THE DYNAMIC RESPONSE OF THE ONE DEGREE OF FREEDOM BILINEAR HYSTERETIC SYSTEM

By
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ABSTRACT

An exact solution for the steady state response of the one degree of freedom bilinear hysteretic system is presented and this solution is compared with results obtained using a popular approximate technique. The existence of an unbounded resonance of the steady state response of the hysteretic system is verified analytically and some general conclusions are made about the transient response of the system.

INTRODUCTION

Examples of physical systems which exhibit some form of hysteresis are numerous and in many cases the hysteretic behavior may be adequately described by the general bilinear characteristic. This is particularly true of systems which contain coulomb damping and systems which contain one or more elasto-plastic elements. Systems of the first type include some built-up structures of rivited, bolted, or damped construction in which the combined effect of friction and elastic forces may result in a bilinear hysteretic restoring force. Systems of the second type occur when use is made of structural materials for which the elasto-plastic engineering approximation to "yielding" is satisfactory; i.e. some steels, masonry in shear, etc.

Within the past few years there has been considerable interest in the dynamic response of both the general bilinear hysteretic system and the limiting elasto-plastic system. Among the earliest treatments of the subject is that due to L. S. Jacobsen (1, 2, 3) who investigated the transient response of the general system by means of graphical techniques and also did work in developing a mechanical analog capable of representing hysteretic behavior. More recently, L. E. Goodman and J. H. Klumpp (4, 5) have done both analytic and experimental work on the dynamic properties of a laminated beam with a slip interface which is an example of a system having the general bilinear hysteretic character. The response of the limiting elasto-plastic system was considered graphically for transient pulses and square wave excitation by R. Tanabashi (6) and later, the same author (7) studied the transient response of the general system using graphical techniques and an electric analog computer. A similar investigation was also made by W. T. Thompson (8) who employed electric analog methods to solve for the response of the general system to a unidirectional force excitation. J. F. Ruzicka (9) has used both an approximate analytical theory and electric analog techniques to study the dynamics of a vibration absorber which has the general bilinear hysteresis characteristic, and the transient response of structures which contain one or more elasto-plastic elements has been considered by G. V. Berg (10, 11), T. Kobori and R. Minai (12), R. Tanabashi and K. Kaneta

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(13), and others (14-18). The stability of the steady state motion of the general system has been demonstrated analytically by N. Ando (19) and this same author has formulated an exact analytic solution for the steady state response of the limiting elasto-plastic system and has made extensive analytic studies of the transient response of both one and two degree of freedom systems of the general type. The behavior of the general bilinear hysteretic system has been studied quite thoroughly using approximate analytic techniques in a series of three papers by T. K. Caughey (20, 21, 22). More recently, approximate techniques have been used by P. Jennings (14) to study the steady state response of a general class of hysteretic systems which contains the bilinear hysteretic system as a special case.

The objective of the present paper is to both complement and extend the efforts of other workers. To this end, an exact solution for the steady state response of the general one degree of freedom bilinear hysteretic system is presented and some general conclusions are made about the transient response of the system and the existence of unbounded resonance behavior.

FORMULATION

The most general form of the bilinear hysteretic restoring force is shown in Fig. 1. If a single degree of freedom system having this restoring force characteristic is subjected to a time varying force input $P(\tau)$, the differential equation of motion will be

$$m\ddot{y} + F(y, \dot{y}) = P(\tau) \quad . \tag{1}$$

For the purposes of analysis, this equation may be simplified by the introduction of the dimensionless variables

$$\begin{aligned} x &= (k_1/F_n)y \\ f(x, \dot{x}) &= F(y, \dot{y})/F_n \\ \alpha &= k_2/k_1 \\ t &= \sqrt{k_1/m}\;\tau \\ p(t) &= P(\tau)/F_n \quad . \end{aligned} \tag{2}$$

The differential equation of motion then becomes

$$\ddot{x} + f(x, \dot{x}) = p(t) \tag{3}$$

and the normalized restoring force will have the configuration shown in Fig. 2.

If the system represented by equation (3) is subjected to a trigonometric excitation of amplitude r and frequency ω, the steady state system displacement will have a wave form similar to that shown schematically in Fig. 3. Let x_1 be the maximum positive displacement of the system and let ϕ be the phase angle by which the displacement lags the excitation. Then, beginning at the point where the displacement is equal to x_1, the equation describing subsequent motion of the system will be

$$\ddot{x} + x = (x_1 - 1)(1-\alpha) + r \cos(\omega t + \phi), \tag{4}$$

where

$$x(0) = x_1$$
$$\dot{x}(0) = 0. \tag{5}$$

Equation (1) will remain valid until the displacement has decreased to some value x_2, at which point the slope of the restoring force diagram changes to α. If t_2 is the time required for the system to move from displacement x_1 to x_2 it will be seen from equations (4) and (5) that for $\omega = 1$

$$x_2 = \left[x_1 - (x_1-1)(1-\alpha) - \frac{r}{1-\omega^2}\cos\phi\right]\cos t_2 + \frac{\omega r}{1-\omega^2}\sin\phi \sin t_2 + (x_1-1)(1-\alpha) + \frac{r}{1-\omega^2}\cos(\omega t_2 + \phi) \tag{6}$$

and

$$\dot{x}_2 = -\left[x_1 - (x_1-1)(1-\alpha) - \frac{r}{1-\omega^2}\cos\phi\right]\sin t_2 + \frac{\omega r}{1-\omega^2}\sin\phi\cos t_2 - \frac{r\omega}{1-\omega^2}\sin(\omega t_2 + \phi). \tag{7}$$

However, due to the assumed normalization of the hysteresis loop,

$$x_2 = x_1 - 2.$$

Thus, equation (6) may be rewritten as

$$0 = \left[x_1 - (x_1-1)(1-\alpha) - \frac{r}{1-\omega^2}\cos\phi\right]\cos t_2 + \frac{\omega r}{1-\omega^2}\sin\phi\sin t_2 + (1 + \alpha - \alpha x_1) + \frac{r}{1-\omega^2}\cos(\omega t_2 + \phi). \tag{8}$$

Further motion of the system with negative velocity will take place along the lowermost restoring force branch of slope α. Therefore, the equation governing this motion will be

$$\ddot{x} + \alpha x = (1-\alpha) + r\cos\left[\omega(t + t_2) + \phi\right] \tag{9}$$

where the initial conditions are now

$$x(0) = x_2 = x_1 - 2$$
$$\dot{x}(0) = \dot{x}_2 . \tag{10}$$

If the frequency and amplitude of excitation are such that the effects of ultra harmonics may be neglected, the hysteresis loop will have the configuration shown in Fig. 2 with no zeros of the velocity at other than the points of maximum and minimum displacement. In this case, if x_3 is the value of the minimum displacement and t_3 is the time required for the system to move from x_2 to x_3, it will be seen from equations (9) and (10) that for $\omega^2 = \alpha$,

$$x_3 = \frac{1}{\sqrt{\alpha}}\left[\dot{x}_2 + \frac{\omega r}{\alpha-\omega^2}\sin(\omega t_2 + \phi)\right]\sin\sqrt{\alpha}\, t_3 + \frac{r}{\alpha-\omega^2}\cos\left[\omega(t_3 + t_2) + \phi\right]$$

$$+ \left[x_1 - 2\frac{(1-\alpha)}{\alpha} - \frac{r}{\alpha-\omega^2}\cos(\omega t_2 + \phi)\right]\cos\sqrt{\alpha}\, t_3 + \frac{(1-\alpha)}{\alpha} \tag{11}$$

and

$$\dot{x}_3 = 0 = \left[\dot{x}_2 + \frac{\omega r}{\alpha-\omega^2}\sin(\omega t_2 + \phi)\right]\cos\sqrt{\alpha}\, t_3 - \frac{r\omega}{\alpha-\omega^2}\sin\left[\omega(t_3 + t_2) + \phi\right]$$

$$- \sqrt{\alpha}\left[x_1 - 2 - \frac{(1-\alpha)}{\alpha} - \frac{r}{\alpha-\omega^2}\cos(\omega t_2 + \phi)\right]\sin\sqrt{\alpha}\, t_3. \tag{12}$$

If the analysis is restricted to steady state motion where the hysteresis loop is symmetric, the periodicity conditions on the solution become

$$x_1 = -x_3 \tag{13}$$

and

$$t_2 + t_3 = \pi/\omega. \tag{14}$$

Thus, the problem has been reduced to the solution of a set of six equations [equations (7), (8), and (11) thru (14)] in terms of six unknowns [x_1, ϕ, $\dot{x}_2$, t_2, x_3, and t_3]. For reasons which will soon become apparent, it is convenient to introduce a new variable ϕ' defined as

$$\phi' \equiv \phi + (t_2 + t_3 - \pi/\omega). \tag{15}$$

Then, in terms of this new variable the periodicity requirement, equation (14), may be written as

$$\phi = \phi'. \tag{16}$$

METHOD OF SOLUTION FOR STEADY STATE RESPONSE

Because of the highly transcendental character of the simultaneous equations governing the steady state behavior, direct solution by elimination of variables is impractical. Thus, guided by the results of graphical solutions (23) one turns to an iterative method of solution. The analysis in this case is begun by arbitrary selection of initial values for the two variables x_1 and ϕ. Then, using numerical techniques it is possible to progressively solve for t_2 from equation (8), $\dot{x}_2$ from equation (7), t_3 from equation (12), and finally x_3 and ϕ' from equations (11) and (15). Having thus determined all of the intermediate variables, equations (13) and (16) may be used to calculate new values for x_1 and ϕ. If these new values are the same as the ones assumed initially, the problem is solved; if not, the entire process is begun again using the new values as initial conditions. This procedure is continued until the initial and final values of x_1 and ϕ over a cycle of calculation are equal within some desired limit of accuracy. This method can be made to yield extreme numerical accuracy and is not subject to the limitations associated with approximate numerical techniques for the direct integration of the equations of motion such as the Runge-Kutta method (24).

Convergence of the above procedure is difficult to demonstrate mathematically but may be inferred from the physical nature of the method itself. It will be noted that the present approach is essentially just a formalized mathematical way of constructing the system phase plane contour corresponding to periodic excitation from some arbitrary point in phase space. The only real difference between the method employed here and actual construction of the phase contour is that here only the end points of the contour are evaluated without the explicit determination of all intermediate points. The convergence of phase plane solutions has been demonstrated for a bilinear hysteretic system which is subjected to square wave excitation (23) and there is no reason to believe that the system should behave any differently with trigonometric excitation. However, the best argument for the convergence of such a procedure is that it actually does converge in practice as shown below.

STEADY STATE RESULTS

Figs. 4 and 5 show the results of digital computer solutions in the range of parameters where ultraharmonic behavior is not a predominant factor. It is seen that all of the curves exhibit a characteristic leaning toward low frequency which is typical of so called "Soft" systems. This results in a somewhat gentle slope on the high frequency side of the response curves and a very steep slope on the low frequency side. Numerous determinations of the response were made on the low frequency side of the curve and although convergence was extremely slow, it was possible to obtain sufficiently accurate results to indicate that: 1) within the accuracy of the numerical computations the slope on the low frequency side is never negative, and 2) this slope may approach an infinite limit at its steepest point. Thus, on the basis of these observations it is concluded that there can be no more than one vertical tangency to a given response curve which in turn implies that the response curves are all single valued.

COMPARISON WITH APPROXIMATE SOLUTION RESULTS

Several methods have been used to obtain approximate solutions for the steady state response of hysteretic systems. One popular method which has been employed by T. K. Caughey (20), P. Jennings (14), and the present author (23) is the method of slowly varying parameters or the method of Kryloff and Bogoliuboff (25). This method is based upon the "equivalent linearization" of the system and is most accurate for small system nonlinearities (in this case for α nearly equal to one). Since this method has been used rather extensively to investigate hysteretic systems, the results of the exact solution were compared with those of the approximate solution. The comparison for two different values of α is shown in Figs. 6 and 7. Results (not shown) for values of α closer to unity show almost exact agreement over a very wide range of inputs.

UNBOUNDED RESONANCE BEHAVIOR

Guided by the results of approximate solutions for the steady state response of the one degree of freedom system (20, 23), one is led to consider the special case of the system response when

$$\omega = \sqrt{\alpha} \tag{17}$$

and

$$\phi = 0. \tag{18}$$

However, in specifying ϕ and ω it is no longer possible to concurrently specify r. Thus, in this special case r must be looked upon as one of the variables of the problem.

Equations (11) and (12) for x_3 and $\dot{x}_3$ were derived under the restriction that $\omega = \sqrt{\alpha}$. Thus, for the particular conditions specified by (17) and (18) these equations must be replaced by

$$x_3 = \left(\frac{\dot{x}_2}{\sqrt{\alpha}} - \frac{r}{2\alpha} \sin \sqrt{\alpha}\, t_2\right) \sin \sqrt{\alpha}\, t_3 + \left[x_1 - 2 - \frac{(1-\alpha)}{\alpha}\right] \cos \sqrt{\alpha}\, t_3 + \frac{(1-\alpha)}{\alpha} + \frac{r}{2\sqrt{\alpha}}\, t_3 \cos \sqrt{\alpha}\, (t_3 + t_2) \tag{19}$$

and

$$\dot{x}_3 = 0 = \left(\dot{x}_2 - \frac{r}{2\sqrt{\alpha}} \sin \sqrt{\alpha}\, t_2\right) \cos \sqrt{\alpha}\, t_3 - \sqrt{\alpha} \left[x_1 - 2 - \frac{(1-\alpha)}{\alpha}\right] \sin \sqrt{\alpha}\, t_3 + \frac{r}{2\sqrt{\alpha}} \cos \sqrt{\alpha}\, (t_3 + t_2) - \frac{r}{3}\, t_3 \sin \sqrt{\alpha}\, (t_3 + t_2). \tag{20}$$

These two equations along with equations (7), (8), (13), and (14) when $\phi = 0$ and $\omega = \sqrt{\alpha}$ are the six equations which determine the system behavior. x_3 and t_3 can be eliminated from these six equations using equations (13) and (14), and $\dot{x}_2$ can be eliminated from the resulting equations reducing the problem to the solution of three equations in three unknowns. The three equations are

$$x_m = \frac{(1-\alpha)\cos t_2 + \frac{r}{(1-\alpha)}(\sqrt{\alpha}\sin t_2 - \sin\sqrt{\alpha}\, t_2) + (1+\alpha)}{\alpha(1-\cos t_2)} \tag{21}$$

$$r = \frac{x_m(\cos\sqrt{\alpha}\, t_2 - 1) + \left[2 + \frac{(1-\alpha)}{\alpha}\right](\sin t_2 \sin\sqrt{\alpha}\, t_2 + \cos^2\sqrt{\alpha}\, t_2 + \frac{(1-\alpha)}{\alpha}\cos\sqrt{\alpha}\, t_2)}{\frac{1}{2\alpha}\sin\sqrt{\alpha}\, t_2 + \frac{1}{2\sqrt{\alpha}}\left(\frac{\pi}{\sqrt{\alpha}} - t_2\right)\cos\sqrt{\alpha}\, t_2} \tag{22}$$

$$0 = -\left\{-\left[x_m - (x_m - 1)(1-\alpha)\right]\sin t_2 + \frac{\sqrt{\alpha}\, r}{1-\alpha}(\cos t_2 - \cos\sqrt{\alpha}\, t_2) - \frac{r}{2\sqrt{\alpha}}\cos\sqrt{\alpha}\, t_2\right\}$$
$$\cos\sqrt{\alpha}\, t_2 - \sqrt{\alpha}\left[x_m - 2 - \frac{(1-\alpha)}{\alpha}\right]\sin\sqrt{\alpha}\, t_2 - \frac{r}{2\sqrt{\alpha}} \tag{23}$$

where the particular value of x_1 which corresponds to conditions (17) and (18) has been denoted by x_m.

Formally, equations (21) and (22) may now be used to solve for r and x_m in terms of t_2. These results in conjunction with equation (23) would then give one equation in the one unknown t_2. However, it is clearly seen that this would lead to such a complicated expression that practically speaking the problem could not be solved. Thus, the procedure followed here will be to select a reasonable value for t_2 and then demonstrate that this value along with the values it predicts for x_m and r are all consistent with the statements of equations (21), (22), and (23). Assume that the desired solution is $t_2 = 0$; or, to be more correct both physically and mathematically, assume that t_2 approaches zero. Then, if r is no less than zeroeth order in t_2, taking the limit of both sides of equation (21) as t_2 approaches zero gives

$$\lim_{t_2 \to 0}(x_m) = \lim_{t_2 \to 0}\left[4/\alpha t_2^2 + \alpha/3 + O(t_2^2)\right] \tag{24}$$

and

$$x_m \to \infty \text{ as } t_2 \to 0. \tag{25}$$

The assumption that r is no less than zeroeth order in t_2 may be verified from equation (22). After considerable manipulation it can be shown that

$$\lim_{t_2 \to 0} (r) = \lim_{t_2 \to 0} \left[4(1-\alpha)/\pi + O(t_2) \right] \quad (26)$$

and hence that

$$r \to 4(1-\alpha)/\pi \text{ as } t_2 \to 0 . \quad (27)$$

Thus, r is indeed zeroeth order as assumed. It now remains only to show that the values of t_2, r, and x_m obtained above satisfy the third equation (23). Taking the limit of both sides of this equation it may be shown that the right hand side is of order t_2. Thus, as t_2 approaches zero the equation is satisfied and the assumed values of t_2, r, and x_m must represent the true limiting solution of the problem.

Summarizing, it has been shown that the exact equations for the steady state motion predict an amplitude and phase resonance ($x_m \to \infty$, $\phi = 0$) which occurs with finite amplitude of excitation at a frequency $\omega = \sqrt{\alpha}$. The amplitude of excitation which yields this behavior may be looked upon as a critical parameter of the particular system under consideration and will be denoted by

$$r_c = 4(1-\alpha)/\pi . \quad (28)$$

TRANSIENT RESPONSE

One of the more important questions which arises in connection with the transient response of the general hysteretic system concerns the nature of the final state of the system upon completion of the excitation. In some cases it will be found that the system develops a certain permanent offset, while in other cases the system will return to oscillate about its original point of zero displacement[2]. If the excitation is reasonably symmetric and of low enough level that the response is due primarily to resonance effects or, if the excitation is of very short duration, any final offset will most likely result from the behavior of the system after the excitation has ceased. Thus, in order to better understand the mechanism by which an offset may occur, it is instructive to actually follow the motion of the bilinear hysteretic system during the period from the end of the excitation until a steady state condition is reached.

Define a "maxima of the displacement" as any displacement x_i which satisfies the conditions

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[2] If there is no viscous damping in the system, the final state will in general be oscillatory.

$$x_i = 0$$
$$|x_i| > 1 \qquad (29)$$
$$|f(x_i, 0)| = \alpha|x_i| + (1-\alpha) \ .$$

Then, x_i is the displacement which corresponds to a reversal in the sign of the velocity. If x_i is a maxima which occurs after the system excitation has ceased, it may be shown that the next maxima x_{i+1} will be given by the equation

$$x_{i+1} = (\text{sgn } x_i)\left\{- \left[(|x_i| - 1 - 1/\alpha)^2 + 4(|x_i| - 1)\right]^{1/2} + (1-\alpha)/\alpha\right\} \qquad (30)$$

where necessary conditions for the validity of this equation are

$$-x_i \ (\text{sgn } x_{i-1}) > 1 \qquad (31)$$

and

$$|x_i| > 1. \qquad (32)$$

When either of the conditions (31) or (32) is not satisfied it means that the system is energetically unable to move into a restoring force region of slope α and the system will merely oscillate periodically along a restoring force segment of unity slope. Using equation (30), the course of the system motion upon completion of the excitation may now be followed from maxima to maxima until condition (31) is no longer satisfied. At this point the system will begin to exhibit undamped periodic oscillations about a mean displacement (offset) δ_s given by

$$\delta_s = (1-\alpha)\left[(1)(\text{sgn } x_{k-1}) + x_k\right] \qquad (33)$$

where x_k is the last maxima which can be obtained from equation (30).

The information contained in equation (30) and conditions (31) and (32) may be interpreted graphically as shown in Fig. 8. This figure gives a family of curves which may be used to obtain the $(i + 1)^{th}$ maxima of the displacement in terms of the i^{th} maxima for various values of the parameter α. If, for a particular maxima x_j, the point (x_{j+1}, x_j) lies within the cross-hatched area of the figure, then x_j is the last maxima which will satisfy condition (31). Hence, x_{j+1} equals x_k and subsequent motion of the system will take place along a single unity slope segment of the restoring force diagram with an offset δ_s given by equation (33).

It will be noted from Fig. 8 that the absolute offset $|\delta_s|$ has a definite upper bound which depends only on the value of the hysteresis loop parameter α. In general, this upper bound may be obtained by setting

$$\frac{dx_k}{dx_{k-1}} = 0 \tag{34}$$

Performing the indicated differentiation and using equation (33), one finds that

$$|\delta_s|_{max} = \begin{cases} (1-\alpha)\left\{1/\alpha - 2\left[(1-\alpha)/\alpha\right]^{1/2}\right\} & ; \alpha < 1/2 \\ 0 & ; \alpha > 1/2 \end{cases} \quad . \tag{35}$$

Equation (35) makes it possible to draw several interesting conclusions about the transient response of the bilinear hysteretic system. Based only on the assumption that a "maxima of the displacement" exists after completion of the excitation, equation (35) says that the maximum final offset will be bounded for all $\alpha > 0$ and may become unbounded only if $\alpha = 0$. More important, however, is the observation that if $\alpha > 1/2$ there can be no permanent offset for the system regardless of the value of the initial maxima x_1.

SUMMARY AND CONCLUSIONS

The results of the preceding analysis may be summarized as follows: 1) An "exact" solution for the steady state response of the one degree of freedom bilinear hysteretic system has been obtained and the results of this solution have been compared with the results of an approximate solution obtained by the method of equivalent linearization. This comparison indicates that the approximate method is capable of giving very accurate results for cases where the slope of the restoring force diagram in the "yielded" region is nearly the same as that in the "elastic" region (α nearly equal to 1). Furthermore, where the approximate solution does differ significantly from the exact solution the present analysis indicates that the approximate solution will always be conservative from a design point of view; i.e. the approximate solution will predict an amplitude of response which is greater than or equal to the actual value. 2) Studies of the transient response of the bilinear hysteretic system have shown that the system will in general exhibit no permanent offset if the ratio of the slope of the restoring force diagram in the "yielded" region to the slope in the "elastic" region is greater than or equal to 1/2 ($\alpha > 1/2$).

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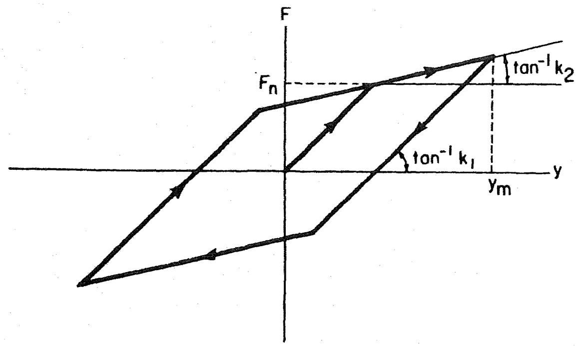

FIGURE 1. GENERAL BILINEAR HYSTERETIC RESTORING FORCE.

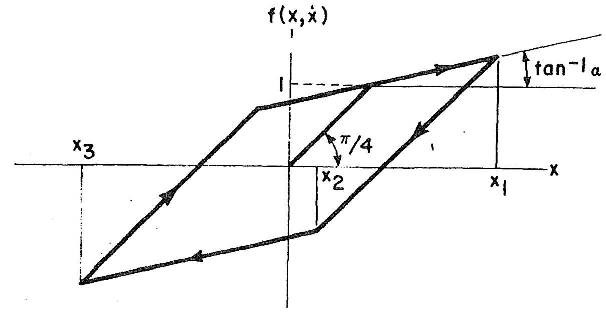

FIGURE 2. NORMALIZED BILINEAR HYSTERETIC RESTORING FORCE.

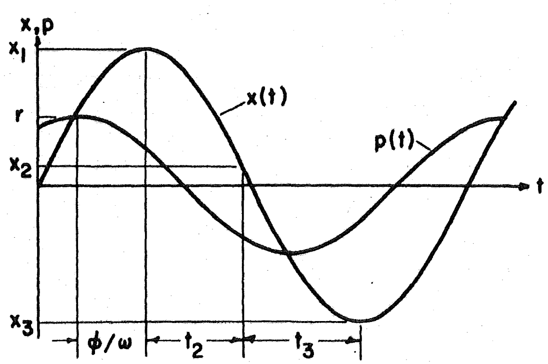

FIGURE 3. DISPLACEMENT WAVE FORM.

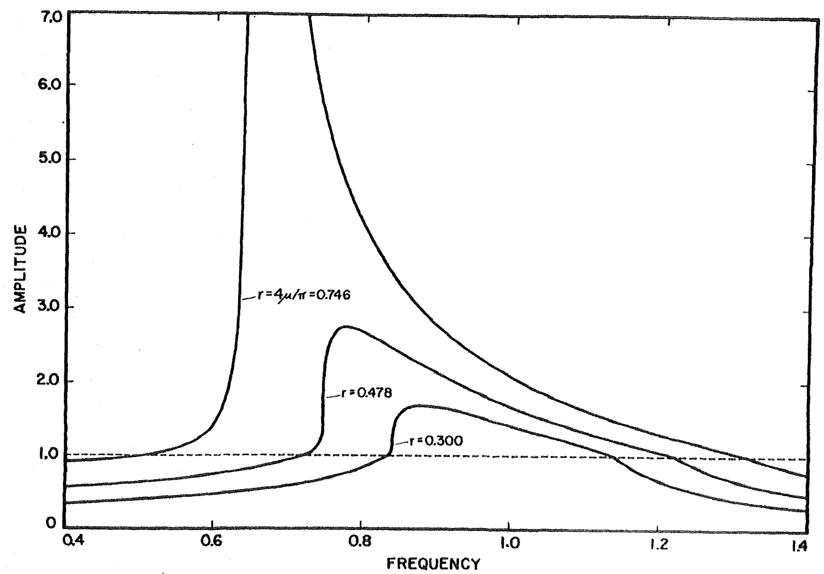

FIGURE 4. STEADY STATE RESPONSE, $\alpha = \tan \pi/8 = 0.414$

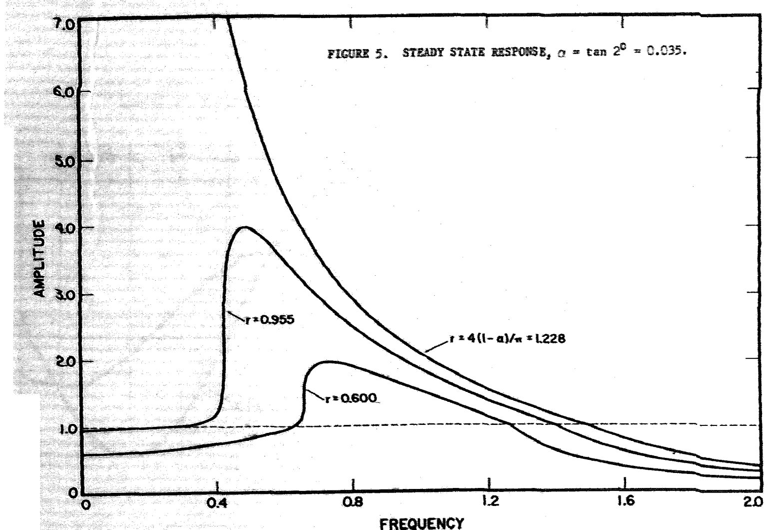

FIGURE 5. STEADY STATE RESPONSE, $\alpha = \tan 2^0 = 0.035$.

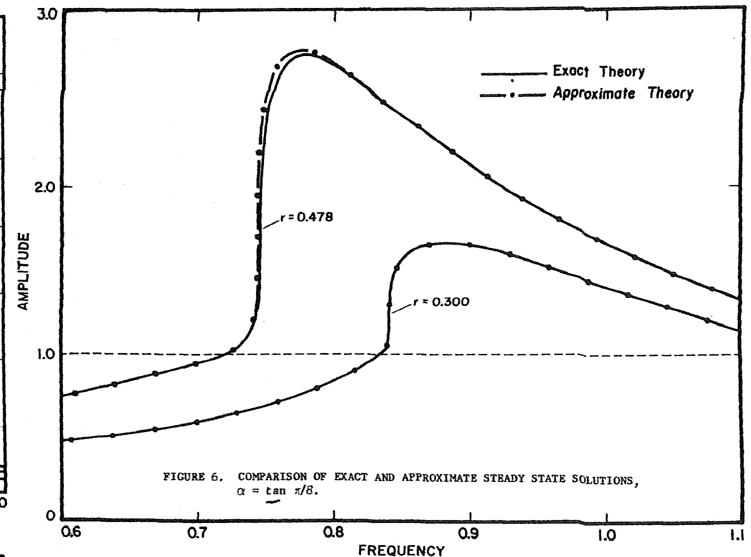

FIGURE 6. COMPARISON OF EXACT AND APPROXIMATE STEADY STATE SOLUTIONS, $\alpha = \tan \pi/8$.

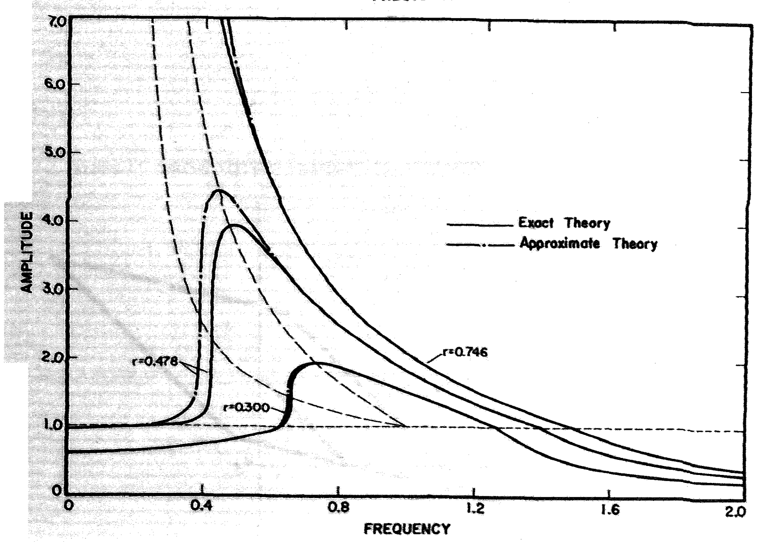

FIGURE 7. COMPARISON OF EXACT AND APPROXIMATE STEADY STATE SOLUTIONS, $\alpha = \tan 2^0$ (NEARLY ELASTO-PLASTIC).

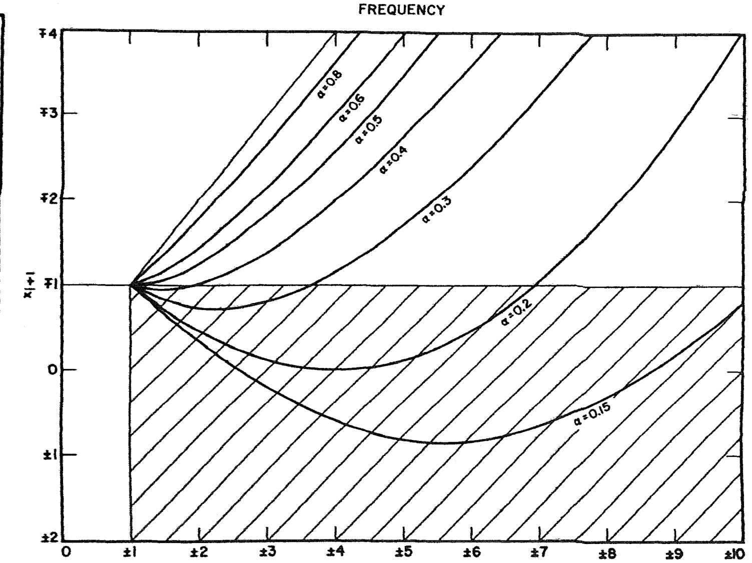

FIGURE 8. x_{i+1} vs x_i AS A FUNCTION OF α.