TORSIONAL VIBRATION OF MULTI-STORIED BUILDINGS

by Toshio Shiga*

Abstract

In this paper an attempt is made to investigate the nature of the torsional vibration of multi-storied buildings, in each story of which the centers of rigidity fall upon a single point as with those of gravity.

Such points as follows are discussed: The case of single-storied buildings is with the nature of the torsional vibration of multi-storied buildings mentioned above; the torsion is more serious when the torsional rigidity is less than the lateral rigidity; the vibration pattern is remarkably varied by slight shift of ground motion, when the building is not rigid.

Introduction

The unsymmetrical plan or elevation of a building or the irregular arrangement of walls lead to a discord between the center of gravity and that of rigidity, which is inevitably followed by the torsional vibration.

An attempt is to be made here to describe the nature of the torsional vibration of multi-storied buildings, as shown in Fig. 1, where not only the centers of rigidity fall upon a single point as with those of gravity, but also in each story, the radius of gyration both of mass and of stiffness and the rigidity ratio of the transverse direction to the longitudinal are all equal.

Since the ordinary multi-storied buildings have such conditions as above-mentioned, following natures are supposed to be generally applicable.

Fundamental assumptions and the method of describing the vibration

Following fundamental assumptions are established in dealing logically with such vibration as mentioned above.

(1) To neglect the deformation of floor slabs and to regard them as rigid bodies.
(2) To assume that all the mass of each part of a building is concentrated upon the position of the floor slab.
(3) To assume that the floor slab can move only on the horizontal plane, neglecting the effects of axial deformations of the columns.
(4) To deal with the vibration of multi-storied buildings assumed by the above-mentioned assumptions as an elastic vibration system of multiple particles having rotational inertia.
(5) To neglect the torsional rigidity of members.

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Dealing with the vibration of multi-storied buildings as mentioned above, the vibration is to be described as shown in Fig. 2.

In other words, the horizontal displacement of each story is described as x,y and φ , where x and y are the displacement of the center of gravity and φ is the gyration angle around the center of gravity. For convenience of analysis, the gyration angle φ is described as $\varphi = z/i$, where z is the displacement following the gyration of a point at a distance of 'i' from the center of gravity. The sign of these symbols is taken as indicated in Fig. 2, z is plus when it is opposed to the turning of clockwise.

Influence coefficients

According to these assumptions mentioned above, the natures of the frame i,e, the displacement influence should be decided in order to introduce the equation of vibration. This is called the influence coefficient and described by following symbols.

The reaction of the center of gravity in the k-th story is described as C_{ik} when the unit displacement is yielded to the center of gravity of the i-th story; the direction of displacement and reaction is suffixed as follows:

$$C^{\text{reaction direction}}_{\text{displacement direction } ik}$$

Since the equation of vibration is described as the combined pattern of displacement, as we shall see later, the influence coefficient is described by the reaction against the unit displacement.

It should be noted, in addition, that every influence coefficient related to the gyration around the gravity center is described by the displacement z, not by the gyration angle φ .

The reciprocal theory is to be established between the influence coefficients, because the influence coefficient is a reaction against the unit displacement.

The influence coefficients of the multi-storied building being descussed in this paper are shown as in Fig. 3, and therefore following relations are established.

Since the rigidity ratio of the transverse direction to the longitudinal is equal in each story,

$$C^{y}_{yik} = \alpha C^{x}_{xik}$$

Where α is the rigidity ratio of the transverse direction to the longitudinal.

Since the axis x and y correspond to respectively the principal elastic axes,

$$C^{y}_{xik} = C^{x}_{yik} = 0$$

Since the centers of rigidity in each story fall upona single point as with those of gravity, and the radius of gyration of mass and that of stiffness are equal,

$$C^{z}_{xik} = C^{x}_{zik} = -\, e'_{y}\, C^{x}_{xik}$$

$$C^{z}_{yik} = C^{y}_{zik} = \quad e'_{x}\, C^{y}_{yik} = \alpha\, e'_{x}\, C^{x}_{xik}$$

$$C^{z}_{zik} = c\, C^{x}_{xik}$$

$$e'_{x} = \frac{e_{x}}{i} \qquad e'_{y} = \frac{e_{y}}{i}$$

$$c = \alpha\, e'^{2}_{x} + e'^{2}_{y} + j'^{2} \qquad j'^{2} = \frac{\mathfrak{m}_{ii}}{i^{2}\, C^{x}_{xii}}$$

Where e_x and e_y are the eccentricity, 'i' the radius of gyration, $\mathfrak{m}_{ii}$ the torsional rigidity around the rigidity center of the i-th story, C^{x}_{xii} the lateral rigidity in the x direction of the i-th story.

Equations of vibration

The displacement and the influence coefficient being described as above, the equations of vibration are as follows:

$$\left.\begin{aligned}
\sum_{i=1}^{m} C^{x}_{xik}X_i + \sum C^{x}_{yik}y_i + \sum C^{x}_{zik}Z_i &= -M_k(\ddot{X}_k + \ddot{X}_0) \\
\sum_{i=1}^{m} C^{y}_{xik}X_i + \sum C^{y}_{yik}y_i + \sum C^{y}_{zik}Z_i &= -M_k(\ddot{y}_k + \ddot{y}_0) \\
\sum_{i=1}^{m} C^{z}_{xik}X_i + \sum C^{z}_{yik}y_i + \sum C^{z}_{zik}Z_i &= -M_k(\ddot{Z}_k + \ddot{Z}_0)
\end{aligned}\right\} \quad (1)$$

Where C is the influence coefficient, x,y and z the relative displacement, x_0,y_0 and z_0 the ground movement, 'M' the mass, i and k the suffix indicating story position, both from the first to the m-th.

Equation (1) is just the same in form as the equation of plane vibration (vibration without eccentricity) of 3m degree-of-freedom system, provided that x,y and z are regarded as variables of a system.

Therefore, the solution can be got, if we deal a single-storied building a three degree-of-freedom system and a m-storied building as a 3m degree-of-freedom system.

Property of natural vibration

Since the above-mentioned relation is applied to the influence coefficient, the equation (1) in this case is as follows:

$$\left.\begin{array}{l} \sum_{i=1}^{m} C_{ik}U_i \qquad\qquad - e_y' \sum C_{ik}W_i = n^2 M_k U_k \\ \qquad \sum_{i=1}^{m} C_{ik}V_i + \alpha e_x' \sum C_{ik}W_i = n^2 M_k V_k \\ -e_y' \sum_{i=1}^{m} C_{ik}U_i + \alpha e_x' \sum C_{ik}V_i + c \sum C_{ik}W_i = n^2 M_k W_k \end{array}\right\} \quad (2)$$

$$X = U_i q \qquad Y = V_i q \qquad Z = W_i q$$

$$\ddot{q} = -n^2 q \qquad q = A\cos nt + B\sin nt$$

$$C_{ik} = C^{x}_{xik}$$

Where n is the cicular frequency.

It is proved by the property of the coefficient matrix in the equation above-mentioned that the circular frequencies and the vibration mode shapes have such properties as follows:

The circular frequency ranges from the first to the 3m-th mode, and is described by following formula,

$$n = \bar{n} \times n'_s / n_0 \qquad (S = 1 \sim 3) \qquad (3)$$

Where n is the circular frequency of the multi-storied building, $\bar{n}$ that of the plane vibration system (the eccentricities of the building are neglected), n' the circular frequency of tosional vibration system (the building is assumed single-storied) and n_o that of the system neglected the eccentricity in the assumed single-storied building.

The mode shapes are to be classified into three, two translational and one rotational in general, and the vertical configulation in x and y direction is both equal to that of vibration from the first to the m-th mode in the plane vibration system. Moreover, the twisting centers of each story fall upon a single point and coincide with that of the vibration from the first to the third mode of the torsional vibration system. That is, the mode shapes are described by following formulas:

$$\left.\begin{aligned} U_i &= \bar{U}_i \times U' \\ V_i &= \bar{U}_i \times V' \qquad (i = 1 \sim m) \\ W_i &= \bar{U}_i \times W' \end{aligned}\right\} \quad (4)$$

Where u_i, v_i and w_i are the normal function of the multi-storied building (from the first to the 3m-th mode), $\bar{u}_i$ that of the plane vibration system (from the first to the m-th mode), and u,v and w that of the torsional vibration system (from the first to the third mode).

The natures of the plane vibration and of the torsional vibration of single-storied building, in taking advantage of such properties as mentioned above, inable us easily to know the nature of the torsional vibration of the multi-storied buildings.

This nature, if only it satisfied such necessary conditions as mentioned above, is to be inevitably established, independently of the shape of plan or the positional relation between the centers of rigidity and of gravity.

The nature of the torsional vibration in single-storied buildings, which have such natures as above, enables us to know the nature in the case of multi-storied buildings discussed here. Then, the nature of the single-storied buildings is shown as follows:

The equation of the circular frequency and the normal function in this case may be described as follows:

the equation of circular frequency:

$$\left(\frac{n}{n_o}\right)^6 - (1+\alpha+C)\left(\frac{n}{n_o}\right)^4 + (\alpha + C + C\alpha - \alpha^2 e_x'^2 - e_y'^2)\left(\frac{n}{n_o}\right)^2$$

$$- \alpha(C - \alpha e_x'^2 - e_y'^2) = 0 \qquad (5)$$

the normal function:

$$U = 1 \qquad V = -\alpha \frac{e_x'}{e_y'} \cdot \frac{1-\left(\frac{n}{n_o}\right)^2}{\alpha - \left(\frac{n}{n_o}\right)^2} \qquad W = \frac{1-\left(\frac{n}{n_o}\right)^2}{e_y'} \qquad (6)$$

the distance of the twisting center from the center of gravity:

$$\frac{\rho_Y}{i} = \frac{1}{W} \qquad \frac{\rho_x}{i} = \frac{V}{W} \qquad \frac{\rho}{i} = \frac{\sqrt{\rho_x^2 + \rho_Y^2}}{i} \qquad (7)$$

Where n is the circular frequency, n_o equal to $\sqrt{G/M}$; the frequency of the plane vibration (in the x direction), and G the rigidity in the x direction.

Fig. 4,5 and 6 show how the circular frequency and the distance of the twisting center from the center of gravity vary in proportion to the eccentricity and the radius of gyration of stiffness; Fig. 7 an example of the vibration pattern.

As shown in these figures, the translational vibration increases its torsion and decreases its circular frequency in proportion as the eccentricity increases, when the torsional rigidity is higher than the lateral rigidity as an ordinary building. On the contrary, the rotational vibration decreases the torsion and increases the circular frequency.

When the torsional rigidity is less than the lateral rigidity, the translational vibration invariably increases both the torsion and the circular frequency; the rotational vibration, on the contrary, decreases both of them. That is, the circular frequency in the latter case reverses its nature in the former.

It should be noted that the fundamental vibration becomes rotational and the torsion increases when the torsional rigidity is less than lateral rigidity, for instance as shown in Fig. 8, when the walls are concentrated around the center of the plan.

Now, let's make concrete the general attribute of the above-mentioned vibration. The natural vibration of Fig. 9 is shown in Fig. 10.

Forced vibration

If the system is subject to sinusoidal ground motion in the x direction, the solution is found from equation (1) by substituting $x_0 = a_0 \cos pt$, $y_0 = z_0 = 0$.

relative displacements:

$$\left.\begin{aligned} X_i &= \sum e_s \beta_s u_{si}\, a_0 (\cos pt - \cos n_s t) \\ Y_i &= \sum e_s \beta_s v_{si}\, a_0 (\cos pt - \cos n_s t) \\ Z_i &= \sum e_s \beta_s w_{si}\, a_0 (\cos pt - \cos n_s t) \end{aligned}\right\} \qquad (8)$$

accelerations:

$$\left.\begin{aligned} \ddot{X}_i &= \sum f_s \beta_s u_{si}\, a_0 \left(-\cos pt + \frac{e_s}{f_s} \cos n_s t\right) \\ \ddot{Y}_i &= \sum f_s \beta_s v_{si}\, a_0 \left(-\cos pt + \frac{e_s}{f_s} \cos n_s t\right) \\ \ddot{Z}_i &= \sum f_s \beta_s w_{si}\, a_0 \left(-\cos pt + \frac{e_s}{f_s} \cos n_s t\right) \end{aligned}\right\} \qquad (9)$$

wherein:

$$\beta_s = \frac{\sum_{i=1}^{m} M_i U_{si}}{\sum_{i=1}^{m} M_i (U_{si}^2 + V_{si}^2 + W_{si}^2)}, \qquad \begin{aligned} \sum_s \beta_s U_{si} &= 1 \\ \sum \beta_s V_{si} &= 0 \\ \sum \beta_s W_{si} &= 0 \end{aligned}$$

$$e_s = \frac{1}{\varphi_s^2 - 1} \qquad f_s = \frac{\varphi_s^2}{\varphi_s^2 - 1} \qquad \varphi_s = \frac{n_s}{p}$$

Where n_s is the circular frequency of the natural vibration, 'p' the circular frequency of ground motion, 'i' suffixe indicating the story location, and 's' suffixe indicating the vibration order.

Let's examine, concerning the simple case, how the location of the twisting center is varied as the period of the ground motion varies.

Let's take an example in a case when the building shown in Fig. 11 is subjected to the ground motion in the x direction and, moreover the free vibration has completely damped.

The distance of the twisting center from the center of gravity is found directly from the original equation of vibration, which can also be solved through formula (1), and is described as follows:

first story:

$$\frac{\rho_1}{i} = \frac{1}{e'} \cdot \frac{2c(c-e'^2) - (c^2+6c-5e'^2)\frac{1}{\varphi_0^2} + (3c+2)\frac{1}{\varphi_0^4} - \frac{1}{\varphi_0^6}}{2(c-e'^2) - (c+1)\frac{1}{\varphi_0^2} + \frac{1}{\varphi_0^4}} \tag{10}$$

second story:

$$\frac{\rho_2}{i} = \frac{1}{e'} \cdot \frac{3c(c-e'^2) - (c^2+9c-8e'^2)\frac{1}{\varphi_0^2} + 3(c+1)\frac{1}{\varphi_0^2} - \frac{1}{\varphi_0^6}}{3(c-e'^2) - (c+1)\frac{1}{\varphi_0^2}}$$

wherein:

$$C = e'^2 + j'^2 \qquad j'^2 = \left(\tfrac{j}{i}\right)^2 = \frac{a^2}{12\, i^2}\left\{\frac{n+1}{n-1} + \frac{m+1}{m-1}\,\alpha\,\lambda^2\right\}$$

According to formula (10), wherein $m = n = 3$, $e' = 0.3$, $\alpha = 1$, $\lambda = 1$, $i/_a = 0.4$ of the building in Fig. 11, the distance of the twisting center from the center of gravity is just as shown in Fig. 12.

Fig. 12 shows that the distance of the twisting center from the center of gravity of both the first and the second story are almost equal and become large in proportion to the calue of φ , but less than the value calculated by the static analysis, when the value of φ is large i.e. the building is rigid against the ground motion.

Eighteen percent is the most increasing rate of shearing force wrought by the torsion of the exterier column of both the first and the second story wherein φ is 2.

When φ is infinit, the distance of the twisting cinter from the center of gravity of both stories becomes equal and coincides with the value calculated by the static analysis, $\rho_1 = \rho_2 = \frac{e^2 + j^2}{e} = 2.898a$.

When the building is not rigid against the ground motion i.e. the value of φ is small, the vibration pattern is remarkably varied by the slight shift of φ and becomes complicated. It may be easily inferred that the more stories has a building,the more remarkable becomes this tendency.

Summary

The nature of the torsional vibration of the multi-storied buildings has so far been discussed, in which not only the centers of gravity of each story fall upon a single point as with those of rigidity, but also the radius of gyration both of mass and of stiffness, and the rigidity ratio of the transverse direction to the longitudinal, are all equal.

The natural vibration of such buildings as mentioned above can be found by combining thepplane vibration with the torsional vibration of single-storied buildings is the same as that of the single-storied buildings.

It should be noted that when the torsional rigidity is less than the lateral rigidity, the fundamental vibration is rotational and the torsion is serious; when the building is not rigid against the ground motion, the torsion becomes more serious in general and the vibration pattern is remarkably varied by the slight shift of the ground motion and the vibration pattern becomes more complicated.

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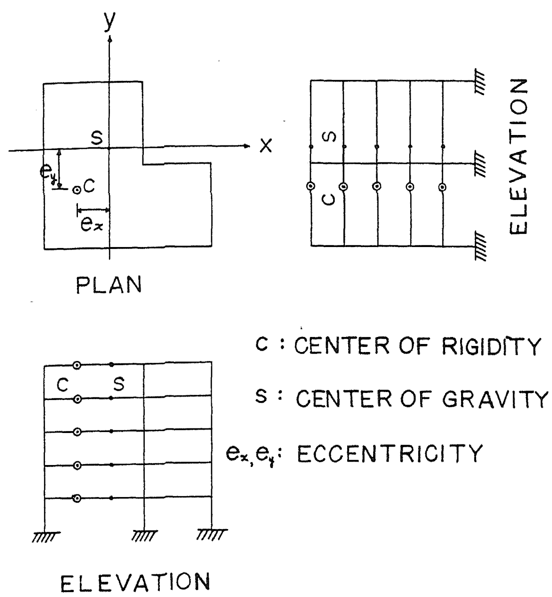


FIG. 1

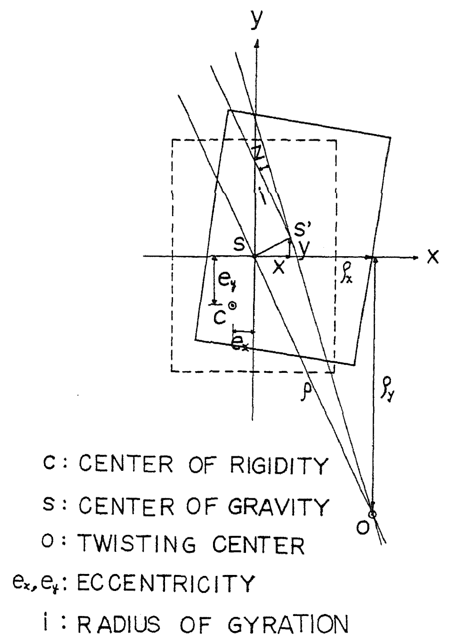


FIG. 2 DESCRIBTION OF THE VIBRATION

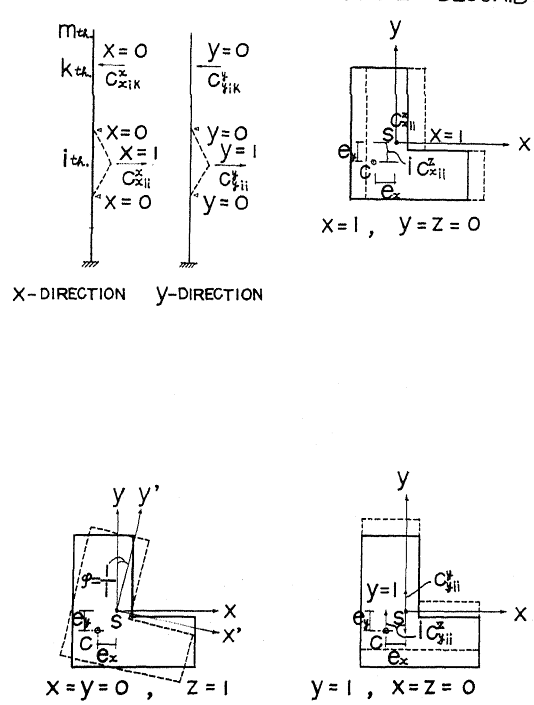


FIG. 3 INFLUENCE COEFFICIENTS

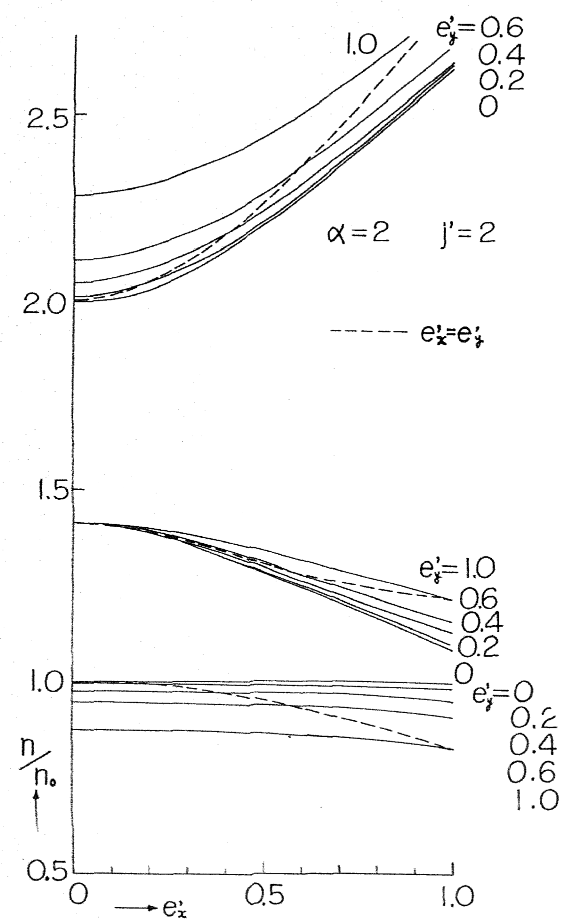


FIG.4 (a) NATURAL FREQUENCIES

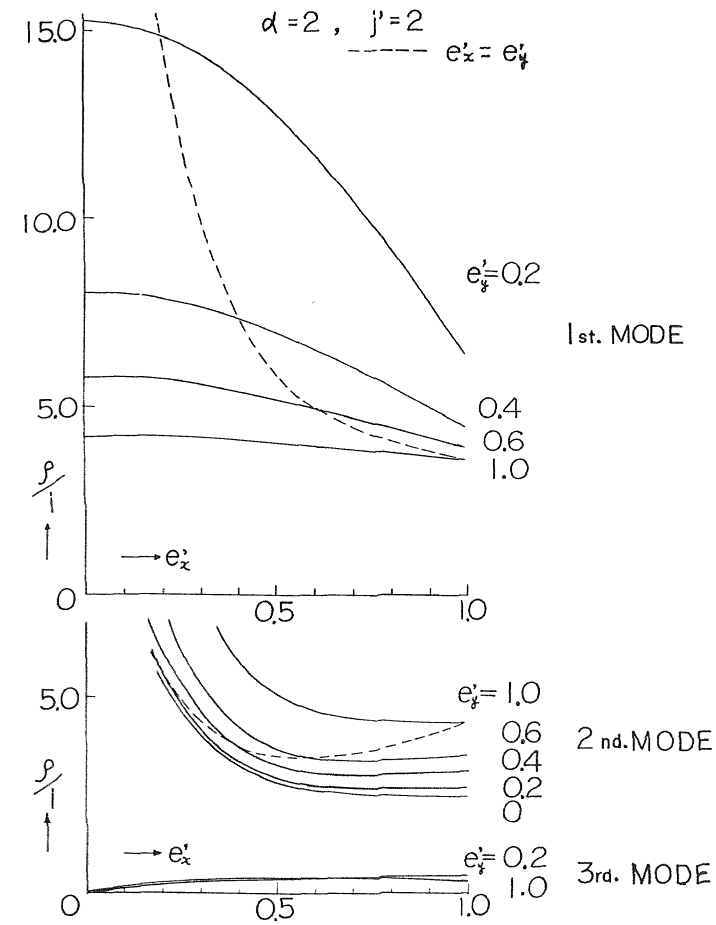


FIG. 4 (b) THE DISTANCE OF THE TWISTING CENTER FROM THE CENTER OF GRAVITY

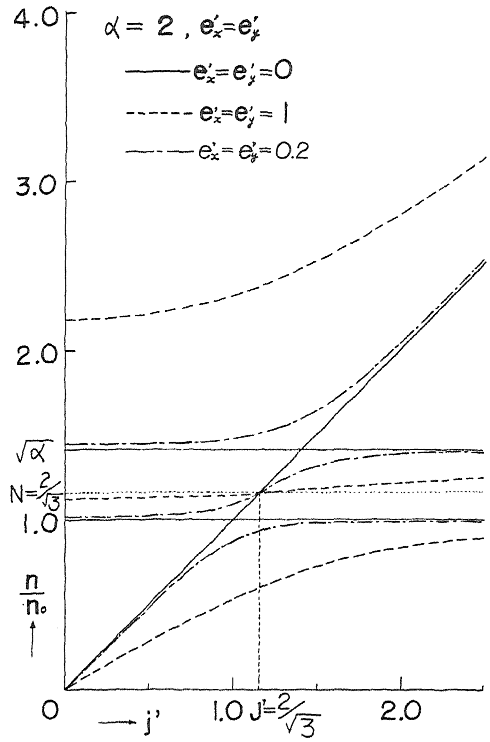


FIG. 5 (a) NATURAL FREQUENCIES

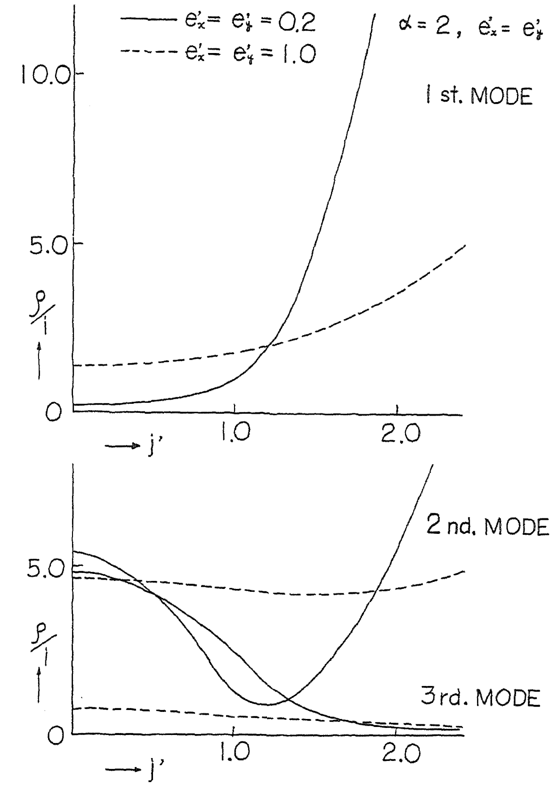


FIG. 5 (b) THE DISTANCE OF THE TWISTING CENTER FROM THE CENTER OF GRAVITY

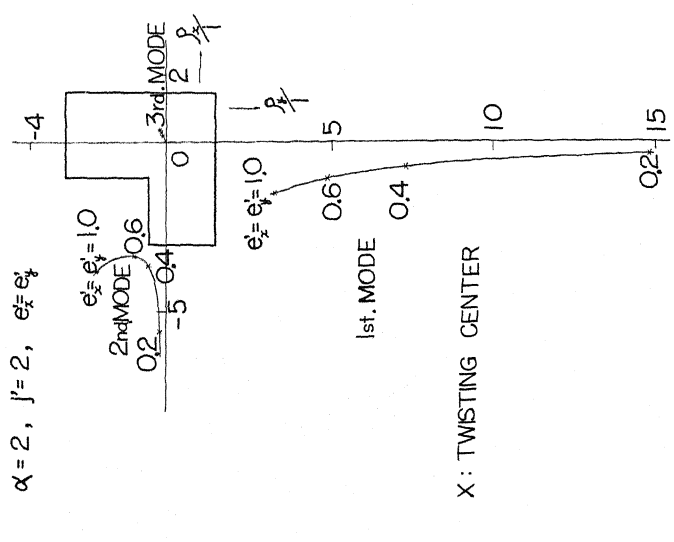


FIG. 6 LOCATION OF TWISTING CENTER

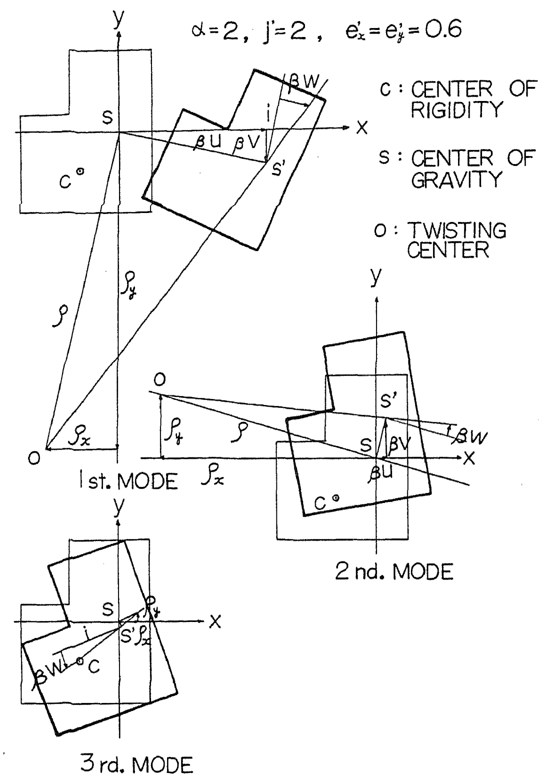


FIG. 7

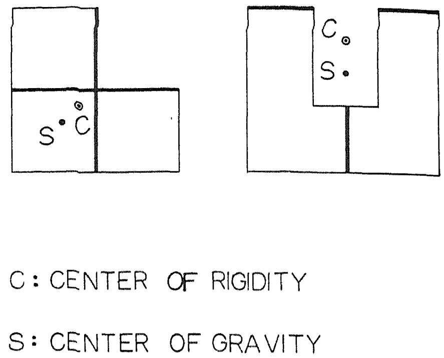


FIG. 8

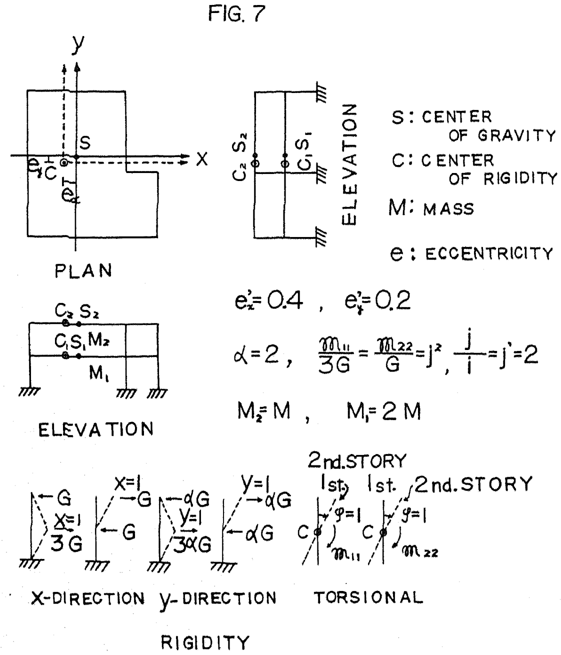


FIG. 9

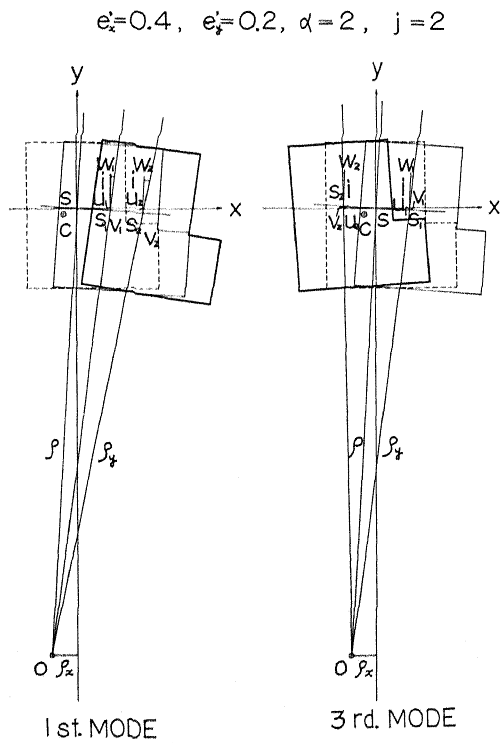


FIG. 10 (a)

$e'_x = 0.4$, $e'_y = 0.2$, $\alpha = 2$
$j' = 2$

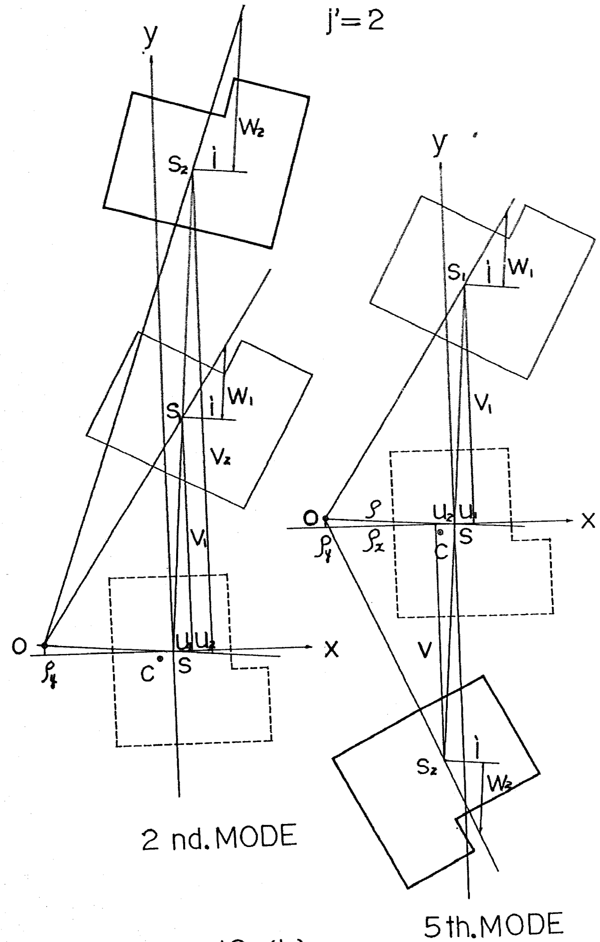
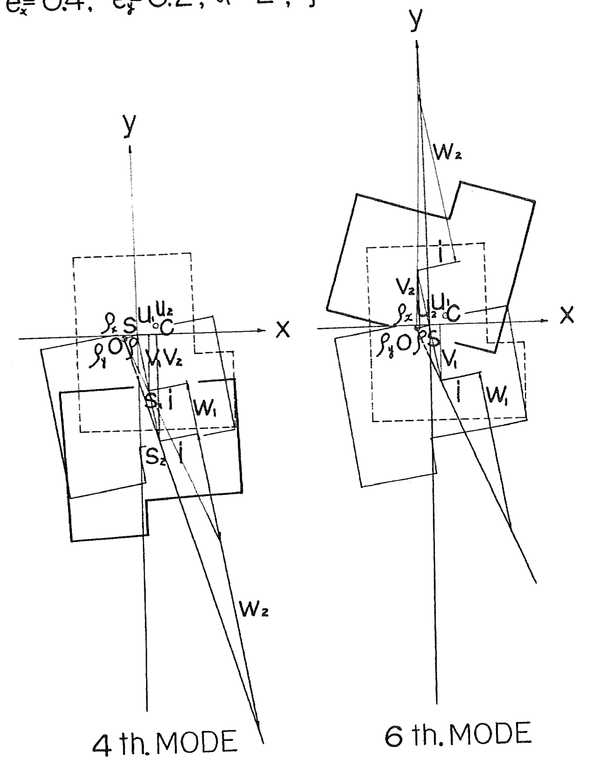


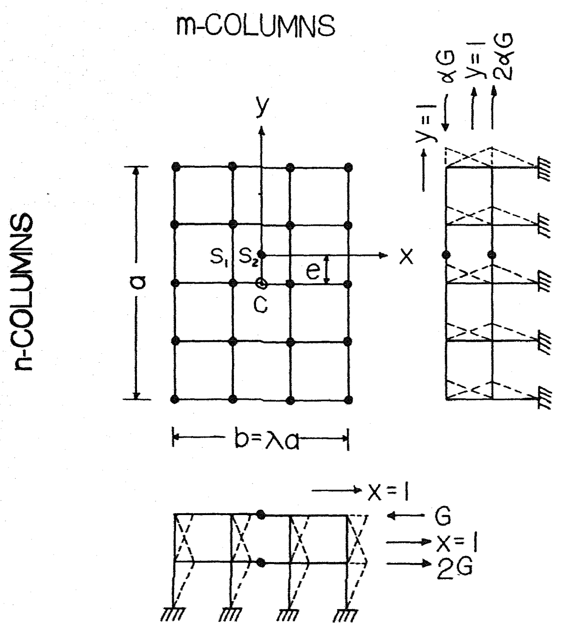
FIG. 10 (b)

$e'_x = 0.4$, $e'_y = 0.2$, $\alpha = 2$, $j' = 2$



c : CENTER OF RIGIDITY
s : CENTER OF GRAVITY
o : TWISTING CENTER

FIG. 10 (c)



s : CENTER OF GRAVITY

c : CENTER OF RIGIDITY

e : ECCENTRICITY

FIG. 11

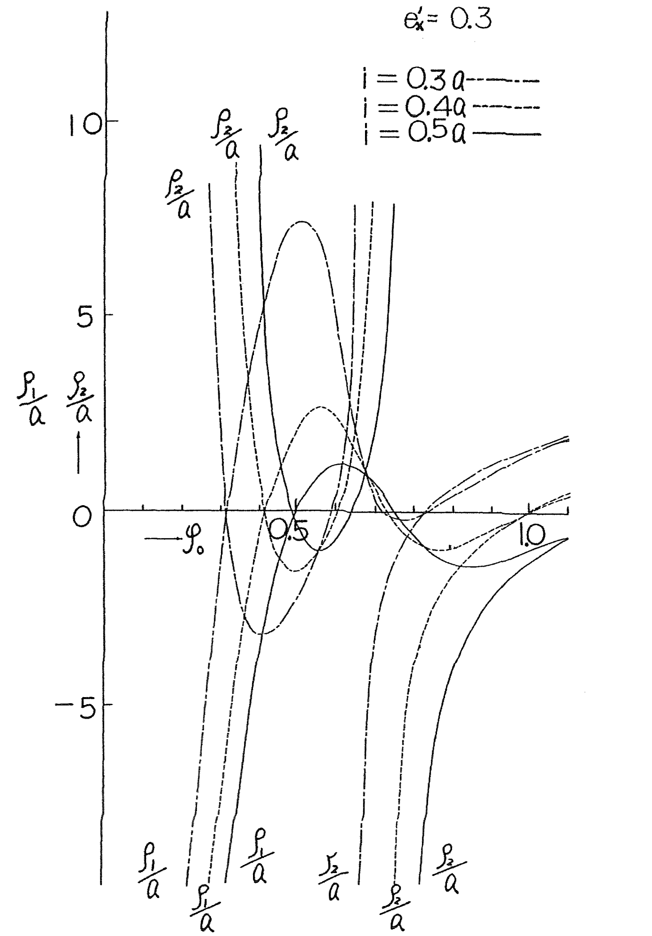


FIG. 12 THE DISTANCE OF THE TWISTING CENTER FROM THE CENTER OF GRAVITY

TORSIONAL VIBRATION OF MULTI-STORIED BUILDINGS

BY: T. SHIGA

QUESTION BY: D.A. LAWS - NEW ZEALAND.

In considering the building with one storey (three degrees of freedom) we can simplify the picture in two important special cases:

The case where the stiffness is the same in all directions, and the case where the centre of mass lies on one of the principal axes of stiffness. In either of these cases the translational motion along the line of centres (CS, Figure 1) can proceed independently of the other degrees of freedom, so that one of the three modes is of this character. The other two degrees of freedom - translation transverse to the line CS, and rotation interact to give two rotational modes. It seems to me that the centres of rotation of both these modes must lie on the line CS. Their positions along this line should be as given by the equations or figures in the paper by Skinner et al.

AUTHOR'S REPLY: As you mentioned, we can simplify the picture in two special cases. In either of these cases, the translational motion along the line CS (Fig.1) can proceed independently of the other degrees of freedom. The other two degrees of freedom interact to give two torsional modes and the centers of rotation of both these modes lie on the line CS.

In my paper, general cases are treated, where the stiffness of transverse direction is not equal to that of longitudinal direction and the center of gravity does not lie on the principal axes of stiffness.

In the cases mentioned above, three degrees of freedom interact to give three torsional modes, and the centers of rotation of these three modes do not lie on the CS line.