RESPONSE ANALYSIS OF FRAMED STRUCTURES

by

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FORWORD

This report treats a calculation method for the seismic response of the arbitrary framed structures which have hinges, plastic hinges in the members, and springs at the supporting points, and also states some calculated results by means of the author's own method.

NOMENCLATURE

First of all, we have to assign a number to each panel point of the given framed structure and each member of it respectively. This numbering is arbitrary, but once done, the sequence should always be observed. For this becomes the sequence of elements of all vectors and diagonal matrices.

Secondly in order to distinguish each other let us call both ends of each member "a" and "b" respectively and take b-bound direction as the member direction. But in the figures of this report "+" is adhered to the end of "a" instead of writing "a" for the temporary convenience.

a) Vector

Na, Sa, Ma:	Normal forces, shearing forces and moment at the end "a" of each member. (Fig. 2)
Nb, Sb, Mb:	Normal forces, shearing forces and moments at the end "b" of each member. (Fig. 2)
Nfa, Sfa, Mfa:	Fixed forces at each member's end "a", due to external loads.
Nfb, Sfb, Mfb:	Fixed forces at each member's end "b", due to external loads.
Na´, Sa´, Ma´:	Na - Nfa, Sa - Sfa, Ma - Mfa.
Nb´, Sb´, Mb´:	Nb - Nfb, Sb - Sfb, Mb - Mfb.
Ua, Va, θa:	Displacements of U, V directions and displacement angle of each member at the end "a" (Fig. 2).
Ub, Sb, θb:	Displacements of U, V directions and dis-

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placement angle of each member at the end " b" (Fig. 2).

H_p, V_p, M_p: External forces of X, Y directions and external moments at each panel point.

H_r, V_r, M_r: Reactions of X, Y directions and reaction moments at each supporting point.

X_p, Y_p, θ_p: Displacements of X, Y directions and displacement angle of each panel point.

X_r, Y_r, θ_r: Displacement of X, Y directions and rotational angle of each supporting point.

ε_{ea}, ε_{eb}: Rotational angle differences between the panel point and the member end "a" or "b".

ε_{xr}, ε_{yr}, ε_{er}: Displacements of X, Y directions and rotational angle at each supporting point, due to the plastic phenomena.

b) Diagonal Matrix

F, J, L: Cross sectional areas, sectional moduli and lengths of members.

X, Y: X and Y components of distance of each member, measured from "a" to "b".

λ_x, λ_y, λ_e: X and Y directions' spring constants and rotational spring constants of each supporting point's spring.

I: Unit matrix

c) Configuration Matrix

α, β: Matrices showing which member is connected with which panel point. For example, when the end "a" or "b" of member "k" is connected with the panel point "i", $\alpha_{ik} = 1$ or $\beta_{ik} = 1$, otherwise $\alpha_{ik} = 0$ or $\beta_{ik} = 0$.

γ: Matrix showing on what panel point the reaction is acting. Consider, for example, when the reaction "k" is acting on the panel point "i", $\gamma_{ik} = 1$, otherwise $\gamma_{ik} = 0$.

d) Others

E: Young's modulus

$\bar{A}$: Transposed one of matrix A

INDUCTION OF FUNDAMENTAL FORMULA

When we use Castgliano's theorem after having calculated the strain energy due to the external forces which is stored in all members of the structure, the following two equations will be obtained. And those indicates the relations either between the member forces and the member end displacements or between the reactions and the acting points' displacements of them.

$$\begin{pmatrix} N_a' \\ S_a' \\ M_a' \end{pmatrix} = \begin{bmatrix} EF/L, & 0 & , & 0 \\ 0 & , 12EJ/L^3, & 6EJ/L^2 \\ 0 & , 6EJ/L^2, & 4EJ/L \end{bmatrix} \begin{pmatrix} u_a - u_b \\ v_a - v_b + L\theta_b \\ \theta_a - \theta_b \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} H_r \\ V_r \\ M_r \end{pmatrix} = \begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_\theta \end{bmatrix} \begin{pmatrix} x_r \\ y_r \\ \theta_r \end{pmatrix} \tag{2}$$

Next just imagine that we cut all the members at the panel points and think of the equilibrium equations of each member and panel point. Then we will get the following two equations.

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -L & I \end{bmatrix} \begin{pmatrix} N_a' \\ S_a' \\ M_a' \end{pmatrix} + \begin{pmatrix} N_b' \\ S_b' \\ M_b' \end{pmatrix} = 0 \tag{3}$$

$$\begin{bmatrix} \alpha\frac{X}{L}, & -\alpha\frac{Y}{L}, & 0 \\ \alpha\frac{Y}{L}, & \alpha\frac{X}{L}, & 0 \\ 0, & 0, & \alpha \end{bmatrix} \begin{pmatrix} N_a \\ S_a \\ M_a \end{pmatrix} + \begin{bmatrix} \beta\frac{X}{L}, & -\beta\frac{Y}{L}, & 0 \\ \beta\frac{Y}{L}, & \beta\frac{X}{L}, & 0 \\ 0, & 0, & \beta \end{bmatrix} \begin{pmatrix} N_b \\ S_b \\ M_b \end{pmatrix} + \begin{bmatrix} \bar{\gamma} & 0 & 0 \\ 0 & \bar{\gamma} & 0 \\ 0 & 0 & \bar{\gamma} \end{bmatrix} \begin{pmatrix} H_r \\ V_r \\ M_r \end{pmatrix} = \begin{pmatrix} H_p \\ V_p \\ M_p \end{pmatrix} \tag{4}$$

Finally as we consider the compatible conditions, that is to say, the deformation of the member ends and supports are equal to the sum of the deformation of the corresponding panel points and the plastic deformation, we obtain

$$\begin{pmatrix} u_a \\ v_a \\ \theta_a \end{pmatrix} = \begin{bmatrix} \frac{X}{L}\bar{\alpha}, & \frac{Y}{L}\bar{\alpha}, & 0 \\ -\frac{Y}{L}\bar{\alpha}, & \frac{X}{L}\bar{\alpha}, & 0 \\ 0, & 0, & \bar{\alpha} \end{bmatrix} \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \varepsilon_{\theta a} \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} u_b \\ v_b \\ \theta_b \end{pmatrix} = \begin{bmatrix} \frac{X}{L}\bar{\beta}, & \frac{Y}{L}\bar{\beta}, & 0 \\ \frac{-Y}{L}\bar{\beta}, & \frac{X}{L}\bar{\beta}, & 0 \\ 0, & 0, & \bar{\beta} \end{bmatrix} \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \varepsilon_{\theta b} \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} x_r \\ y_r \\ \theta_r \end{pmatrix} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix} + \begin{pmatrix} \varepsilon_{xr} \\ \varepsilon_{yr} \\ \varepsilon_{\theta r} \end{pmatrix} \tag{7}$$

After having applied eqs. (5), (6) and (7) to the eqs. (1) and (2), if we refer to the eq. (3), the following result will be obtained.

$$\begin{pmatrix} N_a' \\ S_a' \\ M_a' \end{pmatrix} = \begin{bmatrix} EF/L, & 0, & 0 \\ 0, & 12EJ/L^3, & 0 \\ 0, & 6EJ/L^2, & 2EJ/L^2 \end{bmatrix} \begin{bmatrix} \frac{X}{L}\bar{\delta}, & \frac{Y}{L}\bar{\delta}, & 0 \\ \frac{-Y}{L}\bar{\delta}, & \frac{X}{L}\bar{\delta}, & \frac{L}{2}\bar{\mu} \\ 0, & 0, & \frac{L}{2}\bar{\delta} \end{bmatrix} \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix} + \begin{bmatrix} 0, & 0 \\ 6EJ/L^2, & 6EJ/L^2 \\ 4EJ/L, & 2EJ/L \end{bmatrix} \begin{pmatrix} \varepsilon_{\theta a} \\ \varepsilon_{\theta b} \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} N_b' \\ S_b' \\ M_b' \end{pmatrix} = \begin{bmatrix} -EF/L, & 0, & 0 \\ 0, & 12EJ/L^3, & 0 \\ 0, & 6EJ/L^2, & 2EJ/L^2 \end{bmatrix} \begin{bmatrix} \frac{X}{L}\bar{\delta}, & \frac{Y}{L}\bar{\delta}, & 0 \\ \frac{-Y}{L}\bar{\delta}, & \frac{X}{L}\bar{\delta}, & \frac{L}{2}\bar{\mu} \\ 0, & 0, & \frac{L}{2}\bar{\delta} \end{bmatrix} \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix} + \begin{bmatrix} 0, & 0 \\ 6EJ/L^2, & 6EJ/L^2 \\ 2EJ/L, & 4EJ/L \end{bmatrix} \begin{pmatrix} \varepsilon_{\theta a} \\ \varepsilon_{\theta b} \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} H_r \\ V_r \\ M_r \end{pmatrix} = \begin{bmatrix} \lambda_x\gamma, & 0, & 0 \\ 0, & \lambda_y\gamma, & 0 \\ 0, & 0, & \lambda_\theta\gamma \end{bmatrix} \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix} + \begin{bmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_\theta \end{bmatrix} \begin{pmatrix} \varepsilon_{xr} \\ \varepsilon_{yr} \\ \varepsilon_{\theta r} \end{pmatrix} \tag{10}$$

And the substitutions (4) for (8), (9) and (10) will be the following formula.

$$[\delta'][K'][\bar{\delta}'](\eta) + [\delta'][K'][\bar{\alpha}'](\varepsilon) = (Q_p') \tag{11}$$

Subsequently if we make use of the last rows of eqs. (8) and (9) and eq. (10), the result may be;

$$[\alpha'][K'][\bar{\delta}'](\eta) + [\alpha'][K'][\bar{\alpha}'](\varepsilon) = (Q_m) \tag{12}$$

where

$$[\delta'] = \begin{bmatrix} \delta\frac{X}{L}, & -\delta\frac{Y}{L}, & 0, & \bar{\gamma}, & 0, & 0 \\ \delta\frac{Y}{L}, & \delta\frac{X}{L}, & 0, & 0, & \bar{\gamma}, & 0 \\ 0, & \mu\frac{L}{2}, & \delta\frac{L}{2}, & 0, & 0, & \bar{\gamma} \end{bmatrix}$$

$$[d'] = \begin{bmatrix} 0, L/2, L/2, 0, 0, 0 \\ 0, L/2, -L/2, 0, 0, 0 \\ 0, 0, 0, I, 0, 0 \\ 0, 0, 0, 0, I, 0 \\ 0, 0, 0, 0, 0, I \end{bmatrix}, \quad [K'] = \begin{bmatrix} EF/L & & & & & \\ & 12EJ/L^3 & & & & \\ & & 4EJ/L^3 & & & \\ & & & \lambda x & & \\ & & & & \lambda y & \\ & & & & & \lambda\theta \end{bmatrix}$$

$$(\varepsilon) = \begin{pmatrix} \varepsilon_{\theta a} \\ \varepsilon_{\theta b} \\ \varepsilon_{xr} \\ \varepsilon_{yr} \\ \varepsilon_{\theta r} \end{pmatrix}, \quad (Q_m) = \begin{pmatrix} M'_a \\ M'_b \\ H_r \\ V_r \\ M_r \end{pmatrix}, \quad (\eta) = \begin{pmatrix} x_p \\ y_p \\ \theta_p \end{pmatrix}$$

$$(Q'_p) = \begin{pmatrix} H_p \\ V_p \\ M_p \end{pmatrix} - \begin{bmatrix} \alpha\frac{X}{L}, -\alpha\frac{Y}{L}, 0 \\ \alpha\frac{Y}{L}, \alpha\frac{X}{L}, 0 \\ 0, 0, \alpha \end{bmatrix} \begin{pmatrix} N_{fa} \\ S_{fa} \\ M_{fa} \end{pmatrix} - \begin{bmatrix} \beta\frac{X}{L}, -\beta\frac{Y}{L}, 0 \\ \beta\frac{Y}{L}, \beta\frac{X}{L}, 0 \\ 0, 0, \beta \end{bmatrix} \begin{pmatrix} N_{fb} \\ S_{fb} \\ M_{fb} \end{pmatrix}$$

Eqs. (11) and (12) are the common ones for the solution of the framed structures with plastic hinges. In case of framed structures with no plastic hinges, (ε) is zero. Therefore the eq. (11) will make as follows:

$$[\delta'][K'][\bar{\delta}'](\eta) = (Q'_p) \qquad (13)$$

The above is the common formula to solve the elastic displacements of framed structures.

If there are hinges at some member end "a" or "b", it is necessary for us to put zero into the corresponding elements of the diagonal matrix 12EJ/L or 4EJ/L .

When some supporting points are fixed vertically, horizontally or rotationally, the corresponding elements of the diagonal matrix λx, λy or $\lambda \theta$ will become infinite.

Hence the corresponding elements of the vector xp, yp, or θp will make zero. Therefore it is necessary for us to exclude the corresponding rows of matrix $[\delta']$ and the corresponding elements of λx, λy or $\lambda \theta$ and Xp, Yp or θp.

SOLUTION OF FRAMED STRUCTURES

Using eqs. (11) and (12), we can solve the framed structures which may have plastic hinges in their members. In case of the static problem, it is common that the external forces Qp is given and η, ε and Qm are to be estimated. And to attain the expected result we usually transform the eqs. (11) and (12). But in this report this problem is beyond discussion.

While in case of the dynamic problem, it is required to solve Qp and Qm for given η. For this purpose it is convenient to rewrite the eq. (12) as follows:

$$(Q_{me})=[d'][K'][\bar{\delta}'](\eta)+\{[d'][K'][\bar{d}']-[D]\}(\varepsilon) \qquad (14)$$

$$(Q_{mp})=[D](\varepsilon) \qquad (15)$$

$$(Q_{me})+(Q_{mp})=(Q_m)\leq(Q_Y) \qquad (16)$$

where

D: Diagonal matrix consisting of the diagonal elements of matrices $[d'][k'][\bar{d}']$.

Qme: Member forces under the condition that the plastic hinges do not increase in number.

Qmp: Additional member forces which are to be corrected by the existence of plastic hinges.

Qy: Yield moments or yield forces.

The existing member forces cannot be able to exceed the yield forces, the yield forces are not necessarily to be constants. For instance, they may be the function of η. The relations of Qme, Qmp and Qm are as shown in Fig. (6).

Just imagine that Qme, Qmp and ε vary ΔQme, ΔQmp and $\Delta\varepsilon$, when the deformation increases $\Delta\eta$. The eq. (6) may be written as follows:

$$\left.\begin{array}{lll} \text{if } (Q_m)+(\Delta Q_{me})>(|Q_Y|) & \text{then } (\Delta Q_{mp})=(|Q_Y|)-(Q_m)-(\Delta Q_{me}) \\ \text{if } |(Q_m)+(\Delta Q_{me})|\leq(|Q_Y|) & \text{then } (\Delta Q_{mp})=0 \\ \text{if } (Q_m)+(\Delta Q_{me})<-(|Q_Y|) & \text{then } (\Delta Q_{mp})=-(|Q_Y|)-(Q_m)-(\Delta Q_{me}) \end{array}\right\} \qquad (17)$$

where /Qy/ indicates the absolute value of the yield force Qy. If we get ΔQmp from eq. (17), $\Delta\varepsilon$ can be calculated from eq. (15). But we have to use the iteration method to obtain $\Delta\varepsilon$, for ΔQme includes $\Delta\varepsilon$ (Eq. 14). Fig. 7 shows the flow chart for the calculation of $\Delta\varepsilon$. If we get $\Delta\varepsilon$ by the above-mentioned method, ΔQp will be calculated by inserting $\Delta\varepsilon$ and assumed $\Delta\eta$ into the eq. (11).

Imagine that masses of framed structure are concentrated at the panel points, then we will be able to induce the following equations of motion.

$$[M](\ddot{\eta}) + [C](\dot{\eta}) + (Q_P) = -[M](a) \qquad (18)$$

where

$$[M] = \begin{bmatrix} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_\theta \end{bmatrix}, \quad [C] = \begin{bmatrix} C_x & 0 & 0 \\ 0 & C_y & 0 \\ 0 & 0 & C_\theta \end{bmatrix} \qquad (a) = \begin{pmatrix} a_x \\ a_y \\ a_\theta \end{pmatrix}$$

Mx, My, Mb: Diagonal matrices of masses or moments of inatias of panel points.

Cx, Cy, Ce: Diagonal matrices of damping coefficients.

$\dot{\eta}$, $\ddot{\eta}$: First or second derivative coefficent of η with respect to time.

a_x, a_y, a_θ: Vectors of acceleration by earthquakes.

In this equation (18), Qp may be calculated by the eqs. (17) and (11), using the above-mentioned method.

If we restrict the motion to the horizontal direction only, the second vector of Qp and the third one, namely Hp and Mp, become zero.

If we delete Yp and Qp from eqs. (11) and (12) by making use of this result, they will become as follows:

$$[\delta_1'][K_1'][\bar{\delta}_1'](x_P) + [\delta_1'][K_1'][\bar{d}'](\varepsilon) = (H_P) \qquad (19)$$

$$[d'][K_1'][\bar{\delta}_1'](x_P) + [d'][K_1'][\bar{d}'](\varepsilon) = (Q_m) \qquad (20)$$

where

$$[K_1'] = [K'] - [K'][\bar{\delta}_2'][\delta_2' K' \bar{\delta}_2']^{-1}[\delta_2'][K']$$

$$[\delta_1'] = \left[\delta\frac{X}{L}, -\delta\frac{Y}{L}, 0, \bar{\gamma}, 0, 0\right]$$

$$[\delta_2'] = \begin{bmatrix} \delta\frac{Y}{L}, & \delta\frac{X}{L}, & 0, & 0, & \bar{\gamma}, & 0 \\ 0, & \mu\frac{L}{2}, & \delta\frac{L}{2}, & 0, & 0, & \bar{\gamma} \end{bmatrix}$$

Furthermore it is needed to change δ' and K' in eqs. (4) and (5) into δ_1' and K_1' respectively, and this is followed by the rewriting of the motion's equation. $[M_x](\ddot{x}_P) + [C_x](\dot{x}_P) + (H_P) = -[M_x](a_x)$ (21)

The author has made some programs which calculate the seismic response of the framed structures by means of the eqs. (19), (20) and (21), and has made a trial on some structures. Some of them are shown in Fig. 8.

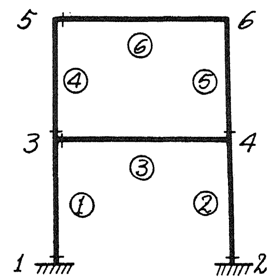
CONCLUSION

The deformations of the arbitrary framed structure are solved by the eq. (18) or eqs. (11) and (12).

If there are no plastic hinges, the eq. (13) is useful, otherwise eqs. (11) and (12). And the member forces can be solved by the eqs. (8) and (9), using the precedingly solved deformation .

The seismic response of the framed structures can be be solved by the eq. (18) or (21), in which (Qp) or (Hp) is the restrained forces and can be calculated by the eq. (11) or (19) using the condition of eq. (16).

In the long run it might be said that the plastic property of the ground has influence mostly upon the vibrational movements of the framed structures. However this will be the future work to be done hereafter.



γ

	1	2
1	1	
2		1
3		
4		
5		
6		

α MATRIX

	①	②	③	④	⑤	⑥
1	1					
2		1				
3			1	1		
4					1	
5						1
6						

β MATRIX

	①	②	③	④	⑤	⑥
1						
2						
3	1					
4		1	1			
5				1		
6					1	1

Fig. 1 EXAMPLE OF CONFIGURATION MATRICES

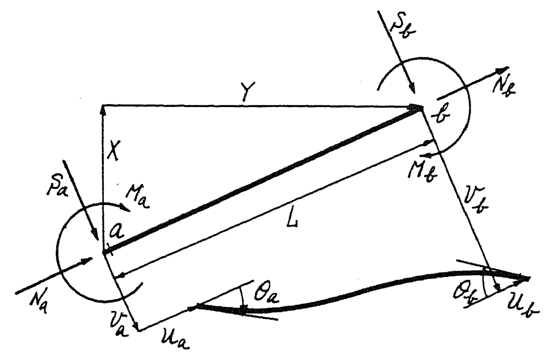


Fig. 2 EXPLANATION OF MEMBER FORCES AND MEMBER DEFORMATIONS

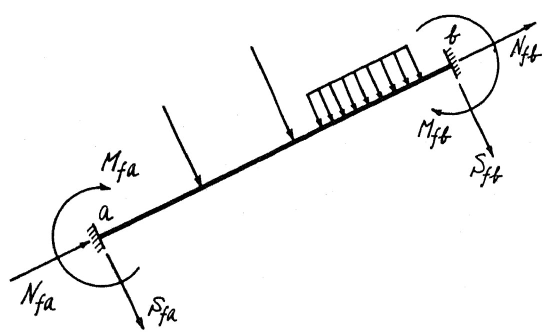


Fig. 3 EXPLANATION OF FIXED END FORCES

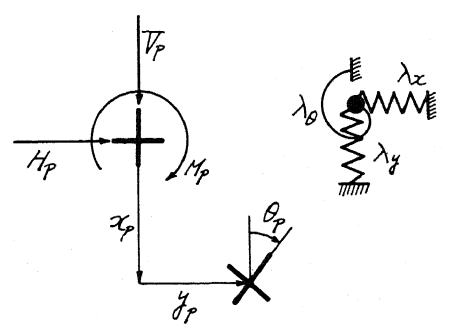


Fig. 4 EXPLANATION OF EXTERNAL FORCES, PANEL DEFORMATIONS AND SPRING CONSTANTS

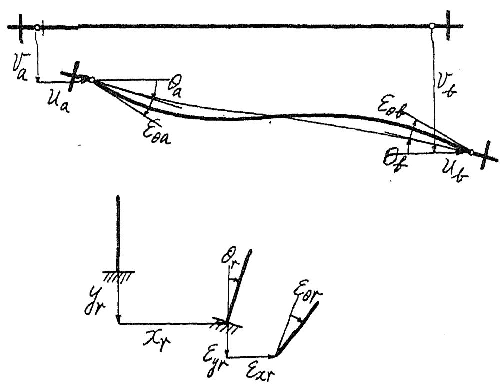


Fig. 5 EXPLANATION OF $\varepsilon_{\theta a}$, $\varepsilon_{\theta b}$, ε_{xr}, ε_{yr} AND $\varepsilon_{\theta r}$

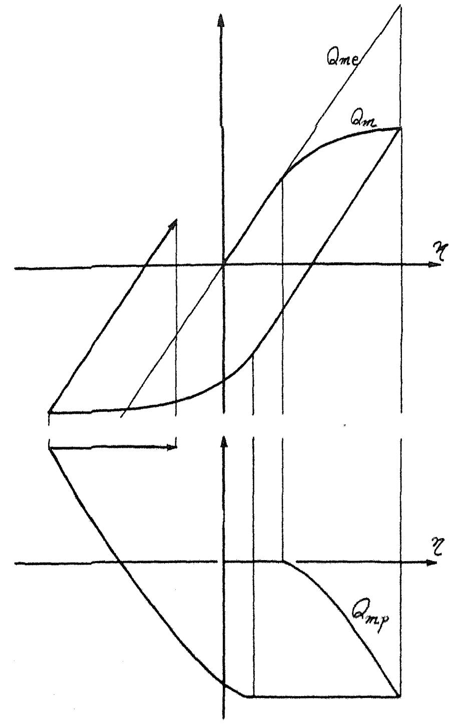


Fig. 6 RELATION AMONG Qme, Qmp AND Qm

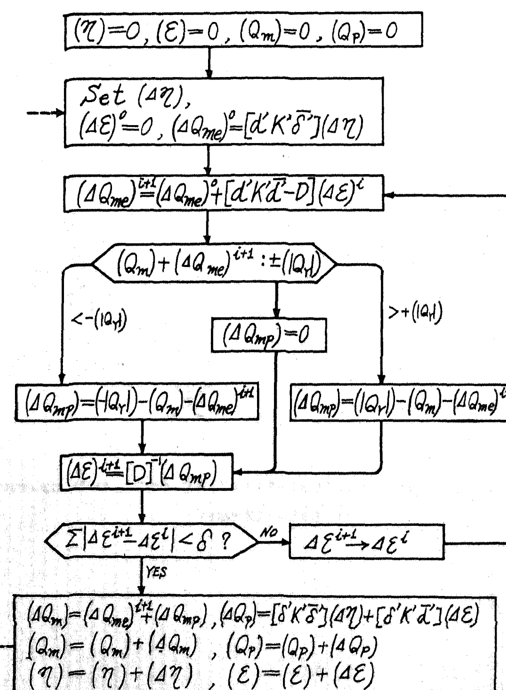


Fig. 7 FLOW DIAGRAM FOR EQUATION (17)

STRUCTURE DATA

MEMBER	1	2	3	4	5	6
L (cm)	98	98	520	980	980	520
J ($cm^4 \times 10^6$)	0.003	0.003	95	3.4	3.4	9.5
YIELD MOMENT'a'(tm)	10	10	-	100	100	-
YIELD MOMENT'b'(tm)	1000	1000	-	100	100	-
WEIGHT (ton)		132			66	
DAMPING (%)		0.3			0.04	

MAXIMUM DEFLECTION (cm)

MAXIMUM ACCELERATION OF EARTHQUAKE	PANEL POINT 3 - 4	PANEL POINT 5 - 6	YIELDED POINT
100 gal	1.399	1.838	End 'b' of member 1, 2
200 gal	3.375	3.959	
300 gal	5.408	6.553	End 'b' of member 1, 2 End 'a' of member 4, 5
400 gal	7.667	9.311	

Fig. 8 SEISMIC RESPONSE OF THE STRUCTURE SHOWN IN Fig.1
(input seismic waves are the linearly corrected El Centro earthquakes)