AN ENGINEERING APPROACH TO COMPUTING THE NATURAL MODES AND FREQUENCIES OF A TALL BUILDING

by

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ABSTRACT

The first n natural frequencies and mode shapes of an N degree of freedom structure ($n < N$) are derived from the solution of a reduced eigenvalue problem of order smaller than N. The reduced eigenvalue problem is formulated by using experience to select approximations to the first n modes desired. Accuracy is improved when more than n modes are selected. The method is illustrated by a study on an 18 story building.

INTRODUCTION

The analysis of a tall building subjected to dynamic excitations requires the knowledge of its natural frequencies and corresponding modes. The computations for these frequencies and modes can be accomplished by formulating an eigenvalue problem representing the undamped free vibration of the building and solving for the eigenvalues and corresponding eigenvectors. The order of the eigenvalue problem is equal to the number of degrees of freedom of the building. Thus, for a 20 story building with one degree of freedom per floor, the resulting eigenvalue is of order 20.

In most cases the response studies of a tall building are acceptable with only the first few modes being considered. If, for instance, only the first 4 natural frequencies and modes of a 20 degree of freedom structure are considered sufficient for purposes of analysis, then it is shown in this paper that using experience to predict the first 5 mode shapes the eigenvalue problem of order 20 can be reduced to one of order 5. The solution of the reduced eigenvalue problem of order 5 yields the first 4 natural frequencies and mode shapes with sufficient accuracy to be acceptable for all practical purposes.

In general a reduced eigenvalue problem of order n will yield acceptable results for the first (n-1) natural frequencies and modes. This greatly reduces the computational efforts and permits experience in predicting behavior to be entered as part of the problem solution. This should prove very useful in design as well as analysis.

A study of an 18 story building illustrates the method. The periods, frequencies and modes computed from reduced eigenvalue problems of orders 3,4,5 and 6 are compared in tables and figures with those obtained from the solution of the eigenvalue problem of order 18.

FREE VIBRATION OF A TALL BUILDING

Consider the N story building of Fig. 1 with the mass lumped at the floor levels. For free vibration in the north-south direction, the building has N degrees of freedom; one at each floor level. In free vibration the only forces acting on the masses are the elastic spring forces

$$\{F\} = -[k]\{u\} \tag{1}$$

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in which {F} and {u} are respectively the vectors of forces and displacements at the floor levels and [k] is the corresponding stiffness matrix. The minus sign in Eq. (1) results from the fact that the forces acting on the masses are opposite in sign and equal in magnitude to the forces [k]{u} that act on the building frames at the floor levels, and cause a displacement {u} (see Fig. 1b). According to Newton's 2nd law of motion the forces -[k]{u} acting on the masses are equated to the product of the mass matrix ⌈ m ⌋ and the floor accelerations vector $\{\ddot{u}\}$

$$[k]\{u\} = \lceil m \rfloor \{\ddot{u}\} \tag{2}$$

where ⌈ ⌋ designates a diagonal matrix with non-zero elements on the principal diagonal only.

For the N degree of freedom building of Fig. 1a, Eq. (2) forms a set of N linear second order differential equations. The solution is given by

$$\{\ddot{u}\} = -\omega^2\{u\} \tag{3}$$

in which ω is the angular natural frequency of the building vibrating in a natural mode represented by {u} . Substituting Eq. (3) in (2)

$$[k]\ \{u\} = \omega^2 \lceil m \rfloor \{u\} \tag{4}$$

Premultiplying each side of Eq. (4) by $[k]^{-1}$ results in

$$\{u\} = \omega^2 [k]^{-1} \lceil m \rfloor \{u\} \tag{5}$$

Since $[k]^{-1} = [a]$ where [a] is the flexibility matrix corresponding to the lateral forces and displacements at the floor levels, Eq. (5) becomes

$$[D]\ \{u\} = \frac{1}{\omega^2}\{u\} \tag{6}$$

where

$$[D] = [a]\ \lceil m \rfloor$$

Eq. (6) represents an eigenvalue problem. The quantities $1/\omega^2$ are the eigenvalues. The number of eigenvalues satisfying Eq. (6) is equal to the number of degrees of freedom. To each eigenvalue $1/\omega_i^2$ $(i = 1,2,\ldots,N)$ there corresponds an eigenvector $\{u^{(i)}\}$ which represents the i^{th} natural mode of the vibrating structure.

SOLUTION OF THE EIGENVALUE PROBLEM

1. Classical Approach

Eq. (6) can be written in the form

$$\left([D] - \frac{1}{\omega^2}[I]\right)\{u\} = \{o\} \tag{7}$$

in which [I] is the unit matrix. For a non-trivial solution the determinant

$$\left|\ [D] - \frac{1}{\omega^2}[I]\ \right| = \Delta$$

must vanish, i.e.,

$$\Delta = 0 \tag{8}$$

Eq. (8) is an Nth order polynomial in $1/\omega^2$ from which the eigenvalues $1/\omega_i^2$ $(i = 1,2,\ldots,N)$ can be solved for. The mode shapes $\{u^{(i)}\}$ are found by substituting the corresponding eigenvalue $1/\omega_i^2$ in

$$\left([D] - \frac{1}{\omega_i^2}[I]\right)$$

and computing any column of the adjoint matrix of $\left([D] - \frac{1}{\omega_i^2}[I] \right)$. Any such column is proportional to the desired mode shape $\{u^{(i)}\}$.

2. Matrix Iteration

The natural frequencies and mode shapes can also be found from Eq. (6) by matrix iterations.[1] [2] The iteration begins by selecting a trial column $\{u\}_{t1}$ for the first mode and carrying out the multiplication $[D]\{u\}_{t1}$. The resulting column vector $\{u\}_{t2}$ is used as a second trial column in carrying out the multiplication $[D]\{u\}_{t2}$. This operation is continued until a new trial column is proportional to the preceding one, that is until

$$[D]\{u\}_{tr} = \{u\}_{t(r+1)} = \lambda \{u\}_{tr}$$

where $\lambda = \frac{1}{\omega_1^2}$, and tr designates the r^{th} trial column.

Using matrix iteration, Eq. (6) will converge to the first mode. Using the condition of orthogonality of the natural modes, the iteration procedure applied to Eq. (6) will yield the higher modes in ascending order.[1]

When each side of Eq. (6) is premultiplied by $\omega^2[D]^{-1}$, then

$$[D]^{-1}\{u\} = \omega^2 \{u\} \tag{9}$$

where

$$[D]^{-1} = \lceil m \rfloor^{-1}[a]^{-1} = \lceil m \rfloor^{-1}[k]$$

Matrix iteration applied to Eq. (9) yields the highest frequency and corresponding mode first. The lower modes can be obtained in descending order by applying the conditions of orthogonality.[1]

GENERALIZED STIFFNESS MATRIX

When a structure, such as the one shown in Fig. 1, is subjected to external forces $\{F\}$ the strain energy stored in the structure can be expressed by

$$U = \frac{1}{2} \sum_{i=1}^{N} F_i u_i \tag{10}$$

in which $u_i (i = 1,2,\ldots,N)$ are the displacements on which forces F_i $(i = 1,2,\ldots,N)$ do work. Eq. (10) can be written in matrix form as

$$U = \frac{1}{2}\{F\}^T\{u\} \tag{11}$$

Using the stiffness matrix $[k]$, force vector $\{F\}$ and displacement vector $\{u\}$ can be related by the equation

$$\{F\} = [k]\{u\} \tag{12}$$

Recalling that $[k]$ is symmetric the transpose of Eq. (12) is

$$\{F\}^T = \{u\}^T[k] \tag{13}$$

Substituting Eq. (13) into Eq. (11)

$$U = \frac{1}{2} \{u\}^T [k] \{u\} \tag{14}$$

Applying a linear coordinate transformation

$$\{u\} = [\phi] \{q\} \tag{15}$$

where displacements $u_i (i=1,2,...,N)$ are expressed in terms of generalized coordinates $q_i (i=1,2,...,n)$, with $n \leq N$, then Eq. (14) becomes

$$U = \frac{1}{2} \{q\}^T [k]_q \{q\} \tag{16}$$

in which

$$[k]_q = [\phi]^T [k] [\phi] \tag{17}$$

Matrix $[k]_q$ is the generalized stiffness matrix of order nxn in the q coordinate system. The strain energy in the u and q coordinates, expressed by Eqs. (14) and (16) respectively, is identical because it is invariant under a coordinate transformation.

GENERALIZED MASS MATRIX

The kinetic energy, T, of the structure in Fig. 1 can be written as

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{u}_i^2 \tag{18}$$

where $\dot{u}_i$ is the velocity of the i^{th} floor mass. Eq. (18) has the matrix form

$$T = \frac{1}{2} \{\dot{u}\}^T \lceil m \rfloor \{\dot{u}\} \tag{19}$$

Applying a linear coordinate transformation as expressed by Eq. (15)

$$\{u\} = [\phi] \{q\}$$

then

$$\{\dot{u}\} = [\phi] \{\dot{q}\} \quad \text{and} \quad \{\dot{u}\}^T = \{\dot{q}\}^T [\phi]^T$$

Substituting for $\{\dot{u}\}$ and $\{\dot{u}\}^T$ from the above in Eq. (19) gives

$$T = \frac{1}{2} \{\dot{q}\}^T [m]_q \{\dot{q}\} \tag{20}$$

in which

$$[m]_q = [\phi]^T \lceil m \rfloor [\phi] \tag{21}$$

is the generalized mass matrix of order nxn in the q coordinate system. The kinetic energy in the u and q coordinates expressed by Eqs. (19) and (20) respectively, is identical because it is invariant under a coordinate transformation.

GENERALIZED STIFFNESS MATRIX IN FREE VIBRATION

Consider the structure of Fig. 1 undergoing free vibration in the north-south direction. The strain energy is given by Eq. (10) which is written as

$$U = \frac{1}{2} \{u\}^T \{F\} \tag{22}$$

Using the flexibility matrix [a] vectors {F} and {u} can be related by

$$\{u\} = [a]\{F\}$$

Recalling that [a] is symmetric the transpose of the last equation is

$$\{u\}^T = \{F\}^T[a] \tag{23}$$

Substituting Eq. (23) into Eq. (22) gives

$$U = \frac{1}{2}\{F\}^T[a]\{F\} \tag{24}$$

The forces {F} in Eq. (24) are inertial forces in the case of free vibration and are given by (see Eq. 4)

$$\{F\} = \omega^2 [\, m \,] \{u\} \tag{25}$$

Transposing each side of Eq. (25) and recalling that $[\, m \,]$ is diagonal, then

$$\{F\}^T = \omega^2 \{u\}^T [\, m \,] \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (24) gives

$$U = \frac{1}{2}\omega^4 \{u\}^T [\, m \,][a][\, m \,]\{u\} \tag{27}$$

Applying a linear coordinate transformation as expressed by Eq. (15) and substituting for {u} and $\{u\}^T$ in terms of {q} in Eq. (27) yields

$$U = \frac{1}{2}\omega^4 \{q\}^T [\phi]^T [\, m \,][a][\, m \,][\phi]\{q\} = \frac{1}{2}\{q\}^T [K]_q \{q\}$$

in which

$$[K]_q = \omega^4 [\phi]^T [\, m \,][a][\, m \,][\phi] = \omega^4 [G] \tag{28}$$

is the generalized stiffness matrix in the q coordinates.

THE REDUCED EIGENVALUE PROBLEM

Consider the structure of Fig. 1 subjected to a dynamic excitation. Let the response of the structure be determined by the normal mode method using the first (n-1) natural modes and frequencies of the building. Using the Rayleigh Ritz method[1][3] the first (n-1) mode shapes are assumed, and the actual mode shapes can be approximated from an eigenvalue problem formulated through the use of the assumed modes. The accuracy of the results is improved when more modes are assumed than the number of modes desired. Let n modes be assumed; then the displacement vector {u} can be written as in Eq. (15) where each column $\{\phi^{(i)}\}$ of $[\phi]$ represents an assumed mode shape. (See Fig. 2.) Matrix $[\phi]$ is of order Nxn with $n \leq N$. The q's are the generalized coordinates associated with the assumed mode shapes. The assumed modes can be based on experience in predicting behavior or on the knowledge of the natural modes of structures similar to the one under consideration.

The equations of motion for free vibration in the q coordinates are

$$-[K]_q \{q\} = [m]_q \{\ddot{q}\} \tag{29}$$

This equation is identical to Eq. (2) except that here the equations are formulated in the q coordinates while Eq. (2) holds true in the u coordinate system. The generalized mass matrix $[m]_q$ and the generalized stiffness $[K]_q$ which appear in Eq. (29) are given by Eqs. (21) and (28) respectively. Substituting from Eq. (28) the value $\omega^4[G]$ for $[K]_q$ into Eq. (29) and using the relation

$$\{\ddot{q}\} = -\omega^2 \{q\}$$

which holds true in free vibration, Eq. (29) becomes

$$[m]_q\{q\}= \omega^2[G]\{q\} \quad (30)$$

or

$$[\Phi]\ \{q\}= \frac{1}{\omega^2}\{q\} \quad (31)$$

where

$$[\Phi] = [m]_q^{-1}[G]$$

Eqs. (30) and (31) are analogous to Eqs. (4) and (6) respectively. Note that $[m]_q$ in Eq. (30) is analogous to [k] in Eq. (4), and [G] in Eq. (30) is analogous to [m] in Eq. (4). Eqs. (30) and (31) represent the formulation of an eigenvalue problem of order $n < N$. For instance, if only the first four modes and frequencies are desired, the first five mode shapes may be assumed so that $[\phi]$ in Eq. (15) is of order Nx5 and the resulting eigenvalue problem of Eq. (31) is of order 5 instead of order N as it appears in Eqs. (4) and (6) in the u coordinates. The solution of Eq. (30) or (31) can be accomplished by the classical method or by matrix iteration as discussed earlier. Matrix iteration applied to Eq. (31) will yield the lowest frequency first and then higher frequencies. If convergence to the n^{th} (highest assumed mode) mode is desired first, then the iteration should be applied to the equation

$$[\Phi]^{-1}\{q\} = \omega^2\ \{q\} \quad (32)$$

Eq. (32) is obtained from Eq. (31) by premultiplying each side by $\omega^2[\Phi]^{-1}$.

Once the n natural frequencies and corresponding eigenvectors $\{q^{(i)}\}$ $(i = 1,2,\dots,n)$ are computed from Eqs. (30), (31) or (32), the mode shapes in the u coordinates are obtained from Eq. (15). The first mode $\{u^{(1)}\}$ is obtained by substituting $\{q^{(1)}\}$ in Eq. (15); similarly $\{u^{(2)}\}$ is obtained from $\{q^{(2)}\}$ and finally $\{u^{(n)}\}$ from $\{q^{(n)}\}$. A comparative study of the natural modes of a tall building obtained by this method follows.

NATURAL MODES OF A TALL BUILDING COMPUTED FROM A REDUCED EIGENVALUE PROBLEM

The reduced eigenvalue problem expressed by Eq. (31) is now used to compute the natural frequencies and modes in the north-south direction for the 18 story framed building of Figs. 3 and 4. The following cases are studied:

Case 3	The first 3 modes are computed from a reduced eigenvalue problem of order 3.
Case 4	The first 4 modes are computed from a reduced eigenvalue problem of order 4.
Case 5	The first 5 modes are computed from a reduced eigenvalue problem of order 5.
Case 6	The first 6 modes are computed from a reduced eigenvalue problem of order 6.
Case a	All 18 modes are computed from an eigenvalue problem of order 18.

The case numbers 3,4,5 and 6 are chosen the same as the number of modes computed, for ease of reference. The 18x18 flexibility matrix [a] of the building in Figs. 3 and 4 was derived accounting for joint rotation as well as axial deformation in the columns.[4]

The assumed shapes used in Cases 3,4,5 and 6 were those of a uniform slender cantilever beam which are given by[1]

$$u(\frac{y}{H})=\frac{[\cosh\beta H+\cos\beta H][\sinh\beta H(\frac{y}{H})-\sin\beta H(\frac{y}{H})]-[\sinh\beta H+\sin\beta H][\cosh\beta H(\frac{y}{H})-\cos\beta H(\frac{y}{H})]}{2(\sinh\beta H\cos\beta H-\cosh\beta H\sin\beta H)} \quad (33)$$

where $u(\frac{y}{H})$ is the displacement at a distance y from the base of the building relative to the displacement at the top of the building (taken as unity for convenience). H is the height of the building. The value of βH in Eq. (33) was approximately taken as

$$\beta H = \frac{2\alpha-1}{2}\pi \quad (34)$$

where α is the mode number. Thus to determine the first assumed mode shape $\{\phi^{(1)}\}$ the value $\beta H = \pi/2$ was substituted in Eq. (33). For the second assumed mode $\{\phi^{(2)}\}$, $\beta H = 3/2\pi$ was substituted in Eq. (33), and for the sixth assumed mode $\{\phi^{(6)}\}$, $\beta H = 11/2\pi$ was substituted in Eq. (33). The six assumed modes are given in Table 1.

The matrices $[m]_q$ and [G] as defined by Eqs. (21) and (28) respectively have the following forms for Case 6:

$$[m]_q=\begin{bmatrix} 7831.65 & 1086.43 & 1116.54 & 1158.80 & 1179.00 & 1231.71 \\ 1086.43 & 7897.29 & 1170.97 & 1151.92 & 1262.03 & 1188.56 \\ 1116.54 & 1170.97 & 7916.26 & 1290.92 & 1125.88 & 1420.42 \\ 1158.80 & 1151.92 & 1290.92 & 7883.85 & 1468.90 & 1036.87 \\ 1179.00 & 1262.03 & 1125.88 & 1468.90 & 7786.28 & 1696.46 \\ 1231.71 & 1188.56 & 1420.42 & 1036.87 & 1696.46 & 7643.92 \end{bmatrix}$$

$$[G]=\begin{bmatrix} 2756.40 & 238.08 & 337.67 & 328.96 & 371.16 & 392.40 \\ 238.08 & 357.78 & 73.93 & 61.78 & 65.03 & 69.97 \\ 337.67 & 73.93 & 155.49 & 71.70 & 60.78 & 64.30 \\ 328.96 & 61.78 & 71.70 & 97.72 & 65.99 & 58.74 \\ 371.16 & 65.03 & 60.78 & 65.99 & 85.56 & 69.96 \\ 392.40 & 69.97 & 64.30 & 58.74 & 69.96 & 80.26 \end{bmatrix}$$

The matrices $[m]_q$ and [G] for Case 5 are obtained from the above by striking out the 6th row and 6th column of each matrix. Similarly for Case 4 the last 2 rows and columns are eliminated from the above matrices, and for Case 3 the last 3 rows and columns are eliminated.

The $[\phi]$ matrix as defined by Eq. (15) has the form of Table 1 for Case 6. Matrix $[\phi]$ for Cases 3,4 and 5 is also obtained from Table 1 by using respectively the first 3,4 and 5 columns corresponding to the first 3,4 and 5 assumed mode shapes.

The natural vibration periods and frequencies computed for Cases 6,5, 4 and 3 are listed in Tables 2 and 3. The first (n-1) are found to be in good agreement with the same quantities found from the solution of the eigenvalue problem of order 18, (Case a). The mode shapes computed for each case are compared with the corresponding modes as computed from the eigenvalue problem of order 18 in Figs. 5 to 22. Here again, in general, the first (n-1) modes appear to agree very well with those computed from the eigenvalue problem of order 18 (Case a).

CONCLUSIONS

An examination of Tables 2,3 and Figs. 5 through 22 indicates that using assumed modes, the first n natural frequencies and modes of a tall, slender framed building can be obtained with acceptable accuracy through the solution of a reduced eigenvalue problem of order (n+1). The assumed modes in the example of this study are those of a uniform slender cantilever beam, although the actual stiffness of the building varies greatly with height, (Fig. 4). This did not affect the computed periods and frequencies very much. The computed modes, however, are more sensitive to the choice of assumed modes and will be more accurate when more care is taken in selecting the appropriate assumed shapes.

Because it is generally acceptable to consider only the first few modes when the response of a tall building is studied, there are two important features in the approach presented here:

1. The analyst or designer has an opportunity to enter his experience in predicting behavior as part of the problem solution.
2. The order of the resulting eigenvalue problem is greatly reduced, thus simplifying computations.

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3. Ritz, W., Gesammelte Werke, Gauthier-Villars, Paris, 1911.
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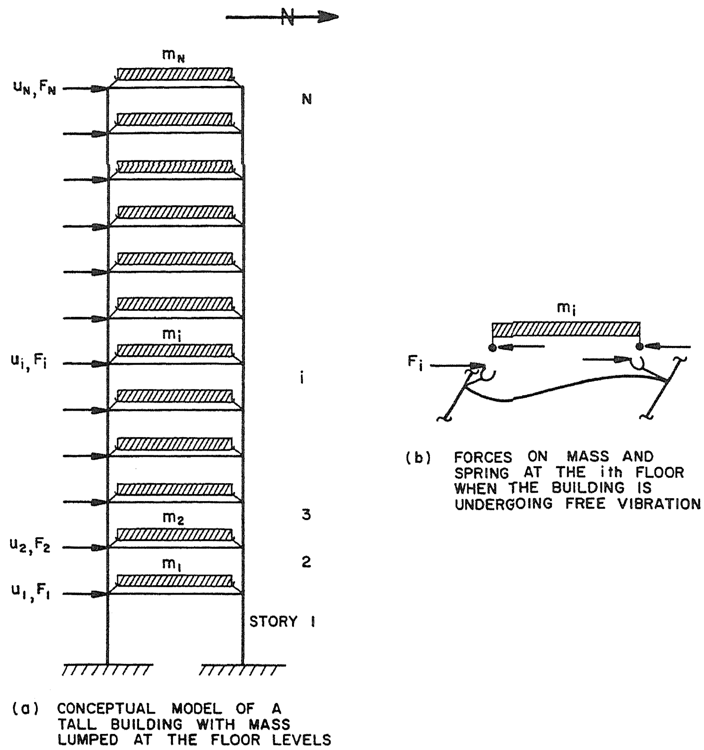


FIGURE 1

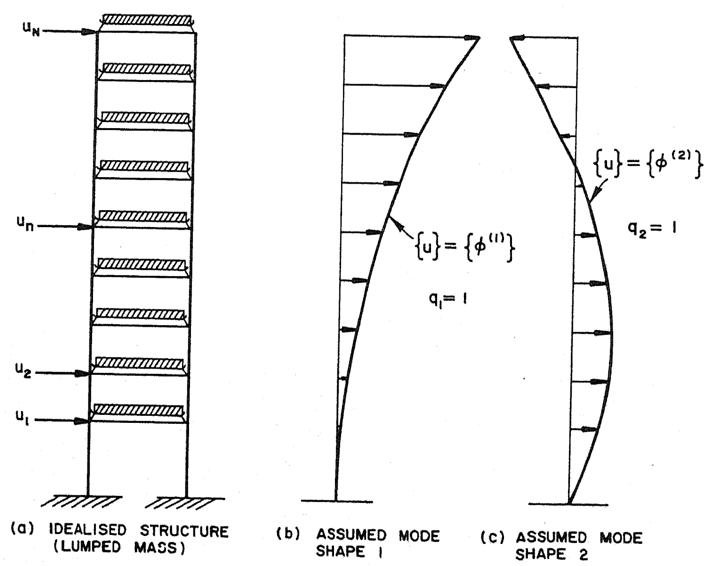


FIGURE 2

TABLE 1. Assumed Modes for the Building of Figs. 3 & 4 Computed from Eq. 33.

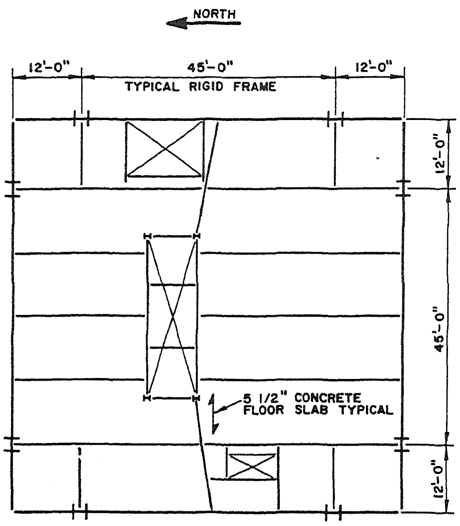
Level	1st Mode	2nd Mode	3rd Mode	4th Mode	5th Mode	6th Mode
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
18	0.9254	0.7412	0.5763	0.4091	0.2465	0.0903
17	0.8559	0.5016	0.1943	-0.0947	-0.3388	-0.5219
16	0.7866	0.2678	-0.1484	-0.4725	-0.6421	-0.6356
15	0.7176	0.0453	-0.4213	-0.6520	-0.5744	-0.2423
14	0.6492	-0.1594	-0.5971	-0.6031	-0.1979	0.3461
13	0.5817	-0.3399	-0.6586	-0.3544	0.2891	0.6961
12	0.5156	-0.4901	-0.6027	0.0114	0.6414	0.5519
11	0.4511	-0.6051	-0.4426	0.3796	0.6832	0.0144
10	0.3887	-0.6812	-0.2057	0.6366	0.3925	-0.5326
9	0.3290	-0.7171	0.0697	0.7032	-0.0891	-0.6985
8	0.2725	-0.7133	0.3405	0.5579	-0.5263	-0.3646
7	0.2196	-0.6726	0.5652	0.2428	-0.7049	0.2297
6	0.1710	-0.6001	0.7107	-0.1490	-0.5365	0.6589
5	0.1272	-0.5029	0.7574	-0.5033	-0.1013	0.6145
4	0.0889	-0.3902	0.7033	-0.7189	0.3919	0.1253
3	0.0566	-0.2726	0.5643	-0.7405	0.7111	-0.4659
2	0.0310	-0.1621	0.3739	-0.5784	0.7184	-0.7537
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 2. Periods for the Building of Figures 3 and 4.

Mode Number	Periods (in seconds) computed from an eigenvalue problem (Eq. 6) of order 18.	Periods (in seconds) computed from a reduced eigenvalue problem (Eq. 31) of order			
		6	5	4	3
	CASE (a)	CASE 6	CASE 5	CASE 4	CASE 3
1	3.7348	3.7346	3.7346	3.7346	3.7338
2	1.3115	1.3111	1.3109	1.3096	1.3089
3	.7596	.7500	.7580	.7580	.7484
4	.5206	.5202	.5186	.4996	
5	.3900	.3866	.3640		
6	.3057	.2682			

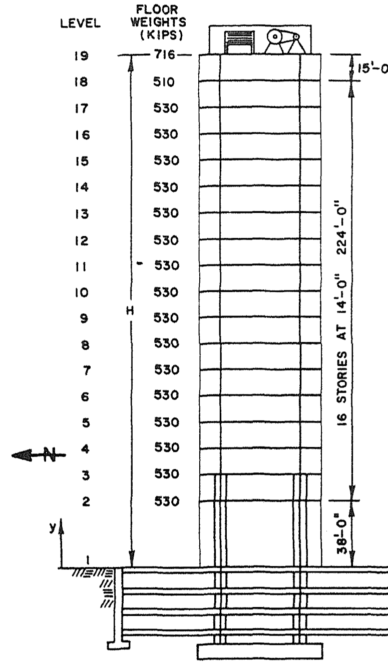
TABLE 3. Frequencies for the Building of Figures 3 and 4.

Mode Number	Frequencies (in c.p.s.) computed from an eigenvalue problem (Eq. 6) of order 18.	Frequencies (in c.p.s.) computed from a reduced eigenvalue problem (Eq. 31) of order			
		6	5	4	3
	CASE (a)	CASE 6	CASE 5	CASE 4	CASE 3
1	.2677	.2678	.2678	.2677	.2678
2	.7624	.7627	.7628	.7635	.7640
3	1.3164	1.3175	1.3192	1.3192	1.3362
4	1.9206	1.9222	1.9283	2.0015	
5	2.5640	2.5364	2.7470		
6	3.2711	3.7289			



TYPICAL FLOOR PLAN OF TALL BUILDING

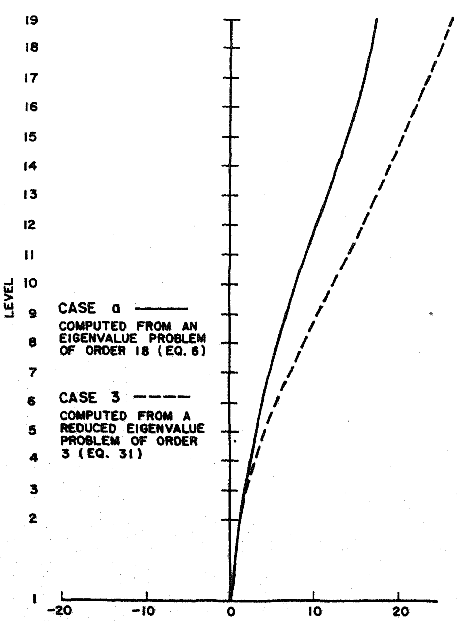
FIGURE 3



LEVEL	FLOOR WEIGHTS (KIPS)	GIRDERS Moment of Inertia (in^4)	COLUMNS Moment of Inertia (in^4)	COLUMNS Area (in^2)
19	716	9,830	4,829	56.3
18	510	9,830	4,829	56.3
17	530	9,830	5,900	57.8
16	530	9,830	5,900	57.8
15	530	9,830	5,900	57.8
14	530	10,428	10,300	77.0
13	530	10,428	10,300	77.0
12	530	10,428	10,300	77.0
11	530	12,480	14,800	94.0
10	530	12,480	14,800	94.0
9	530	12,480	14,800	94.0
8	530	20,450	17,950	122.5
7	530	20,450	17,950	122.5
6	530	20,450	17,950	122.5
5	530	20,450	19,960	132.5
4	530	20,450	19,960	132.5
3	530	42,954	106,930	301
2	530	42,954	106,930	301
1				

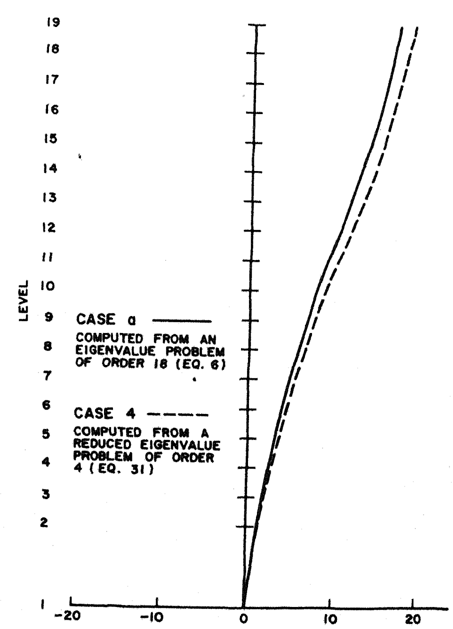
SCHEMATIC ELEVATION SHOWING TYPICAL RIGID FRAME OF BUILDING IN FIGURE 3

FIGURE 4



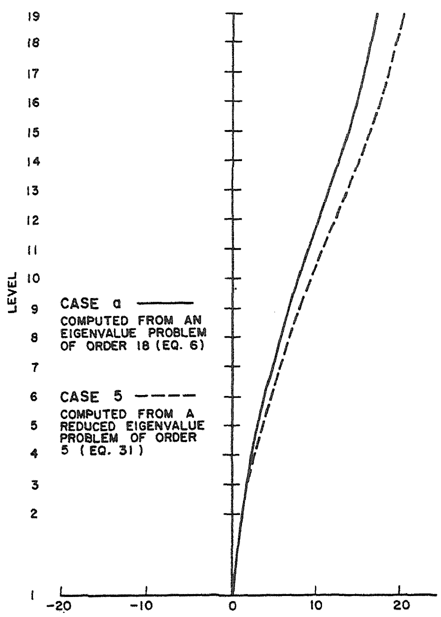
FIRST MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4 COMPUTED FOR CASES a AND 3

FIGURE 5



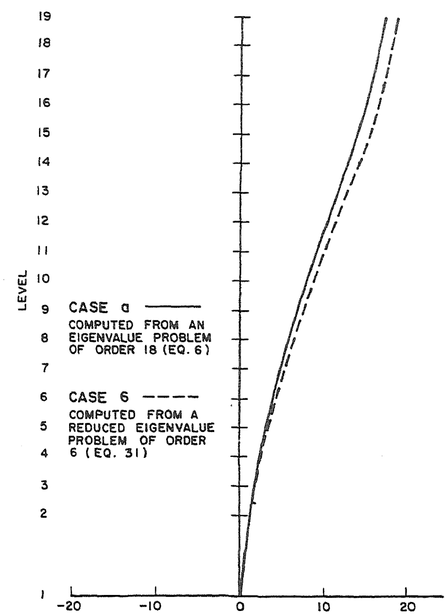
FIRST MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4 COMPUTED FOR CASES a AND 4

FIGURE 6



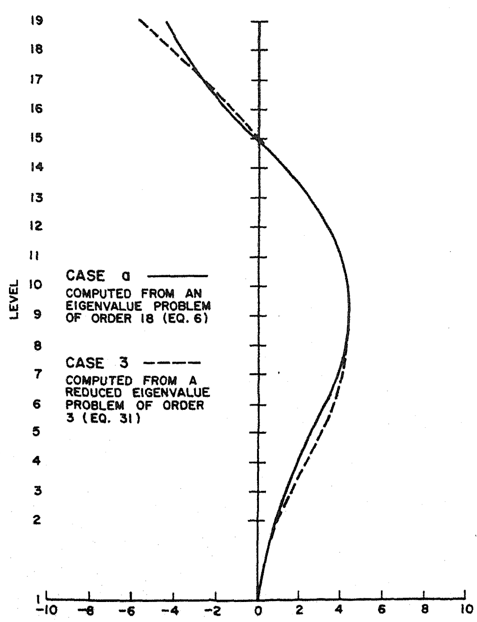
FIRST MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 5

FIGURE 7



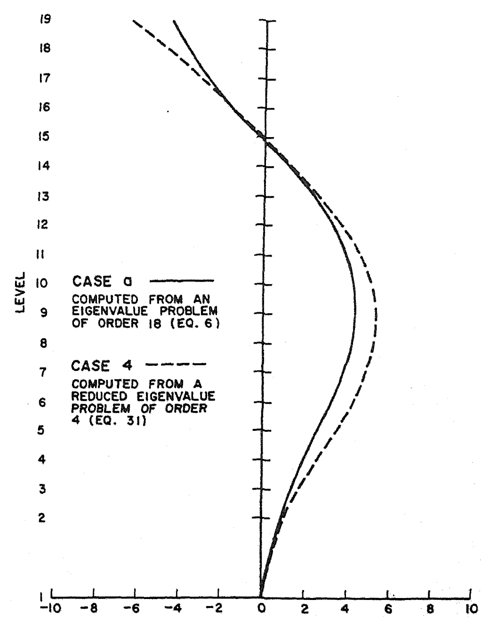
FIRST MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 6

FIGURE 8



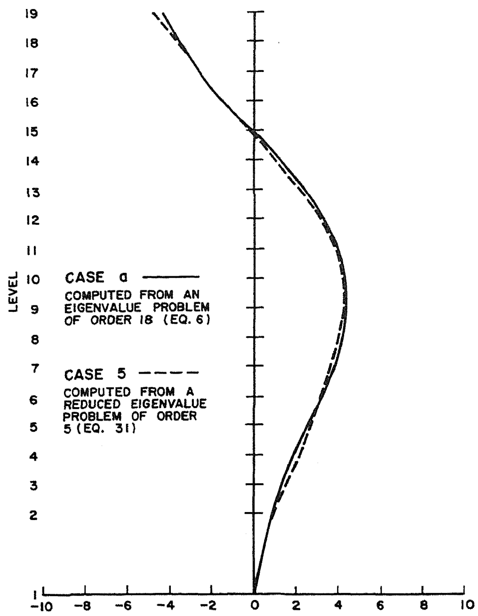
SECOND MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 3

FIGURE 9



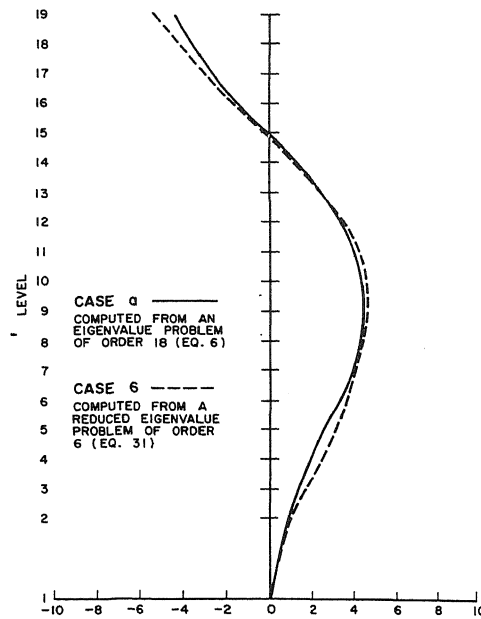
SECOND MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 4

FIGURE 10



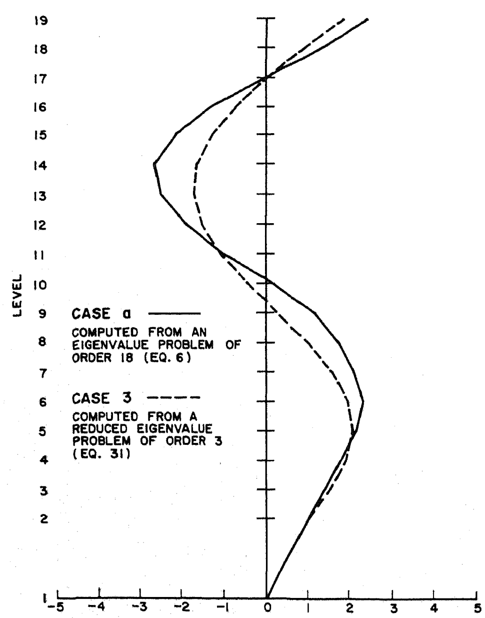
SECOND MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 5

FIGURE 11



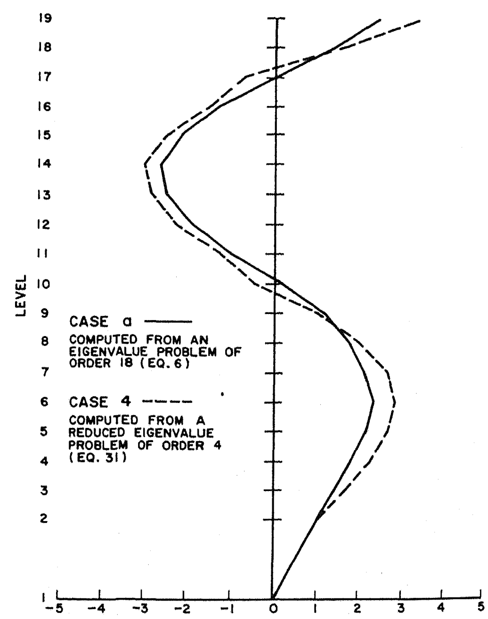
SECOND MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 6

FIGURE 12



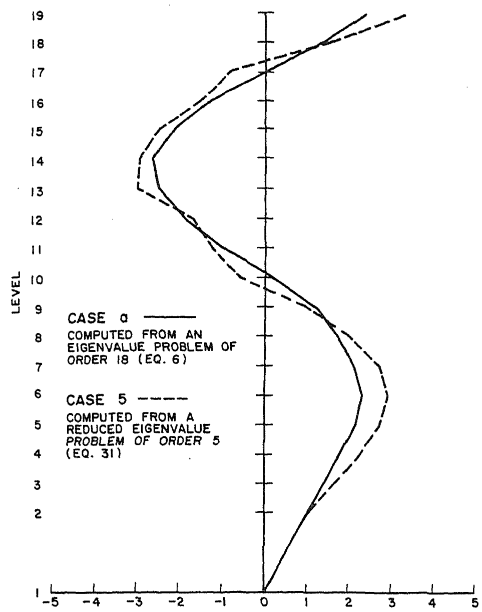
THIRD MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 3

FIGURE 13



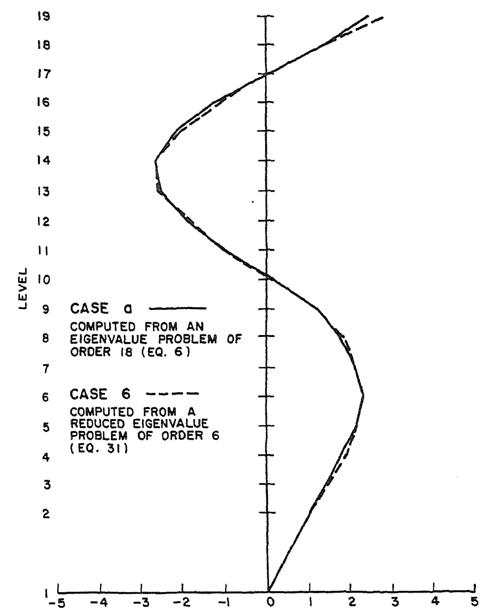
THIRD MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 4

FIGURE 14



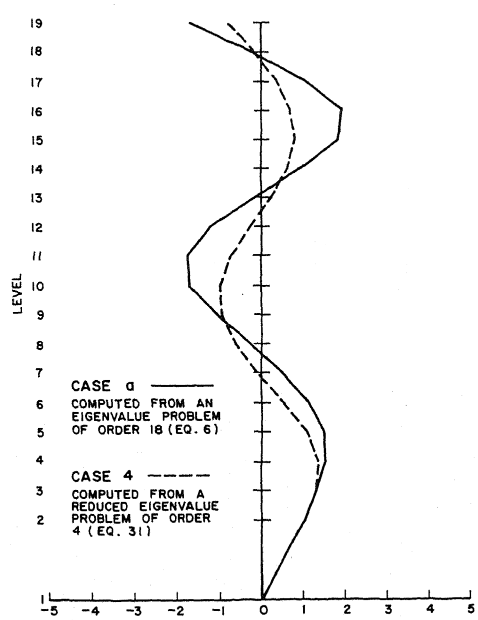
THIRD MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 5

FIGURE 15



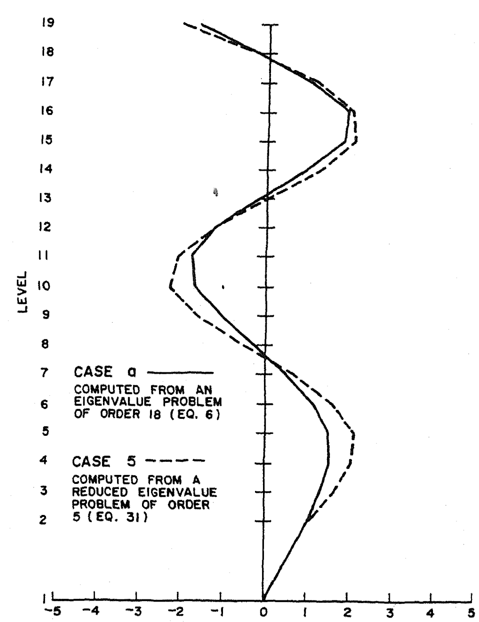
THIRD MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 6

FIGURE 16



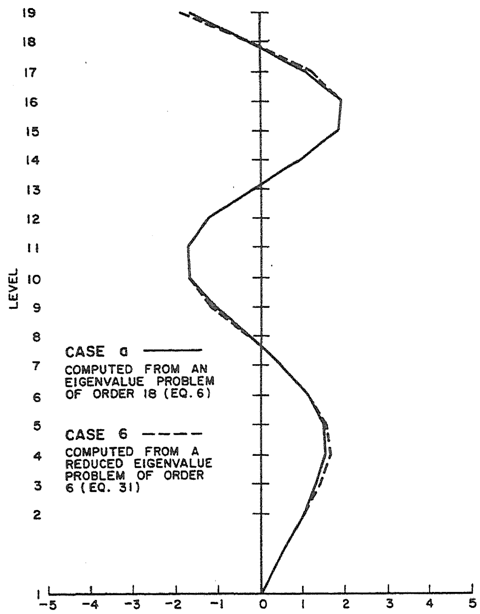
FOURTH MODE SHAPE OF THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 4

FIGURE 17



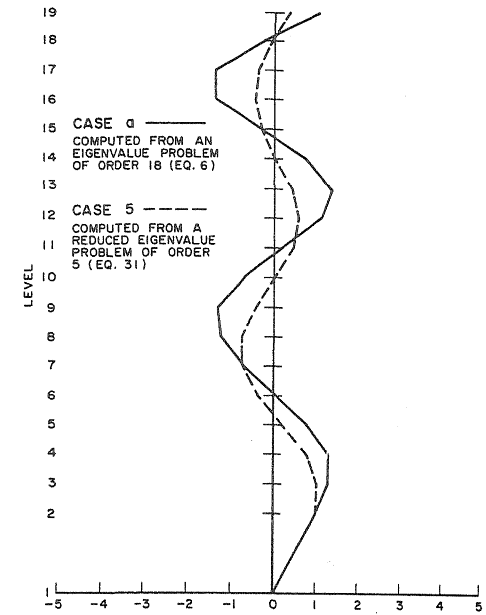
FOURTH MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 5

FIGURE 18



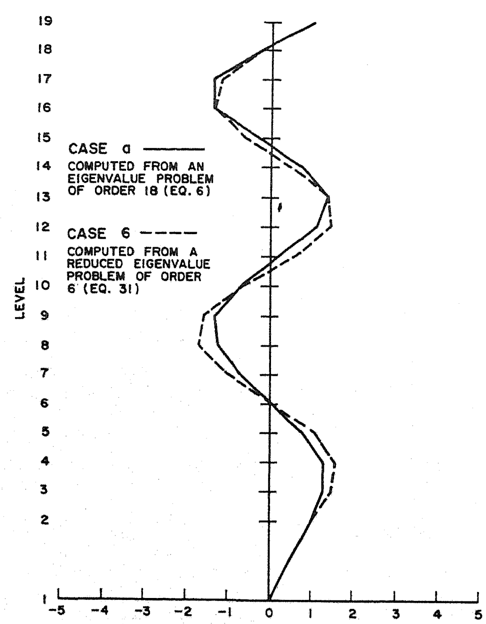
FOURTH MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 6

FIGURE 19



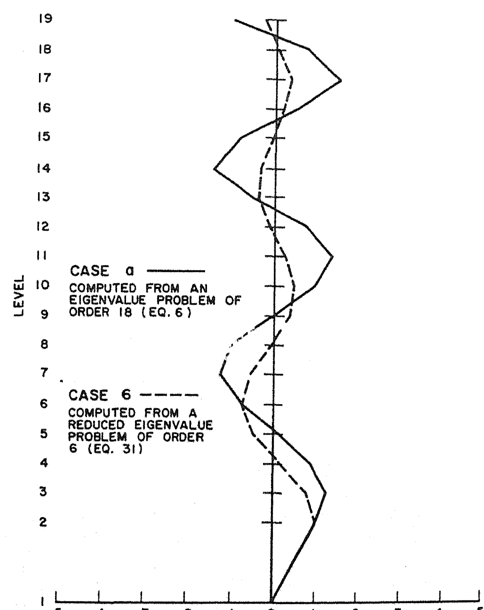
FIFTH MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 5

FIGURE 20



FIFTH MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 6

FIGURE 21



SIXTH MODE SHAPE FOR THE BUILDING OF FIGURES 3, 4
COMPUTED FOR CASES a AND 6

FIGURE 22

AN ENGINEERING APPROACH TO COMPUTING THE NATURAL MODES AND FREQUENCIES OF A TALL BUILDING.

BY M.F. RUBINSTEIN

COMMENT BY: G. HOUSNER - U.S.A.

A study by Dr. E. O'Kelly of the California Institute of Technology shortly to appear in the Bulletin of the Seismological Society of America, shows that when applied the method of Professor Rubinstein does give conveyance to the exact mode shapes.

QUESTION BY: D.A. LAWS - NEW ZEALAND

1. Why not use Equation (17) instead of (28), to substitute into Equation (29)?

2. It would seem possible to give theoretical reasons why Professor Rubinstein's method may be expected to give good results.

The method appears to be equivalent to this: Constrain the building so that the only shapes it can go into are the assumed mode shapes, or superpositions of them; then solve for the actual modes of the constrained building. Thus we must ask, are the first four mode shapes of the unconstrained building likely to be approximately superpositions of the assumed mode shapes?

Where a building has one degree of freedom per floor, the, say, six assumed modes provide for the description of possible building shapes in a certain degree of detail. (If we guessed more modes, their nodes would be closer together and they would represent perhaps excess detail). If we are trying to piece together the true fifth mode shape, say, then perhaps our assumed fifth mode errs in having its nodes too far apart in the top part of the building, so that the assumed sixth mode must be called upon to help describe this part of the true profile; but less likely the assumed seventh or eleventh mode, if we have estimated these. In the same case the assumed fourth mode may help in describing the lower part of the true fifth mode.

Considerations such as these make it plausible that the superposition of six estimated shapes will indeed give us good approximations to four or five true mode shapes. Experiment apparently confirms this.

On the other hand the results obtained, even by repeated application of the method, can never go beyond the range of the linear combinations of the initial input shapes.

I think that perhaps some considerations such as these are implied but not expressed in the paper.

AUTHOR'S REPLY:

1. Equations (17) or (28) may be substituted into E. 29.

   Equation (17) utilizes the stiffness matrix of the system and Equation 28 utilizes it flexibility.

2. The question raised here is certainly a good one and is best answered in a recent paper: "Analysis of the Convergence Properties of Rubinstein's Method for the Determination of the Lower Modes of Vibration of a Multi Degree of Freedom System", by M.E.J. O'Kelly, Bull. Seis. Soc. of Amer. Vol 54 pp 1757-1766, December, 1964.