Vibration of Visco-Elastic Body

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Abstract.

The fundamental equation employed at present in aseismic design of structures is derived from parallel model of spring and dashpot. But if analyzing vibrations of miscellaneous visco-elastic bodies, it is revealed that there are conspicuous differences between vibrational properties, e.g. transmissibility of acceleration, fraction of critical damping etc. On the other hand if compiling observed values of fraction of critical damping on real structures, it becomes clear the treatment as parallel model is not appropriate. Moreover it was verified by the precise indoor tests. As a consequence, the writer reached the conclusion, equation of motion adopted to-date must be modified so as to be consistent with actual behaviors.

1. Foreword

It is a common practice to conceive the parallel combination of elasticity and viscosity as dynamical properties of material of structures and foundations in the event of treating earthquake engineering. If the parallel combination namely Kelvin model is adopted, although there is an advantage of facilitating the treatment of equation, it is doubtful whether the explanation of the vibration of structures can be conducted essentially or not. Natural period and amplitude of structures which are most important in aseismic-design can not be revealed unless elasticity and damping of vibration are accurately assumed. The writer by employing Maxwell-Kelvin model has succeeded in explaining to a considerable extent the relationship amongst stress, strain and time of concrete.[1] Although it is conceivable such explanation also can be made on rock and earth, the necessity of non-linear treatment for the stress condition in excess of a certain extent may arise. Nevertheless since such treatment is one that can be reduced to a technical problem of manipulation of computer, the writer wishes to explain only on the linear range. In the following the attempt is made to solve the vibration of visco-elastic bodies of Maxwell-Kelvin body, Maxwell body etc. in case of forced bending vibration of a bar excited by a sinusoidal displacement. Secondly, by coverting these solutions to the form in one freedom system, the attempt is made to study the vibration properties. Thirdly, upon rearranging the properties of damping, obtained by the previous tests on real structures, the writer wishes to conceive what model should be selected for such real structures. Fourthly, explanation is given on the results of indoor vibration tests of concrete plate and steel plate to supplement the data which were observed on the real structures. Lastly the writer wishes to add the summarization of discussion.

2. The Modal analysis of forced bending vibration of a clamp-free bar excited by a sinusoidal displacement of the clamped end.

As known to-date, the relation between the stress and strain of visco-elastic body which offers the basis of equation of vibration can be written as follows:

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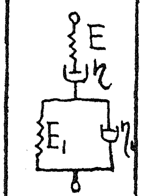
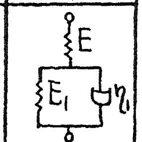
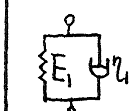
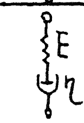
1) Hatano, "Dynamical behaviours of concrete under periodical compressive load". Technical Report C-6104. Central Research Institute of Electric Power Industry, Tokyo.

Table - 1

	Relation between stress σ and strain ε	$E_0=\frac{\sigma}{\varepsilon}$	
Maxwell-Kelvin Model	$\left(1+\frac{E_1}{E}+\frac{\eta_1}{\eta}\right)\sigma+\frac{\eta_1}{E}\frac{d\sigma}{dt}+\frac{E_1}{\eta}\int_0^t\sigma dt=E_1\varepsilon+\eta_1\frac{d\varepsilon}{dt}$	$E_0=E\frac{p+\alpha}{p+\beta p^{-1}+\gamma}$ (1) $\alpha=\frac{E_1}{\eta_1}$, $\beta=\frac{EE_1}{\eta\eta_1}$, $\gamma=\frac{E}{\eta_1}+\frac{E_1}{\eta_1}+\frac{E}{\eta}$, $\left(p=\frac{d}{dt}\right)$	
Simplified M. K. Model	$\left(1+\frac{E_1}{E}\right)\sigma+\frac{\eta_1}{E}\frac{d\sigma}{dt}=E_1\varepsilon+\eta_1\frac{d\varepsilon}{dt}$	$E_0=E\frac{p+\alpha}{p+\gamma}$ (2) $\alpha=\frac{E_1}{\eta_1}$, $\gamma=\frac{E}{\eta_1}+\frac{E_1}{\eta_1}$	
Kelvin Model	$\sigma=E_1\varepsilon+\eta_1\frac{d\varepsilon}{dt}$	$E_0=E_1(1+\delta p)$ (3) $\delta=\frac{\eta_1}{E_1}$	
Maxwell Model	$\frac{1}{E}\frac{d\sigma}{dt}+\frac{\sigma}{\eta}=\frac{d\varepsilon}{dt}$	$E_0=E\frac{p}{p+\gamma}$ (4) $\gamma=\frac{E}{\eta}$	
Elastic Model	$\sigma=E\varepsilon$	$E_0=E$ (5)	

The solution obtained so far on the vibration of a bar is related only to Kelvin body.[2]

Now if describing the equation of vibration by employing the general ratio E_0 between stress and strain, the following can be obtained.

$$E_0 I\frac{\partial^4 y}{\partial x^4}+\frac{\rho A}{g}\frac{\partial^2 y}{\partial t^2}=-\frac{\rho A}{g}\frac{d^2 y_0}{dt^2} \quad \text{..... (6)}$$

Boundary condition is expressed by (7)

$$\left.\begin{array}{lll} x=0 & ---- & y=0,\ \frac{dy}{dx}=0 \\ x=l & ---- & \frac{d^2y}{dx^2}=0,\ \frac{d^3y}{dx^3}=0 \end{array}\right\} \quad \text{..... (7)}$$

Whereas

y : relative displacement $\quad y_0$: motion of foundation
ρ : weight of unit volume $\quad A$: sectional area
I : secondary moment of sectional area

2) Nolle A. W., J. Applied Physics. 753-1948.

Horio M., J. Applied Physics. 977-1951.

(a) Solution to Maxwell-Kelvin body

By inserting (1) to (6), the solution can be obtained as follows, setting $g_0 = Z \sin \omega t$.

Now if setting

$Y = X \sin \omega t + Y \cos \omega t$ .....(8)

the following relation can be obtained.

$$\left.\begin{aligned} &\omega B \frac{d^4X}{dx^4} + B\alpha \frac{d^4Y}{dx^4} - \omega^3 X + \omega\beta X - \gamma\omega^2 Y = Z\omega^3 - \beta Z\omega \\ &-\omega B \frac{d^4Y}{dx^4} + B\alpha \frac{d^4X}{dx^4} + \omega^3 Y - \omega\beta Y - \gamma\omega^2 X = \gamma Z \omega^2 \end{aligned}\right\} \quad \text{.....(9)}$$

Whereas $B = \dfrac{EIg}{\rho A}$

Now if setting as

$$\frac{d^4X}{dx^4} = m^4 X \qquad \frac{d^4Y}{dx^4} = m^4 Y \quad \text{.....(10)}$$

the solution that satisfys (7) can be obtained as follows by using the normal mode $f_n(x)$.

$$\left.\begin{aligned} X &= \sum_{n=1}^{\infty} a_n \left\{A_n(\sin m_n x - \sinh m_n x) + (\cos m_n x - \cosh m_n x)\right\} = \sum_{n=1}^{\infty} a_n f_n(x) \\ Y &= \sum_{n=1}^{\infty} b_n \left\{A_n(\sin m_n x - \sinh m_n x) + (\cos m_n x - \cosh m_n x)\right\} = \sum_{n=1}^{\infty} b_n f_n(x) \end{aligned}\right\} \quad \text{.....(11)}$$

Whereas

$$\left.\begin{aligned} &\cos m_n l \cosh m_n l + 1 = 0 \\ &A_n = \frac{\sin m_n l - \sinh m_n l}{\cos m_n l + \cosh m_n l} \end{aligned}\right\} \quad \text{.....(12)}$$

By inserting (11) in (9), the following equation is obtained by using orthogonality of normal mode.

$$\left.\begin{aligned} a_n &= Z\omega^2 \cdot \frac{(\omega - \frac{\beta}{\omega})(Bm_n^4\omega - \omega^3 + \omega\beta) + \gamma(Bm_n^4\alpha - \gamma\omega^2)}{(Bm_n^4\alpha - \gamma\omega^2)^2 + (Bm_n^4\omega - \omega^3 + \omega\beta)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx} \\ b_n &= Z\omega^2 \cdot \frac{(\omega - \frac{\beta}{\omega})(Bm_n^4\alpha - \gamma\omega^2) - \gamma(Bm_n^4\omega - \omega^3 + \omega\beta)}{(Bm_n^4\alpha - \gamma\omega^2)^2 + (Bm_n^4\omega - \omega^3 + \omega\beta)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx} \end{aligned}\right\} \quad \text{.....(13)}$$

Therefore we obtain the following as the solution.

$$y = \sum_{n=1}^{\infty} f_n(x)(a_n \sin \omega t + b_n \cos \omega t) \quad \text{.....(14)}$$

(b) Solution to Simplified Maxwell-Kelvin Body

By inserting (2) in (6), similar to (a) the following is obtained.

$$
\left.\begin{aligned}
a_n &= Z\omega^2 \cdot \frac{\omega(Bm_n^4\omega-\omega^3)+\gamma(Bm_n^4\alpha-\gamma\omega^2)}{(Bm_n^4\alpha-\gamma\omega^2)^2+(Bm_n^4\omega-\omega^3)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx} \\
b_n &= Z\omega^2 \cdot \frac{\omega(Bm_n^4\alpha-\gamma\omega^2)-\gamma(Bm_n^4\omega-\omega^3)}{(Bm_n^4\alpha-\gamma\omega^2)^2+(Bm_n^4\omega-\omega^3)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx}
\end{aligned}\right\} \quad \cdots\cdots\cdots(15)
$$

(c) Solution to Kelvin Body

By inserting (3) in (6), the following is obtained likewise.

$$
\left.\begin{aligned}
a_n &= Z\omega^2 \cdot \frac{(Bm_n^4-\omega^2)}{(Bm_n^4-\omega^2)^2+(Bm_n^4\delta\omega)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx} \\
b_n &= Z\omega^2 \cdot \frac{-Bm_n^4\delta\omega}{(Bm_n^4-\omega^2)^2+(Bm_n^4\delta\omega)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx}
\end{aligned}\right\} \quad \cdots\cdot(16)
$$

Whereas

$$
B=\frac{E_1 I g}{\rho A}
$$

(d) Solution to Maxwell Body

By inserting (4) in (6), the following is obtained likewise.

$$
\left.\begin{aligned}
a_n &= Z\omega^2 \cdot \frac{\omega(Bm_n^4\omega-\omega^3)-\gamma^2\omega^2}{(\gamma\omega^2)^2+(Bm_n^4\omega-\omega^3)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx} \\
b_n &= Z\omega^2 \cdot \frac{-\omega(\gamma\omega^2)-\gamma(Bm_n^4\omega-\omega^3)}{(\gamma\omega^2)^2+(Bm_n^4\omega-\omega^3)^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx}
\end{aligned}\right\} \quad \cdots(17)
$$

(e) Solution to Perfect Elastic Body

By inserting (5) in (6), the following is obtained.

$$
\left.\begin{aligned}
a_n &= Z\omega^2 \cdot \frac{1}{Bm_n^4-\omega^2} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx} \\
b_n &= 0
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots(18)
$$

All of the above-mentioned solution can be expressed by the form of the product of 3 elements viz. acceleration of foundation $Z\omega^2 = C_1$, resonance factor determined by the physical properties of material, dimension of bar and frequency of vibration of foundation and mode factor determined by the vibration mode.

3. Vibration Characteristics of the Various Visco-Elastic Bodies in One-Freedom System including the Structure and its Foundation.

It is well known that the resonance period and damping of vibration are markedly influenced by the physical properties of the structures as well as by the physical properties of their foundations. Although the solution mentioned in the previous chapter was obtained under the assumption that the foundation is perfectly fixed, such condition does not exist on real structures and the energy of vibration is consumed through the foundation. In this case better solution can be derived by solving the equation in compliance with the boundary conditions that matches with the visco-elastic properties of the foundation. The writer, however, wishes without referring to this method of computation to proceed on the discussion herein by conceiving one freedom system including the structure and the foundation. Because the resonance period, damping of vibration, obtained by the vibration tests on real structures (explained in the chapter hereafter) are entirely values for the case as one freedom system including both the structure and foundation. The dynamical treatment of complicated structure can be extremely simplified by following the above-mentioned procedure.

The solution of vibration in one freedom system is made available by substituting Bm_n^4 in the afore-mentioned solution with ω_0^2 and by setting mode factor as 1. Complying to such procedure, the result computed for, compound amplitude by sinusoidal displacement of foundation, resonance amplitude, resonant circular frequency, displacement in case the inertia force acts statically, characteristic equation, the stableness of vibration, critical condition of vibration etc. are shown in Table 2. It is noted that the vibration characteristics differ conspicuously depending on the kinds of visco-elastic bodies.

As for the Kelvin body, $\omega_0 \leq 2/\xi$ is established and there is a maximum limit on the frequency. Also the resonant amplitude $y_{res.}$ is in inverse proportion to the third power of natural frequency. Therefore the fraction of critical damping $h = y_{st.}/2y_{res.}$ increases in proportion to the natural frequency.

As for the Maxwell body, $\omega_0 \geq \gamma/2$ is established and the minimum limit is set for the frequency. Also $y_{res.}$ is in inverse proportion to the first power of natural frequency. Now if the assumption is made that the fraction of critical damping h which is defined on Kelvin body is computed also on other model by $y_{st.}/2y_{res.}$, h in Maxwell body is in inverse proportion to natural frequency.

Maxwell-Kelvin body and Simplified Maxwell-Kelvin body are provided with both of these properties in common. In case their natural frequency is large compared with their physical constant, the vibration property similar to that of Maxwell body is provided and in the opposite case the vibration property similar to that of Kelvin body is provided.

For the purpose of illustrating these properties, the transmissibility of acceleration $y/y_{st.}$ complying to the natural frequency of the structure were computed respectively for Kelvin body, Maxwell body and Simplified Maxwell-Kelvin body upon assuming the appropriate physical constants. These are shown in Fig. 1-3. These figures show respectively the extremely diversified results depending on the frequency of vibration of foundation. Also upon deriving from these results, Fig. 4 was obtained which show how the values of $y_{st.}/2y_{res.}$ change by the resonant frequency. From these diagrams, it is revealed that the characteristics of vibration differ remarkably depending on the kinds of model body.

4. Damping of vibration of real structure.

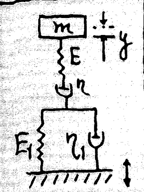
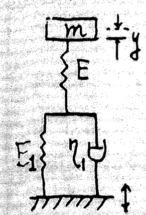
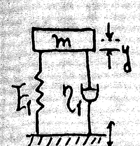
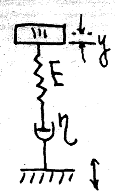
The writer compiled and studied the results of damping of vibration which were so far obtained experimentally on the respective kinds of real structures. All of these results were obtained by the form of fraction of critical damping h which was computed from resonance curve upon conceiving the Kelvin body of one freedom system including the structure and foundation.

Fig. 5 is that obtained with respect to various types of arch dams.

Table 2 - 1

	Amplitude simple harmonic motion $C_1/\omega^2 \cdot \sin\omega t$	Characteristic Equation & Stability (Routh's Method)	Critical Condition	Resonant Circular Frequency ω_{or}
M - K body	$y = C_1 \frac{\sqrt{(\omega - \frac{\beta}{\omega})^2 + \gamma^2}}{\sqrt{(\omega_0^2\alpha - \gamma\omega^2)^2 + (\omega_0^2\omega - \omega^3 + \omega\beta)^2}}$	$\lambda^3 + \lambda^2\gamma + \lambda(\beta + \omega_0^2) + \omega_0^2\alpha = 0$ Stable	$\frac{1}{4}\left\{\frac{2}{27}\gamma^3 + \omega_0^2(\alpha - \frac{\gamma}{3}) - \frac{1}{3}\beta\gamma\right\}^2 + \frac{1}{27}(\omega_0^2 - \frac{1}{3}\gamma^2 + \beta)^3 \geqq 0$	
Simplified M - K body	$y = C_1 \frac{\sqrt{\omega^2 + \gamma^2}}{\sqrt{(\omega_0^2\alpha - \gamma\omega^2)^2 + (\omega_0^2\omega - \omega^3)^2}}$	$\lambda^3 + \lambda^2\gamma + \lambda\omega_0^2 + \omega_0^2\alpha = 0$ Stable	$\frac{1}{4}\left\{\frac{2}{27}\gamma^3 + \omega_0^2(\alpha - \frac{\gamma}{3})\right\}^2 + \frac{1}{27}(\omega_0^2 - \frac{1}{3}\gamma^2)^3 \geqq 0$	$2\omega_{or}^6 - \omega_{or}^4(2\omega_0^2 - 4\gamma^2) - \omega_{or}^2(4\omega_0^2\gamma^2 - 2\gamma^4) + \left\{\omega_0^4(\gamma^2 - \alpha^2) - 2\omega_0^2\alpha\gamma^3\right\} = 0$ $\omega_0 \gg \alpha, \gamma$ .... $\omega_{or} \fallingdotseq \omega_0$ $\omega_0 \ll \alpha, \gamma$ .... $\omega_{or} \fallingdotseq \omega_0\sqrt{\frac{\alpha}{\gamma}}$
Kelvin body	$y = C_1 \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega_0^2\omega\delta)^2}}$	$\lambda^2 + \lambda\delta\omega_0^2 + \omega_0^2 = 0$ Stable	$\omega_0 \leqq \frac{2}{\delta}$	$\omega_{or} = \omega_0\sqrt{1 - \frac{\delta^2}{2}}$ $\delta \ll 1$ .... $\omega_{or} \fallingdotseq \omega_0$
Maxwell body	$y = C_1 \frac{\sqrt{\omega^2 + \gamma^2}}{\sqrt{(\gamma\omega^2)^2 + (\omega_0^2\omega - \omega^3)^2}}$	$\lambda^3 + \lambda^2\gamma + \lambda\omega_0^2 = 0$ Stable	$\omega_0 \geqq \frac{\gamma}{2}$	$2\omega_{or}^6 - \omega_{or}^4(2\omega_0^2 - 4\gamma^2) - \omega_{or}^2(4\omega_0^2\gamma^2 - 2\gamma^4) + \omega_0^4\gamma^2 = 0$ $\omega_0 \gg \gamma$ .... $\omega_{or} \fallingdotseq \omega_0$

Table 2 - 2

	Static Displacement $y_{st.}\ (\omega \to 0)$	Resonant Amplitude $y_{res.}$	$\frac{y_{res.}}{y_{st.}}$	$\frac{y_{st.}}{2\,y_{res.}}$	Natural Circular Frequency ω_0	α, β, γ	Dynamic Modulus of Elasticity E_d
	Theoretical $y_{st.} \to \infty$ Practical $y_{st.} \doteqdot C_1 \frac{1}{\omega_0^2} \frac{\gamma}{\alpha}$	$\omega_0 \gg \alpha, \beta, \gamma$ $y_{res.} \doteqdot C_1 \frac{1}{\omega_0} \frac{1}{(\gamma - \alpha)}$	$\omega_0 \gg \alpha, \beta, \gamma$ $\omega_0 \frac{1}{(\gamma-\alpha)} \frac{\alpha}{\gamma}$	$\omega_0 \gg \alpha, \beta, \gamma$ $\frac{1}{\omega_0} \frac{(\gamma-\alpha)}{2} \frac{\gamma}{\alpha}$	$\omega_0 = \sqrt{\frac{DE}{m}}$ D: Const	$\alpha = \frac{E_1}{\eta_1}$ $\beta = \frac{E}{\eta} \cdot \frac{E_1}{\eta_1}$ $\gamma = \frac{E}{\eta_1} + \frac{E_1}{\eta_1} + \frac{E}{\eta}$	$E_d = \frac{E E_1 (1+\omega^2/\alpha^2)}{E + E_1(1+\omega^2/\alpha^2)}$
	$y_{st.} = C_1 \frac{1}{\omega_0^2} \frac{\gamma}{\alpha}$	$\omega_0 \gg \alpha, \gamma,$ $y_{res.} \doteqdot C_1 \frac{1}{\omega_0} \frac{1}{(\gamma-\alpha)}$ $\omega_0 \ll \alpha, \gamma,$ $y_{res.} \doteqdot C_1 \frac{1}{\omega_0^3} \frac{\gamma}{(1-\frac{\alpha}{\gamma})\sqrt{\frac{\alpha}{\gamma}}}$	$\omega_0 \gg \alpha, \gamma$ $\omega_0 \frac{1}{(\gamma-\alpha)} \frac{\alpha}{\gamma}$ $\omega_0 \ll \alpha, \gamma$ $\frac{1}{\omega_0} \frac{\alpha}{(1-\frac{\alpha}{\gamma})\sqrt{\frac{\alpha}{\gamma}}}$	$\omega_0 \gg \alpha, \gamma$ $\frac{1}{\omega_0} \frac{(\gamma-\alpha)}{2} \frac{\gamma}{\alpha}$ $\omega_0 \ll \alpha\ \gamma$ $\omega_0 \frac{(1-\frac{\alpha}{\gamma})\sqrt{\frac{\alpha}{\gamma}}}{2\alpha}$	$\omega_0 = \sqrt{\frac{DE}{m}}$	$\alpha = \frac{E_1}{\eta_1}$ $\gamma = \frac{E}{\eta_1} + \frac{E_1}{\eta_1}$	$E_d = \frac{E E_1 (1+\omega^2/\alpha^2)}{E + E_1(1+\omega^2/\alpha^2)}$
	$y_{st.} = C_1 \frac{1}{\omega_0^2}$	$\delta \ll 1$ $y_{res.} \doteqdot C_1 \cdot \frac{1}{\omega_0^3 \delta}$	$\delta \ll 1$ $\frac{1}{\omega_0} \frac{1}{\delta}$	$\delta \ll 1$ $\omega_0 \frac{\delta}{2}$	$\omega_0 = \sqrt{\frac{DE_1}{m}}$	$\delta = \frac{\eta_1}{E_1}$	$E_d = E_1\sqrt{1+\delta^2\omega^2}$
	Theoretical $y_{st.} \to \infty$ Practical $y_{st.} \doteqdot C_1 \frac{1}{\omega_0^2}$	$\omega_0 \gg \gamma$ $y_{res.} \doteqdot C_1 \frac{1}{\omega_0} \cdot \frac{1}{\gamma}$	$\omega_0 \gg \gamma$ $\omega_0 \frac{1}{\gamma}$	$\omega_0 \gg \gamma$ $\frac{1}{\omega_0} \cdot \frac{\gamma}{2}$	$\omega_0 = \sqrt{\frac{DE}{m}}$	$\gamma = \frac{E}{\eta}$	$E_d = E \frac{\omega^2/\gamma^2}{1+\omega^2/\gamma^2}$

Although Kamishiiba arch dam[3], Tonoyama arch dam[4], Sasanami arch dam[5] have the dam volume of $35 \times 10^4 - 3 \times 10^4\ m^3$, the values of h were obtained from the resonance curves derived from extremely minute vibrations which were obtained from one point excitation by exciting force of the extent of maximum 10 - 20 ton actuating on these arch dams. The reason of providing several points in the same dam is that the results were obtained for several kinds of cases such as various vibration mode, high water level, low water level etc. With regard to Naramata arch dam[6], the values of "h" were obtained from the resonance curves which were obtained by applying the destructive vibration on the test dam with the dam volume of about 70 m^3 resorting to the exciting force of 300 - 800 Kg. Although 2 points of the mean nature at reservoir full condition and at reservoir empty condition were indicated in the figure, as the exciting force increases from 300 Kg. to 800 Kg, the value of h increases by about 10 - 20%. As indicated by the figure, the relation between h and ω_{or} obviously denys the treatment as Kelvin body. The fact that h increases against large vibration can be presumed that when assuming the Maxwell-Kelvin body or Maxwell body the explanation should be made by inducing the non-linearity in which the values of η_1 or η decrease against large stress.

Fig. 6 shows the values of h[7] obtained by computing from the resonance curves which were obtained by applying vibration produced by exciting machine to concrete piers which were constructed on the foundations provided with various properties. The relation with ω_{or} obviously indicates hyperbola respectively complying to the properties of ground formation, accordingly it is revealed that the treatment as Kelvin body is a mistake. Also it is reported that the value of h for the case the pier laid on clay vibrates with a large amplitude was about 2 times the value of h for the case of the vibration with a small amplitude. For the case of clay it is observed that the non-linearity is indicated to be conspicuous.

Fig. 7 shows the value of h obtained from vibration test by exciting machine regarding steel piping in steam power plant and also the values of h computed from the observed values of the usual micro-vibration.[8] In this case also the relation between h and ω_{or} obviously indicates hyperbola and as a result the previous treatment as Kelvin body must be entirely denied.

3) Takahashi, T. "Behaviour of Vibration of Arch Dam" Technical Report C 5905. Central Research Institute of Electric Power Industry.

4) Okamoto, S. "Observation of Earthquakes on an Arch Dam" Transaction of the Japan Society of Civil Eng. No. 76. 1961.

5) Takahashi, T. "Results of Vibration Tests and Earthquake Observations on Concrete Dams and their Considerations" 8th International Congress on Large Dams 1964.

6) Hatano, T. "The Stability of an Arch Dam against Earthquakes", Technical Report C 5607, Central Research Institute of Electric Power Industry.

7) Kubo, K. "Damping and Vibrational Characteristics of Bridge Piers" Pro. of Japan National Symposium on Earthquake Eng. 1962.

8) Tokyo Electric Power Co. Inc. "Study on the Aseismic Design of Piping in Nuclear Power Station" 1962.

However an extensive change of the h values complying to the magnitude of vibration can not be observed indicating a different result from that of concrete and earth structures. In other words, the non-linearity is considered to be small.

5. Vibration test by concrete plate and steel plate.

For the purpose of comparing precisely the resonance amplitude and damping of vibration, 2 kinds of tests viz. forced vibration test on a shaking table using concrete plate and steel plate and damped free vibration test on concrete floor were conducted. The summary of these tests is as shown in Table 3.

Table 3 - 1

		Forced Vibration	Free Vibration
Concrete plate	Apparatus	Hanged shaking table excited by Electro-magnetic shaker (cap. 150 Kg 5 - 1000 c/s)	
	Measurement	Wire Resistant type Accelometer on the top ofthe plate & on the table	Wire-Resistant type displacement meter between the top of the plate & fixed point
	Vibration	Constant Acceleration Sinusoidal	Damped free vibration given by the hand and by the hammer
	Test piece	(1) 150cm x 5 cm x 20 cm Vertical plate lower end fixed on the table by steel plate, angles, bolts & nuts. (2) 120 cm x 5 cm x 20 cm plate This was made from test piece (1), cutting its upper part. (3) 105 cm x 5 cm x 20 cm plate This was made from test piece (2), cutting its upper part.	(2) 120 cm x 5 cm x 20 cm Vertical plate This was put on the concrete floor attached by the steel plate, angles, bolts & nuts. Natural frequencies were changed by the added weights of steel blocks.

In the forced vibration, for the purpose of obtaining the resonant amplitude, the accelerations of shaking table and the upper end of concrete plate and steel plate were recorded. And the points that indicated 90° phase difference for both of these accelerations were adopted. Fig. 8 and Fig. 9 show respectively the records concerning concrete plate and steel plate. From these results, the amplitude per unit acceleration of table is conceivable to be expressed as $y_{res.\,l}/C_1 = 1/\omega_{or}^2 \cdot \ddot{y}_{res.\,l}/C_1$.

Also from (15) (16) (17) and $y_{st.}$ in Table 2, the following equation can be

Table 3 - 2

		Forced Vibration	Free Vibration
Steel plate	Test piece	(1) 780 mm x 7 mm x 200 mm Vertical plate lower end fixed on the table by steel plate, angles, bolts & nuts. (2) 650 x 7 x 200 plate This was made from test piece (1), cutting its upper part. (3) 530 x 7 x 200 plate This was made from test piece (2), cutting its upper part.	(2) 650 mm x 7 mm x 200 mm Vertical plate This was put on the concrete floor attached by the steel plate, angles bolts & nuts. Natural frequencies were changed by the added weights of steel blocks.

obtained.

$$h = \frac{y_{st.l}}{2 y_{res.l}} = \frac{P}{2 \ddot{y}_{res.l}/C_1} \cdot \frac{\int_0^l f_n(x)dx}{\int_0^l f_n^2(x)dx} \cdot f_n(x=l)$$

Whereas P is the physical constant determined by model. Since the lowest mode of vibration was adopted in order to avoid the range that indicates too large resistance of air in this test, n = 1 was set. Then the following equation is made available.

$$h = \frac{1.565 P}{2 \ddot{y}_{res.l}/C_1}$$

The value of P is 1 or γ/g and Fig. 12 shows the value of h for the case P is conceived as 1. The acceleration of shaking table was about 50 gal for both cases of concrete plate and steel plate and the upper end acceleration resulted in 1000 gal - 5000 gal indicating very large vibrations.

In contrast to the above-mentioned, for the purpose of finding the condition at very minute vibrations, the examples which were obtained for the test of damped free vibration are shown in Fig. 10 and Fig. 11. Although 2 cases, one was by giving light shock by hand and the other by giving somewhat larger shock than the previous-mentioned by hammer are indicated, the maximum amplitude is about 0.1 mm for the former and about 0.3 mm for the latter. The values of h for both cases were obtained from logarithmic decrement and are described in Fig. 12 together with the computed values obtained from forced vibrations.

The following are in general the findings that were made available through these tests.

(a) In case of concrete plate. As large vibrations are produced, the value of h increases and the resonant frequency ω_{or} decreases. Also following exactly the same as observed in real structures, h is in inverse proportion to ω_{or} and as a result the treatment as Kelvin body is absolutely denied. In case of damped free vibration although the decrease of h is not so extreme when ω_{or} is large, such result may perhaps be due to the large air resistance. The trend that h increases as ω_{or} decreases is indicated when ω_{or} is small. In this case since it is conceivable that the air resistance is small and also the dissipation of energy through the foundation is small because of the very small amplitude, it may perhaps be logical to conceive that the damping of vibration is chiefly caused by the internal viscosity. As shown by the writer[1)] since it is better to treat as Maxwell-Kelvin body for the concrete material, the results shown herein can be conceived to be caused essentially from the properties of concrete material. Although the values of h obtained by forced vibration are far larger than the values obtained by damped free vibration, such outcome should be explained by two reasons viz. one is that the energy dissipation through foundation is large and the other is that non linearity of dynamical properties is provided in the concrete. However as shown in Fig. 6 if conceiving the points that the values of h differ conspicuously by the difference of the material of foundations, it may perhaps be logical to consider the dissipation of energy through the foundation as the major cause.

(b) In case of steel plate. As to steel plate, the relation between h and ω_{0r} for the case of forced vibration and for the case of damped free vibration shows a clear difference. For the case of forced vibration while exactly the same trend as that of piping shown in Fig. 7 is indicated, on the other hand for the case of damped free vibration the trend that h increases as ω_{0r} increases is indicated similar to Kelvin body. Furthermore when ω_{0r} becomes large the values of h bear a close resemblance in either case of large amplitude or small amplitude. As for the damped free vibration since it is conceivable that the dissipation of energy through the foundation at very small amplitude is small, if conceiving that the trend similar to that of Kelvin body is due to the internal viscosity of steel material and to the resistance of air, the steel material might be essentially a Kelvin body. Nevertheless in case the energy dissipation through the supporting point as a structure is conceivable, as in Maxwell body it may perhaps be interpreted that a large damping of vibration is caused at low frequency. Also in this case, however, the writer feels that the elements that should be explained by non-linearity are very little when compared with the structures of concrete or earth.

6. Discussion and conclusion

It is revealed that the parallel combination of spring and dash pot which was assumed in entirety with regard to the earthquake engineering of structures heretofore is not appropriate to explain the vibrations of real structures.

Although acceleration spectrum, velocity spectrum etc. which are adopted for dynamic design are computed based on Kelvin body, the writer feels these spectrums must be used after converting to suitable model in entirety.

Also although there is an idea of permitting the aseismic design of structures setting a certain velocity as the basis from the fact that the velocity spectrum value of seismic wave by Kelvin model is approximately constant against a certain value of h, such conception is obviously wrong from the afore-mentioned discussion even though resorting to either Kelvin model or Maxwell model.

Since the material and dynamical properties of concrete, rock, earth etc. can be given a better explanation by treating as Maxwell-Kelvin model as mentioned previously, the structures made of such material and structures set on foundation of such properties can be given a clearer explanation on actual vibrational phenomena by treating as Maxwell-Kelvin model than that done heretofore.

However in case of structures made of steel material especially in case of piping system as mentioned in this paper, even if admitting at least that the explanation as Maxwell body is more suitable as a practical explanation the questions such as why does it essentially admit so? Is it better to conceive as some other better model ? etc. are the remaining problems to be studied hereafter.

The conclusion of the afore-mentioned is summarized as follows:

(a) Although a solution of bending vibration of visco-elastic bar is available with respect to that of Kelvin body, the solution also for Maxwell-Kelvin body, Maxwell body etc. was made obtainable by the modal analysis.

(b) By treating the structure including the foundation as one freedom system and as Maxwell-Kelvin body, Maxwell body and Kelvin body, the characteristics of vibrations were studied utilizing the solution mentioned in (a). As a result it was revealed that while in the Kelvin body which heretofore is assumed in the aseismic design the resonant amplitude y_{res} is in inverse proportion to the third power of natural frequency ω_0 and the fraction of the critical damping h increases in proportion to ω_0 , in the Maxwell body y_{res} is in inverse proportion to the first power of ω_0 and h decreases in inverse proportion to ω_0 and in Maxwell-Kelvin body the properties of both Kelvin body and Maxwell body are indicated.

(c) After examining the tests conducted so far on the real structures of dams, piers of bridges and steel piping systems, if rearranging the relation between h and ω_{or} , all of these relations exist on hyperbola, indicating that the previous treatment as Kelvin body is wrong. Also in concrete structures large values of h against large vibrations were observed and as a result it is presumed that the non-linearity of concrete and the material of foundation earth are the grounds of explanation.

(d) The writer confirmed the explanation of (c) from the results of indoor vibration tests of concrete plate and steel plate. However non-linearity of steel structures is conceived to be small and in case the dissipation of vibration energy through the fixed section is conceived to be small, the value of h indicated the trend to increase together with the increase of ω_{or}. From such trend, the damping characteristics of vibrations, the mechanism of energy dissipation of the steel structures are presumed to involve many subjects that still need to be clarified.

(e) It is necessary to convert the computation of acceleration, velocity, spectrum which heretofore were employed in the dynamic computation according to a proper model.

Also the treatment of non-linearity which was not referred to in this paper is particularly essential for the case of aseismic design that assumes a large seismic acceleration. Concerning this problem, the writer wishes to explain at another opportunity.

Fig. 1

Frequency Response

Kelvin body

(One mass system having the same physical constant δ and variable mass m)

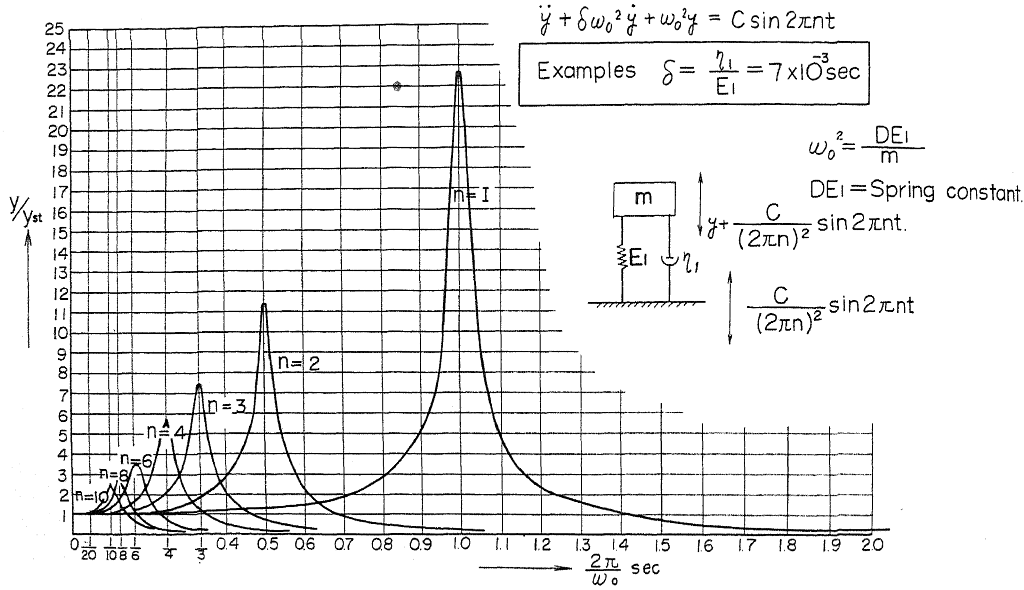


Fig. 2

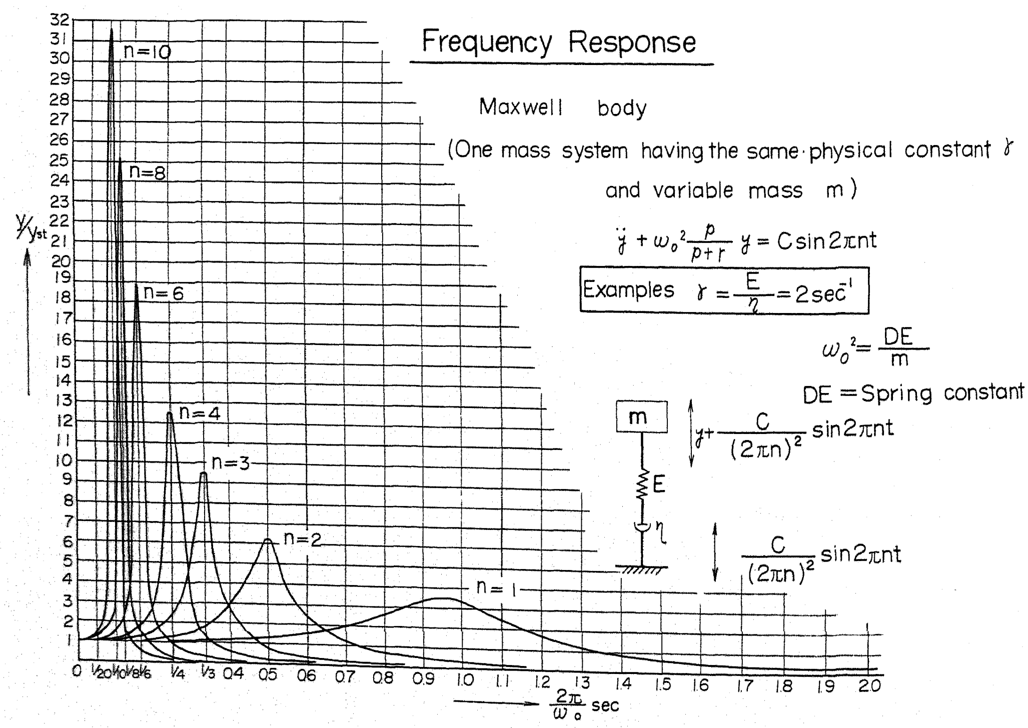


Fig. 3

Frequency Response

Simplified Maxwell Kelvin body

(One mass system having the same physical constants α, γ, and variable mass m)

$$\ddot{y} + \omega_0^2 \frac{p+\alpha}{p+\gamma} y = C \sin 2\pi n t$$

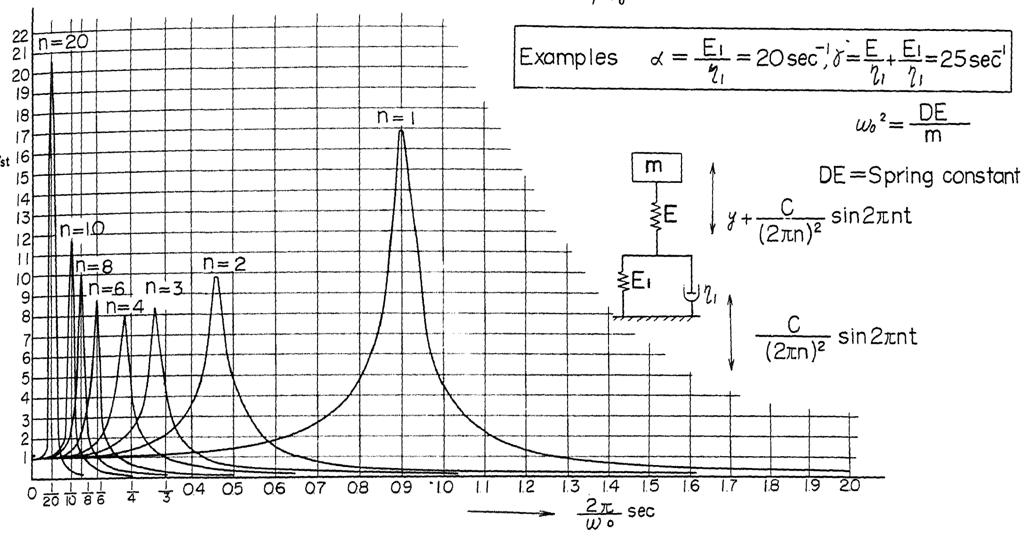


Fig. 4

h — ω_{or} Relation.

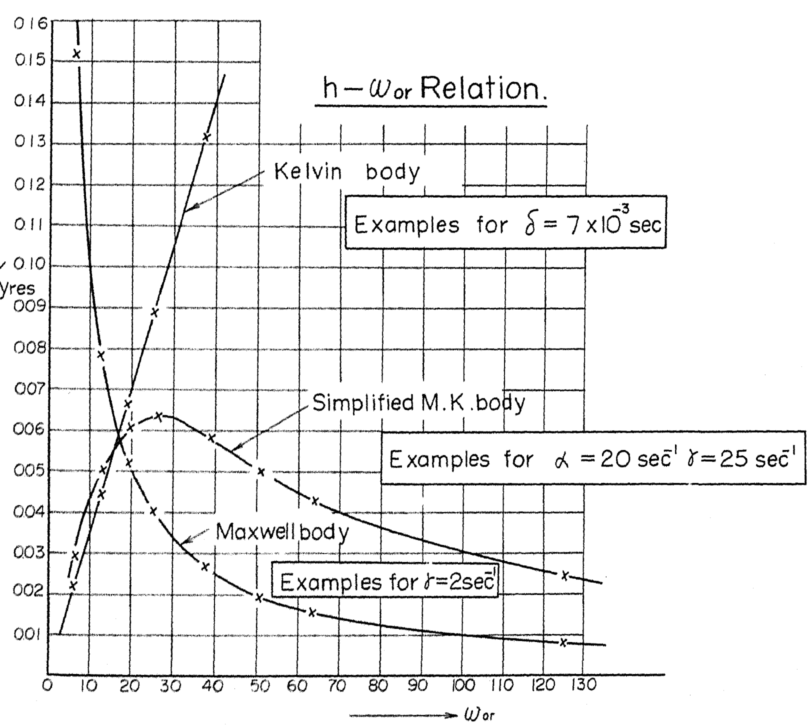
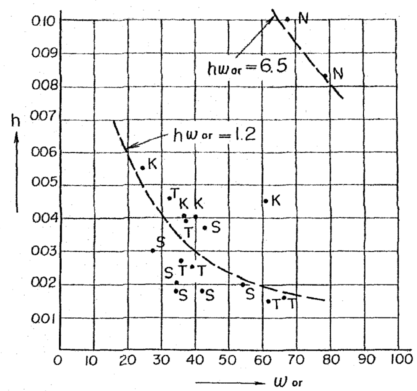


Fig. 5 Relations Between the Fraction of Critical Damping (h) and Natural Circular Frequency (ω_{or}) of the Structure

I. Arch Dam.



N:	Naramata Arch Dam	Height	13 m
K:	Kamishiiba Arch Dam	Height	113 m
T:	Tonoyama Arch Dam	Height	65 m
S:	Sazanami Arch Dam	Height	67 m

Fig. 7 Relations Between the Fraction of Critical Damping (h) and Natural Circular Frequency (ω_{or}) of the Structure

III. Steel Piping in the Steam Power Station

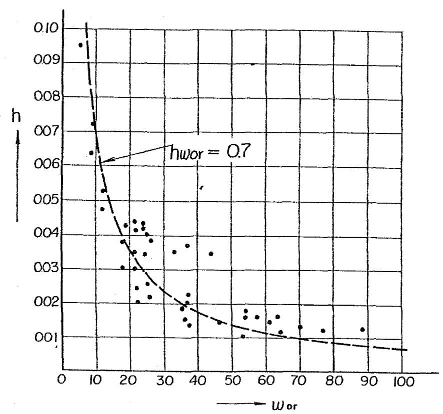
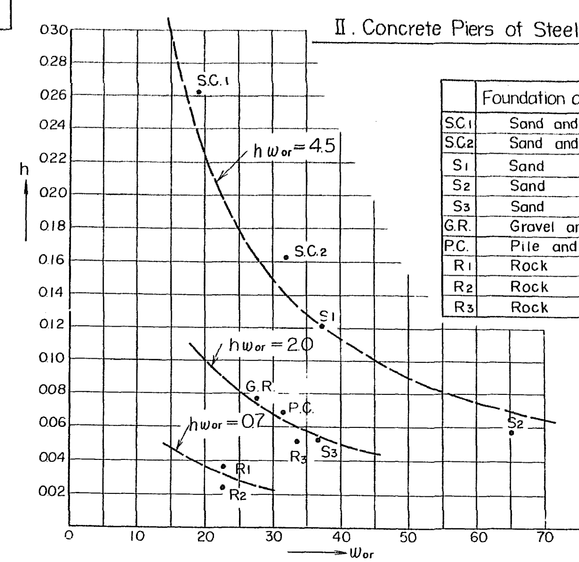


Fig. 6 Relations Between the Fraction of Critical Damping (h) and Natural Circular Frequency (ω_{or}) of the Structure

II. Concrete Piers of Steel Bridges.



	Foundation of Pier	Height of Pier
SC1	Sand and Clay	10.7 m
SC2	Sand and Clay	10.7 m
S1	Sand	12.8 m
S2	Sand	12.8 m
S3	Sand	10.5 m
G.R.	Gravel and Rock	6.4 m
P.C.	Pile and Clay	6.7 m
R1	Rock	19.8 m
R2	Rock	2.1 m
R3	Rock	17.0 m

Fig. 8 Resonance Test of Vertical Concrete Plate on the Shaking Table

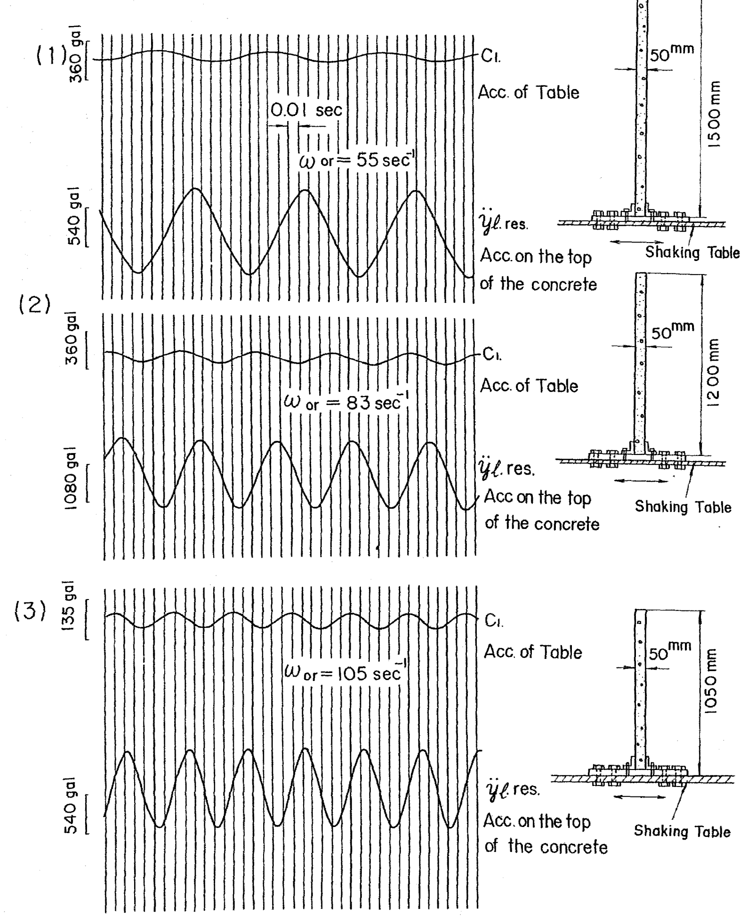


Fig. 9 Resonance Test of Vertical Steel Plate on the Shaking Table

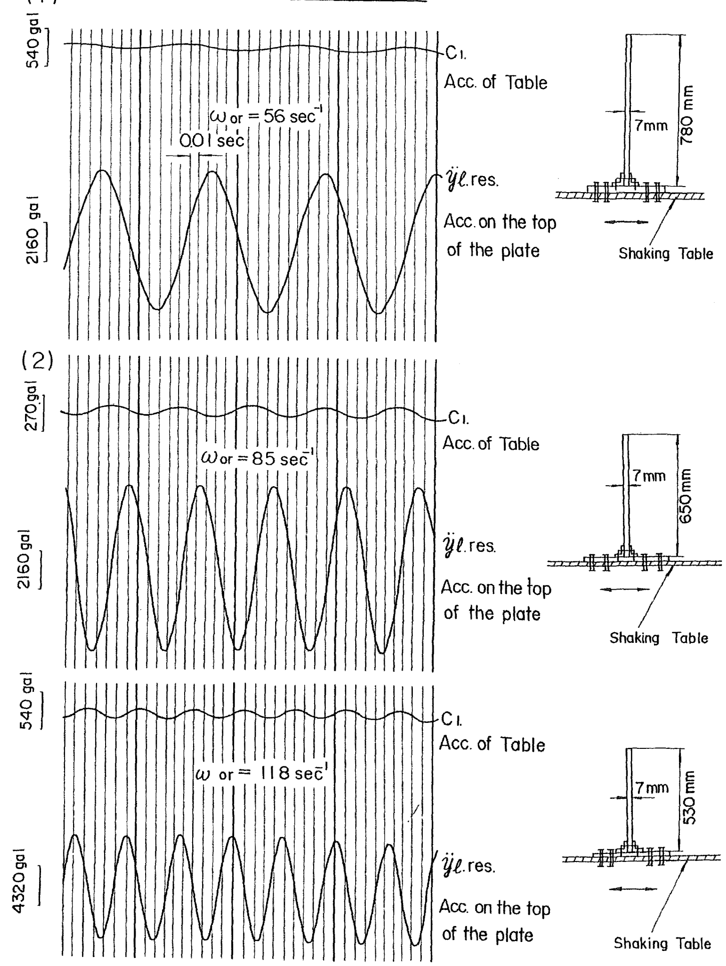
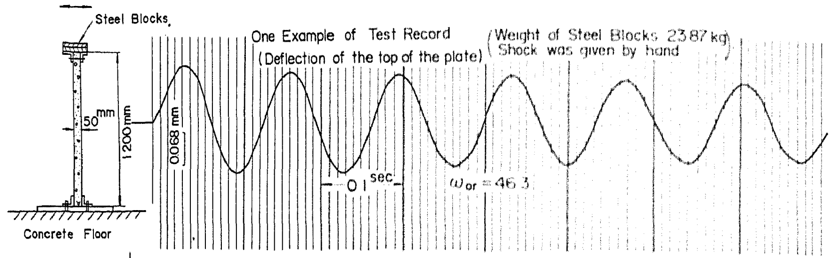
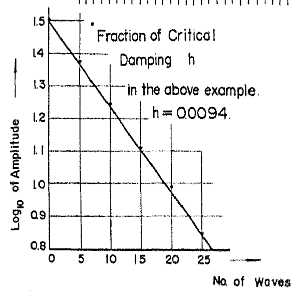


Fig.10 Free Vibration Test of Vertical Concrete Plate on the Floor

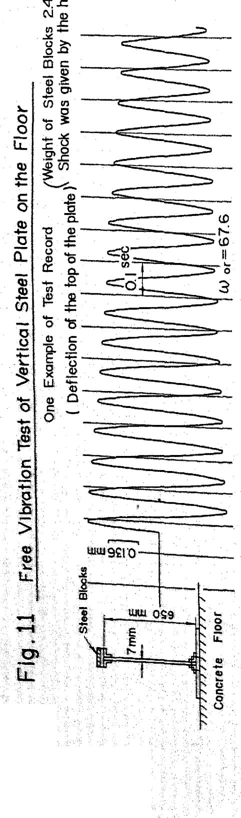




Test Results

No	Weight of Steel Blocks	Shock	ω_{or}	h	No	Weight of Steel Blocks	Shock	ω_{or}	h
1	0	by hand	92.6	0.0092	5	0	by hammer	90.3	0.0125
2	23.87kg	by hand	46.3	0.0094	6	23.87kg	by hammer	44.6	0.0130
3	30.89kg	by hand	41.3	0.0095	7	30.89kg	by hammer	40.2	0.0132
4	38.00kg	by hand	37.7	0.0104	8	38.00kg	by hammer	37.0	0.0133

Fig.11 Free Vibration Test of Vertical Steel Plate on the Floor



Test Results

No.	Weight of Steel Blocks	Shock	ω_{or}	h	No.	Weight of Steel Blocks	Shock	ω_{or}	h
1	0	by hand	108	0.0046	5	0	by hammer	107	0.0057
2	2.46kg	by hand	68	0.0046	6	2.46kg	by hammer	68	0.0048
3	4.89kg	by hand	53	0.0045	7	4.89kg	by hammer	53	0.0049
4	9.61kg	by hand	39	0.0044	8	9.61kg	by hammer	39	0.0045

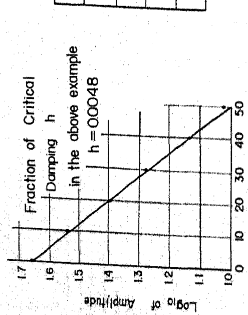
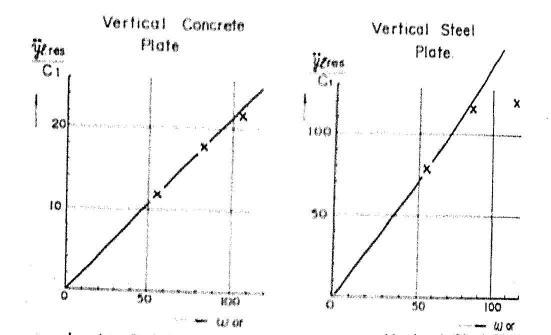
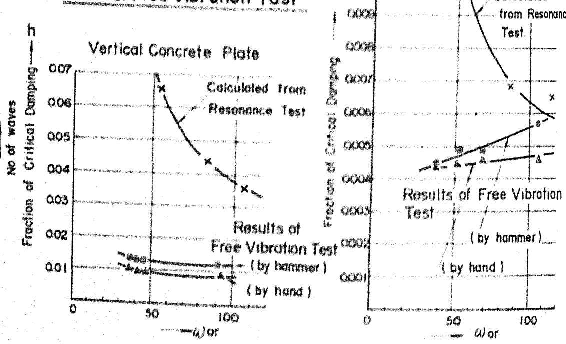


Fig.12 $\ddot{y}_{\ell\,res}/C_1 \sim \omega_{or}$ Relation in the Resonance Test



h~ω_{or} Relation in the Resonance Test & Free Vibration Test



VIBRATION OF VISCO-ELASTIC BODY

BY T. HATANO

QUESTION BY: E.A. SPENCER - NEW ZEALAND

I should like to know if in fig. 12 you show values of damping calculated from a resonance test. When you conducted this resonance curve was the amplitude different for each point on the curve or was the amplitude the same? I think it is possible from results I have found testing reinforced concrete members and trying to establish their damping, that one can get different answers for the % of critical damping at each amplitude. The fact that the amplitude is different in each case would itself introduce a difference that might tend to disguise the effect that you are showing i.e. that it is dependent upon the frequency.

AUTHOR'S REPLY: Relationship between maximum amplitude and fraction of critical damping in the test results shown in Fig. 12 is presented in the following Table 4. Maximum amplitude in this table shows those on the tops of the plates; a concrete plate 1200mm in height and a steel plate 650mm in height, under the various conditions of vibration.

Table 4

Concrete plate (ht. 1200mm)

W_{or} sec^{-1}	h	max. amplitude mm	
83	0.044	1.4	forced vibration
90.3	0.0125	0.3	free vibration
44.6	0.0130	0.3	"
40.2	0.0132	0.3	"
37.0	0.0133	0.3	"
92.6	0.0092	0.1	"
46.3	0.0094	0.1	"
41.3	0.0095	0.1	"
37.7	0.0104	0.1	"

Steel plate (ht.650mm)

W_{or} sec^{-1}	h	max. amplitude mm	
85	0.0068	7.3	forced vibration
107	0.0057	0.3	free vibration
68	0.0048	0.3	"
53	0.0049	0.3	"
39	0.0045	0.3	"
108	0.0046	0.1	"
68	0.0046	0.1	"
53	0.0045	0.1	"
39	0.0044	0.1	"

When these values are plotted in a figure, there seems at a glance to be correlation between max. amplitude and fraction of critical damping, but actually, there is little correlation, for example, in the case of free vibration.

The same analysis on a real structure is mentioned in the following example. This is taken from the result of a field test on Naramata Arch Dam in Fig. 5. Two points of N in Fig. 5 represent the average value of 4 points each; the details of these values are explained in reference 6. These 8 points are the test results in the same mode with different exciting forces and reservoir conditions. Relationship between the fraction of critical damping and amplitude converted into amplitudes on the same point of the dam is shown in Table 5.

Table 5

Naramata Arch Dam

W_{or} sec^{-1}	h	amplitude^{mm}	reservoir	exciting force^{kg}
78.8	0.087	0.33	empty	800
"	0.081	0.24	"	650
"	0.094	0.43	"	800
"	0.068	0.32	"	500
67.7	0.108	0.34	full	"
"	0.101	0.23	"	300
"	0.108	0.32	"	500
"	0.089	0.26	"	300

When these values are plotted in a figure, it might be made clear that there is no correlation between amplitude and fraction of critical damping, and that the relationship between the natural angular frequency W_{or}, and the fraction of critical damping is in inverse proportion, depending upon the magnitude of the exciting force.

Fig. 7 presents the test results in the case of a large amplitude of steel piping by an exciting machine, and in the case of a very small amplitude is usual micro-vibration. It can be considered that there is no correlation between amplitude and fraction of critical damping, on the basis of the fact that test results are plotted on one hyperbola line.

The writer considers that the inverse proportional relationship between W_{or} and fraction of critical damping comes out when the main cause of damping of vibration results from the vibrating energy dissipation from the base of the structure, and that another result would be brought about, when resistance of air and coulomb friction lead to main causes of damping of vibration.