left(\frac{A_{im}A_{is}}{W_i}+\frac{B_{im}B_{is}}{W_i}+\frac{C_{im}C_{is}}{I_i}\right)=\sum_{i=1}^{n}\omega_m^2\left(\frac{A_{is}A_{im}}{W_i}+\frac{B_{is}B_{im}}{W_i}+\frac{C_{is}C_{im}}{I_i}\right)$$

or $$(\omega_s^2-\omega_m^2)\sum_{i=1}^{n}\left[\frac{1}{W_i}(A_{im}A_{is}+B_{im}B_{is})+\frac{1}{I_i}C_{im}C_{is}\right]=0$$

For the case $m=s$, the multiplicant of $(\omega_s^2-\omega_m^2)=0$ is equal to the unity because it represents the normalizing condition (7).

For the case $s \neq m$, $(\omega_s^2-\omega_m^2)\neq 0$, and therefore the orthogomality condition is established:

$$\sum_{i=1}^{n}\left[\frac{1}{W_i}(A_{im}A_{is}+B_{im}B_{is})+\frac{1}{I_i}C_{im}C_{is}\right]=0 \tag{10}$$

4. Forced Vibrations.

The forced vibrations are caused by the earth's tremor, defined by the functions F(t), G(t), and H(t). Normally the tremor is caused by the propagation of a shock wave which has a linear direction and no rotational component H(t). This would imply a condition $F(t)=F(t)$, $G(t)=\lambda F(t)$ (where λ is a constant) and $H(t)=0$. Here, a solution will be worked outfor this case corresponding to an earth's tremor along a given direction defined by the slope λ to the axis 1 of the coordinate system 0 (1,2,3) and by the function F(t). The general case, which is theoretically valid,even though it may not represent the physical problem, can be obtained easily by the procedure to be defined in the discussion which follows the derivation below.

Since the tremor is a physical phenomenon, the function F(t) is continuous, has no points at infinity and has a finite number of maxima and minima points. Therefore F(t) is always expendable into a Fourier series:

$$F(t)=K_o\sum_{m=o}^{\infty}\frac{K_m}{K_o}\cos(mpt+\rho_m) \tag{11}$$

where K_o has dimensions of length.

It can be assumed, without loss to generality, that $t_o=0$, and that $F(t_o)=F(0)=\sum_{m=o}^{\infty}K_m\cos\rho_m = 0$.

Define the movement of the center C_i relative to the ground with w_i, v_i, ψ_i, so that:

$$\left.\begin{aligned} w_i &= \frac{U_i}{W_i} - F(t) = u_i - F(t)\\ v_i &= \frac{V_i}{W_i} - G(t) = v_i - \lambda F(t)\\ \psi_i &= \frac{\Phi_i}{I_i} - 0 = \varphi_i \end{aligned}\right\} \tag{12}$$

With the introduction of these symbols and relations (2), the displacement equations (1) become:

$$
\begin{bmatrix} w_n \\ \vdots \\ w_1 \\ v_n \\ \vdots \\ v_1 \\ \psi_n \\ \vdots \\ \psi_1 \end{bmatrix} = -\begin{bmatrix} a^2_{nn} \dots a^2_{n1} & e^2_{nn} \dots e^2_{n1} & f^2_{nn} \dots f^2_{n1} \\ \vdots & \vdots & \vdots \\ a^2_{n1} \dots a^2_{11} & e^2_{n1} \dots e^2_{11} & f^2_{n1} \dots f^2_{11} \\ e^2_{nn} \dots e^2_{n1} & b^2_{nn} \dots b^2_{n1} & g^2_{nn} \dots g^2_{n1} \\ \vdots & \vdots & \vdots \\ e^2_{n1} \dots e^2_{11} & b^2_{n1} \dots b^2_{11} & g^2_{n1} \dots g^2_{11} \\ f^2_{nn} \dots f^2_{n1} & g^2_{nn} \dots g^2_{n1} & c^2_{nn} \dots c^2_{n1} \\ \vdots & \vdots & \vdots \\ f^2_{n1} \dots f^2_{11} & g^2_{n1} \dots g^2_{11} & c^2_{n1} \dots c^2_{11} \end{bmatrix} \begin{bmatrix} W_n \ddot{w}_n + W_n \ddot{F}(t) \\ \vdots \\ W_1 \ddot{w}_1 + W_n F(t) \\ W_n \ddot{v}_n + W_n \lambda F(t) \\ \vdots \\ W_1 \ddot{v}_1 + W_1 \lambda F(t) \\ I_n \ddot{\psi}_n \\ \vdots \\ I_1 \ddot{\psi}_1 \end{bmatrix} \tag{13}
$$

From (11) : $\ddot{F}(t) = -K_o\rho^2 \sum_{m=0}^{\infty} \frac{K_m}{K_o} m^2 \cos(m\rho t + \rho_m) = -K_o\rho^2 f(t)$

where:
$$f(t) = \sum_{m=0}^{\infty} \frac{K_m}{K_o} m^2 \cos(m\rho t + \rho_m). \tag{14}$$

Introduce the virtual loads:

$$P_i = W_i K_o \rho^2 = W_i \frac{\ddot{F}(t)}{f(t)} \qquad (i=1, 2, 3, \dots n) \tag{15}$$

Determine the participation factors $\phi_\sigma (\sigma=1, 2, \dots 3n)$ fron the system of equations:

$$
\left.\begin{aligned}
P_i = W_i K_o \rho^2 &= \sum_{\sigma=1}^{3n} \phi_\sigma \alpha^1_{i\sigma} = W_i \frac{\ddot{F}(t)}{f(t)} \\
\lambda P_i = \lambda W_i K_o \rho^2 &= \sum_{\sigma=1}^{3n} \phi_\sigma \alpha^2_{i\sigma} = W_i \lambda \frac{F(t)}{f(t)} \\
0 &= \sum_{\sigma=1}^{3n} \phi_\sigma \alpha^3_{i\sigma} = 0
\end{aligned}\right\} \tag{16}
$$

The virtual work corresponding to the loads P_i and λP_i and the displacements caused by the σ mode of free vibration will be:

$$W_\sigma = \sum_{i=1}^{n} P_i \left(\frac{A_{i\sigma} + \lambda B_{i\sigma}}{W_i} + 0 \cdot C_{i\sigma} \right)$$

or: $$W_\sigma = \sum_{i=1}^{n} \left(\frac{A_{i\sigma}}{W_i} \sum_{\sigma=1}^{3n} \phi \alpha^1_{i\sigma} \right) + \sum_{i=1}^{n} \left(\frac{B_{i\sigma}}{W_i} \sum_{\sigma=1}^{3n} \phi_\sigma \alpha^2_{i\sigma} \right) + \sum_{i=1}^{n} \left(\frac{C_{i\sigma}}{I_i} \sum_{\sigma=1}^{3n} \phi_\sigma \alpha^3_{i\sigma} \right)$$

or in view of the orthogonality condition (10): $W_\sigma = \phi_\sigma \omega_\sigma^2$

Therefore:

$$\phi_\sigma \omega_\sigma^2 = \sum_{i=1}^{n} \left(P_i \frac{A_{i\sigma}}{W_i} + \lambda P_i \frac{B_{i\sigma}}{W_i} \right)$$

or $$\phi_\sigma = \frac{1}{\omega_\sigma^2} \sum_{i=1}^{n} \frac{P_i}{W_i} (A_{i\sigma} + \lambda B_{i\sigma}). \tag{17}$$

Solutions of the following type are attempted:

$$
\begin{aligned}
w_i &= \sum_{\sigma=1}^{3n} \mathcal{D}_\sigma(t)\, \phi_\sigma \frac{A_{i\sigma}}{W_i} \\
v_i &= \sum_{\sigma=1}^{3n} \mathcal{D}_\sigma(t)\, \phi_\sigma \frac{B_{i\sigma}}{W_i} \\
\psi_i &= \sum_{\sigma=1}^{3n} \mathcal{D}_\sigma(t)\, \phi_\sigma \frac{C_{i\sigma}}{I_i}
\end{aligned}
\tag{18}
$$

where $\mathcal{D}_\sigma(t)$ represents the dynamic load factor of the σ mode of vibration.

If solutions (18) really exist, then:

$$
\begin{aligned}
\ddot{w}_i &= \sum_{\sigma=1}^{3n} \ddot{\mathcal{D}}_\sigma \phi_\sigma \frac{A_{i\sigma}}{W_i} \\
\ddot{v}_i &= \sum_{\sigma=1}^{3n} \ddot{\mathcal{D}}_\sigma \phi_\sigma \frac{B_{i\sigma}}{W_i} \\
\ddot{\psi}_i &= \sum_{\sigma=1}^{3n} \ddot{\mathcal{D}}_\sigma \phi_\sigma \frac{C_{i\sigma}}{I_i}
\end{aligned}
\tag{19}
$$

Introducing (15), (16), (18), and (19) in the first n equations of the system of equations (13) gives:

$$
\begin{aligned}
\sum_{\sigma=1}^{3n} \mathcal{D}_\sigma \phi_\sigma \frac{A_{i\sigma}}{W_i} = &-\sum_{k=1}^{n} a_{ik}^2 \Big(\sum_{\sigma=1}^{3n} \ddot{\mathcal{D}}_\sigma \phi_\sigma A_{k\sigma}\Big) - \sum_{k=1}^{n} e_{ik}^2 \Big(\sum_{\sigma=1}^{3n} \ddot{\mathcal{D}}_\sigma \phi_\sigma B_{k\sigma}\Big) - \sum_{k=1}^{n} f_{ik}^2 \Big(\sum_{\sigma=1}^{3n} \ddot{\mathcal{D}}_\sigma \phi_\sigma C_{k\sigma}\Big) + \\
&+ \sum_{k=1}^{n} a_{ki}^2 \Big(f(t) \sum_{\sigma=1}^{3n} \phi_\sigma \alpha_{k\sigma}^1\Big) + \sum_{k=1}^{n} e_{ki}^2 \Big(f(t) \sum_{\sigma=1}^{3n} \phi_\sigma \alpha_{k\sigma}^2\Big) + \sum_{k=1}^{n} f_{ki}^2 \Big(f(t) \sum_{\sigma=1}^{3n} \phi_\sigma \alpha_{k\sigma}^3\Big)
\end{aligned}
\tag{20}
$$

or

$$
\sum_{\sigma=1}^{3n} \mathcal{D}_\sigma \phi_\sigma \frac{A_{i\sigma}}{W_i} = -\sum_{\sigma=1}^{3n} \ddot{\mathcal{D}}_\sigma \phi_\sigma \sum_{k=1}^{n} \big(a_{ik}^2 A_{k\sigma} + e_{ik}^2 B_{k\sigma} + f_{ik}^2 C_{k\sigma}\big) + f(t) \sum_{\sigma=1}^{3n} \phi_\sigma \Big[\sum_{k=1}^{n} \big(a_{ik}^2 \alpha_{k\sigma}^1 + e_{ik}^2 \alpha_{k\sigma}^2 + f_{ik}^2 \alpha_{k\sigma}^3\big)\Big]
\tag{21}
$$

But in accordance with (9):

$$
\sum_{k=1}^{n} \big(a_{ik}^2 \alpha_{k\sigma}^1 + e_{ik}^2 \alpha_{k\sigma}^2 + f_{ik}^2 \alpha_{k\sigma}^3\big) = \omega_\sigma^2 \sum_{k=1}^{n} \big(a_{ik}^2 A_{k\sigma} + e_{ik}^2 B_{k\sigma} + f_{ik}^2 C_{k\sigma}\big)
\tag{22}
$$

Therefore, after introducing (22) into (21):

$$
\sum_{\sigma=1}^{3n} \mathcal{D}_\sigma \phi_\sigma \frac{A_{i\sigma}}{W_i} = -\sum_{\sigma=1}^{3n} \Big[\ddot{\mathcal{D}}_\sigma \phi_\sigma \frac{A_{i\sigma}}{W_i \omega_\sigma^2} - f(t)\, \phi_\sigma \frac{A_{i\sigma}}{W_i}\Big]
$$

or

$$
\sum_{\sigma=1}^{3n} \phi_\sigma \Big[\mathcal{D}_\sigma A_{i\sigma} - f(t) A_{i\sigma} + \ddot{\mathcal{D}}_\sigma \frac{A_{i\sigma}}{\omega_\sigma^2}\Big] = 0 \qquad (i = 1, 2, \ldots, n)
\tag{23}
$$

Similarly:

$$
\sum_{\sigma=1}^{3n} \phi_\sigma \Big[\mathcal{D}_\sigma B_{i\sigma} - f(t) B_{i\sigma} + \ddot{\mathcal{D}}_\sigma \frac{B_{i\sigma}}{\omega_\sigma^2}\Big] = 0
\tag{24}
$$

and

$$
\sum_{\sigma=1}^{3n} \phi_\sigma \Big[\mathcal{D}_\sigma C_{i\sigma} - f(t) C_{i\sigma} + \ddot{\mathcal{D}}_\sigma \frac{C_{i\sigma}}{\omega_\sigma^2}\Big] = 0
\tag{25}
$$

Multiply all the terms of the 3n equations (23), (24), (25) respectively with A_{ip}/W_i, B_{ip}/W_i, C_{ip}/I_i (where p is any given integer $1 \leq p \leq n$) and add the respective parts of the equations together:

$$
\sum_{\sigma=1}^{3n} \phi_\sigma \Big[\mathcal{D}_\sigma \sum_{i=1}^{n} \Big(\frac{A_{ip}A_{i\sigma} + B_{ip}B_{i\sigma}}{W_i} + \frac{C_{ip}C_{i\sigma}}{I_i}\Big) - f(t) \sum_{i=1}^{n} \Big(\frac{A_{ip}A_{i\sigma} + B_{ip}B_{i\sigma}}{W_i} + \frac{C_{ip}C_{i\sigma}}{I_i}\Big) + \frac{\ddot{\mathcal{D}}_\sigma}{\omega_\sigma^2} \sum_{i=1}^{n} \Big(\frac{A_{ip}A_{i\sigma} + B_{ip}B_{i\sigma}}{W_i} + \frac{C_{ip}C_{i\sigma}}{I_i}\Big)\Big]
\tag{26}
$$

But according to the normalizing condition (7) and the orthogonality conditions (10):

$$\sum_{i=1}^{n}\left(\frac{A_{i\sigma}A_{ip}+B_{i\sigma}B_{ip}}{W_i}+\frac{C_{i\sigma}C_{ip}}{I_i}\right)=\delta_{\sigma p}$$

where $\delta_{\sigma p}$ is the Kronecker delta.

Therefore equation (26) above is reduced to:

$$\mathcal{D}_\sigma(t)-f(t)+\frac{1}{\omega_p^2}\ddot{\mathcal{D}}_\sigma(t)=0 \tag{27}$$

The solution of the ordinary differential equations (27) provide the dynamic load factors $\mathcal{D}_\sigma(t)$ for all modes of vibration p and verify the existance of solutions of the type (18).

5. Solution of The Differential Equations (27) for $\mathcal{D}_\sigma(t)$.

By multiplying each term by $\sin\omega_p t$ and adding and subtracting the term $\omega_p\dot{\mathcal{D}}_p\cos\omega_p t$, equation (27) is transformed into:

$$\ddot{\mathcal{D}}_\sigma\sin\omega_p t+(\omega_p\dot{\mathcal{D}}_p\cos\omega_p t-\omega_p\dot{\mathcal{D}}_p\cos\omega_p t)+\omega_p^2\mathcal{D}_p\sin\omega_p t=\omega_p^2 f(t)\sin\omega_p t$$

This equations can be rewritten as:

$$\frac{d}{dt}(\dot{\mathcal{D}}_p\sin\omega_p t)-\frac{d}{dt}(\omega_p\mathcal{D}_p\cos\omega_p t)=\omega_p^2 f(t)\sin\omega_p t$$

and by integrating in the interval (0,t):

$$\dot{\mathcal{D}}_p(t)\sin\omega_p t-0-\omega_p\mathcal{D}_p(t)\cos\omega_p t+\omega_p\mathcal{D}_p(0)=\omega_p^2\int_0^t f(t')\sin\omega_p t'\,dt' \tag{28}$$

Similarly it is easy to obtain:

$$\dot{\mathcal{D}}_p(t)\cos\omega_p t-\dot{\mathcal{D}}_p(0)+\omega_p\mathcal{D}_p(t)\sin\omega_p t-0=\omega_p^2\int_0^t f(t')\cos\omega_p t'\,dt' \tag{29}$$

Multiply the terms of (28) by ($-\cos\omega_p t$) and the terms of (29) by $\sin\omega_p t$ and add to obtain:

$$-\dot{\mathcal{D}}_p(t)\sin\omega_p t-\omega_p\mathcal{D}_p(0)\cos\omega_p t+\omega_p\mathcal{D}_p(t)=\omega_p^2\int_0^t f(t')\sin\omega_p(t-t')\,dt'$$

Solving for $\mathcal{D}_p(t)=\mathcal{D}_p$:

$$\mathcal{D}_p=\frac{1}{\omega_p}\dot{\mathcal{D}}_p(0)\sin\omega_p t+\mathcal{D}_p(0)\cos\omega_p t+\omega_p^2\int_0^t f(t')\sin\omega_p(t-t')\,dt' \tag{30}$$

Equations (30) (for $p=1,2,\ldots 3n$), determine all the dynamic load factors $\mathcal{D}_p$ provided the values of $\mathcal{D}_p(0)=\mathcal{D}_{p_0}$ and $\dot{\mathcal{D}}_p(0)=\dot{\mathcal{D}}_{p_0}$ are known. These values are obtained as follows:

By definition:

$$w_i(t=0)=w_{io}=\sum_{\sigma=1}^{3n}\mathcal{D}_{\sigma o}\oint_\sigma\frac{A_{i\sigma}}{W_i} \qquad \dot{w}_i(t=0)=\dot{w}_{io}=\sum_{\sigma=1}^{3n}\dot{\mathcal{D}}_{\sigma o}\oint_\sigma\frac{A_{i\sigma}}{W_i}$$

$$v_i(t=0)=v_{io}=\sum_{\sigma=1}^{3n}\mathcal{D}_{\sigma o}\oint_\sigma\frac{B_{i\sigma}}{W_i} \qquad \dot{v}_i(t=0)=\dot{v}_{io}=\sum_{\sigma=1}^{3n}\dot{\mathcal{D}}_{\sigma o}\oint_\sigma\frac{B_{i\sigma}}{W_i}$$

$$\psi_i(t=0)=\psi_{io}=\sum_{\sigma=1}^{3n}\mathcal{D}_{\sigma o}\oint_\sigma\frac{C_{i\sigma}}{I_i} \qquad \dot{\psi}_i(t=0)=\dot{\psi}_{io}=\sum_{\sigma=1}^{3n}\dot{\mathcal{D}}_{\sigma o}\oint_\sigma\frac{C_{i\sigma}}{I_i}$$

Therefore:

$$w_{io}A_{ip}=\sum_{\sigma=1}^{3n}\mathcal{D}_{\sigma o}\oint_\sigma\frac{A_{i\sigma}A_{ip}}{W_i} \qquad \dot{w}_{io}A_{ip}=\sum_{\sigma=1}^{3n}\dot{\mathcal{D}}_{\sigma o}\oint_\sigma\frac{A_{i\sigma}A_{ip}}{W_i}$$

$$v_{io}B_{ip}=\sum_{\sigma=1}^{3n}\mathcal{D}_{\sigma o}\oint_\sigma\frac{B_{i\sigma}B_{ip}}{W_i} \qquad \dot{v}_{io}B_{ip}=\sum_{\sigma=1}^{3n}\dot{\mathcal{D}}_{\sigma o}\oint_\sigma\frac{B_{i\sigma}B_{ip}}{W_i}$$

$$\psi_{io}C_{ip}=\sum_{\sigma=1}^{3n}\mathcal{D}_{\sigma o}\oint_\sigma\frac{C_{i\sigma}C_{ip}}{I_i} \qquad \dot{\psi}_{io}C_{ip}=\sum_{\sigma=1}^{3n}\dot{\mathcal{D}}_{\sigma o}\oint_\sigma\frac{C_{i\sigma}C_{ip}}{I_i}$$

The addition of each set of these equations (for $i=1,2,\ldots n$) and the introduction of the orthogonality and normalizing conditions, renders:

$$\left.\begin{aligned}&\sum_{i=1}^{n}\left(w_{io}A_{ip}+v_{io}B_{ip}+\psi_{io}C_{ip}\right)=\oint_\sigma\mathcal{D}_{po}\\&\sum_{i=1}^{n}\left(\dot{w}_{io}A_{ip}+\dot{v}_{io}B_{ip}+\dot{\psi}_{io}C_{ip}\right)=\oint_\sigma\dot{\mathcal{D}}_{po}\end{aligned}\right\} \qquad (31)$$

Therefore, depending on the boundary conditions:

$w_i(0)$, $v_i(0)$, $\psi_i(0)$, $\dot{w}_i(0)$, $\dot{v}_i(0)$, and $\dot{\psi}_i(0)$,

both $\mathcal{D}_{po}$ and $\dot{\mathcal{D}}_{po}$ can be obtained from expressions (31). After they are introduced into (30), they fully define the dynamic load factors $\mathcal{D}_p(t)$. Thus for the usual boundary conditions :

$$u_i(0)=\upsilon_i(0)=\varphi_i(0)=u_i(0)=\upsilon_i(0)=\varphi_i(0)=0,$$

and in consideration of (11) and (12):

$$w_{io}=-F(0)=-\sum_{m=0}^{\infty}K_m\cos p_m=0 \qquad \dot{w}_{io}=\sum_{m=0}^{\infty}K_m m p\sin p_m$$

$$v_{io}=-\lambda F(0)=0 \qquad \dot{v}_{io}=\lambda\sum_{m=0}^{\infty}K_m m p\sin p_m$$

$$\psi_{io}=0 \qquad \dot{\psi}_{io}=0$$

Therefore:

$$\left.\begin{aligned}&\oint_\sigma\mathcal{D}_{po}=-\sum_{m=0}^{\infty}K_m\cos p_m\cdot\sum_{i=1}^{n}\left(A_{ip}+\lambda B_{ip}\right)=0\text{, and}\\&\oint_\sigma\dot{\mathcal{D}}_{po}=\sum_{m=0}^{\infty}K_m m p\sin p_m\cdot\sum_{i=1}^{n}\left(A_{ip}+\lambda B_{ip}\right).\end{aligned}\right\} \qquad (32)$$

6. Discussion and Design Procedure.

The most general solution to the dynamic problem of earthquake forces would of course involve the improbable pattern of an earth tremor with all three time functions $F(t)$, $G(t)$, and $H(t)$ different. In such a case for each natural mode of vibration σ, there would be three different participation coefficients $\mathcal{P}_{\sigma 1}$, $\mathcal{P}_{\sigma 2}$, $\mathcal{P}_{\sigma 3}$, and three corresponding dynamic load factors $\mathcal{D}_{\sigma 1}$, $\mathcal{D}_{\sigma 2}$, $\mathcal{D}_{\sigma 3}$. Each of these sets of $(\mathcal{P}_{\sigma 1}, \mathcal{D}_{\sigma 1})$, $(\mathcal{P}_{\sigma 2}, \mathcal{D}_{\sigma 2})$, $(\mathcal{P}_{\sigma 3}, \mathcal{D}_{\sigma 3})$ would be derived independently for the three cases $[F(t) \neq 0, G(t) = H(t) = 0]$, $[F(t) = H(t) = 0, G(t) \neq 0]$, $[F(t) = G(t) = 0, H(t) \neq 0]$.

The participation coefficients $\mathcal{P}$ would be obtained from equations equivalent to (17) and would be different in each case, because, depending on the component of the earth's tremor for which the contributing solution is attempted, the right-hand side of equation (17) would have only A_i, or only B_i, or only C_i terms.

The different values of $\mathcal{D}_\sigma$ would be obtained from (30). In this equation, the function $f(t')$ which appears under the integral, would vary with the component of the earth's tremor $F(t)$, $G(t)$, or $H(t)$ for which the solution would be saught. Thus there would be three such functions $f(t')$, $g(t')$, and $h(t')$, each corresponding to one of the components of the earth's tremor. For each of these functions there would be $3n$ different dynamic load factors $\mathcal{D}_\sigma$, one each for every natural frequency of vibration ω_σ.

Introducing now the following additional symbols:

$$
\begin{aligned}
w_{i1} &= u_{i1} - F(t) & v_{i1} &= \upsilon_{i1} & \psi_{i1} &= \varphi_{i1} \\
w_{i2} &= u_{i2} & v_{i2} &= \upsilon_{i2} - G(t) & \psi_{i2} &= \varphi_{i2} \\
w_{i3} &= u_{i3} & v_{i3} &= \upsilon_{i3} & \psi_{i3} &= \varphi_{i3} - H(t)
\end{aligned}
\tag{33}
$$

it follows that:

$$
\begin{aligned}
u_i &= \sum_{k=1}^{3} w_{ik} + F(t) \\
\upsilon_i &= \sum_{k=1}^{3} v_{ik} + G(t) \\
\varphi_i &= \sum_{k=1}^{3} \psi_{ik} + H(t)
\end{aligned}
\tag{34}
$$

In the practical problem of design, the structure is required to resist a certain tremor. The tremor is either taken from the seismograph of a catastrophic earthquake, or it is an idealized wave of a frequency and amplitude which are believed to be upper boundaries for earthquakes of a certain probability of occurence. Once the design tremor is defined in the form of a function $T(t)$, the designer is asked to assure the safety of the structure for any tremor specified by $T(t)$ acting in any direction horizontally.

To solve this design problem it is helpful to solve two component problems:

(1) $F(t) = T(t)$, and $G(t) = H(t) = 0$.

(2) $F(t) = H(t) = 0$, and $G(t) = T(t)$.

For each of these two tremors, the maximum deformation of vertical components of the structure (columns and shear walls) are determined. This can be achieved by determining the relative displacements of C_i with respect to $C_{(i-1)}$ with the help of expressions (18):

$$(w_i - w_{(i-1)}) = \sum_\sigma \mathcal{D}_\sigma \oint_\sigma \left(\frac{A_{\sigma i}}{W_i} - \frac{A_{\sigma(i-1)}}{W_{(i-1)}} \right)$$

$$(v_i - v_{(i-1)}) = \sum_\sigma \mathcal{D}_\sigma \oint_\sigma \left(\frac{B_{\sigma i}}{W_i} - \frac{B_{\sigma(i-1)}}{W_{(i-1)}} \right)$$

$$(\psi_i - \psi_{(i-1)}) = \sum_\sigma \mathcal{D}_\sigma \oint_\sigma \left(\frac{C_{\sigma i}}{I_i} - \frac{C_{\sigma(i-1)}}{I_{(i-1)}} \right)$$

and applying the procedures developed by Roussopoulos for determining the vectorial deformations $\bar{u}$ of the vertical stiffness elements (columns and shear walls) between the i and (i-1) storys of the structure.

The two component problems above will define two vectorial differential deformations $\bar{u}_1$ and $\bar{u}_2$ for each vertical stiffness element. These will be conjugate radii of the ellipse of deformations (maximum) of each vertical element. From these conjugate radii it is easy by geometric construction to define the principal diameters of respective ellipses of relative displacements of the ends of vertical elements, and hence define the maximum stresses likely to develop as a result of a given tremor T(t) acting in the most adverse direction for each individual vertical member.

The outlined procedure of analysis for design purposes obviously demands massive computations rapidly increasing in number and complexity with the number of storys in a building. To overcome this problem a computer is an absolute necessity. However, the computational effort can be substantially reduced in a high building (say a 30 story-building) if its mass is assumed lumped in distinct storys instead of each slab of the building. If say the mass is assumed lumped at every fifth slab, than the flexibility coefficents will be no more difficult to obtain by the Roussopoulos procedure. The resulting dynamic response will be less accurate, of course, but it will supply the engineer with a good understanding of the order of magnitude of the true natural frequencies of vibration as well as of the true dynamic load factors. This information when available to the designer to even a fair degree of accuracy will be very valuable in guiding the rational distribution of the mass and stiffness of the structure and the provision with extra strength and/or stiffness in critical points.

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