AN EMPIRICAL FORMULA FOR THE SPECTRUM OF STRONG EARTHQUAKE MOTION

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INTRODUCTION

The usual meaning of the term "earthquake motion" is the wave motion at ground surface transmitted from the earthquake origin through the earth's crust. If $O(T)$, $Q(T,q)$ and $G(T,q)$ represent, respectively, the vibrational characteristics of the earthquake origin, of the earth's crust and of the ground at an observing station, the wave form of the earthquake motion at the station may be written as

$$u(T,q) = O(T) \cdot Q(T,q) \cdot G(T,q) \tag{1}$$

in which T and q represent, respectively, wave period and wave attenuation. $O(T)$ depends on both the earthquake magnitude and the mechanism of seismic wave generation, while $Q(T,q)$ is determined by the reflections, refractions, scattering and absorption of waves in the earth's crust.

In this paper there is obtained first an empirical formula for the spectrum of earthquake motions at the ground surface, following which the formula is applied to certain cases for which strong motion earthquake observations are available.

THE SPECTRUM AT DEPTH

Earthquake motions at and near the surface are markedly influenced by the usually complex surface geologic conditions. Thus, in order to study analytically the characteristics of incident seismic waves, it is desirable to work with seismograms obtained at such depth that the surface geologic effects are negligible. The measurements obtained 300 meters underground at Hitachi Mine, 120 km. northeast of Tokyo, over a period of more than ten years, are well adapted to this purpose. All of the displacement-period curves obtained by spectral analysis of these seismograms exhibited a peak at some period.

Statistical analyses of earthquake records have shown that, as epicentral distance increases, the maximum displacement from the displacement-period curve decreases while the corresponding period changes only slightly1) Furthermore, the maximum displacement increases with the

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1) K. Kanai and S. Yoshizawa, "The Amplitude and the Period of Earthquake Motions. II", Bull. Earthq. Res. Inst., 36(1958), 275.
B. Gutenberg and C. F. Richter, "Earthquake Magnitude; Intensity, Energy, and Acceleration", Bull. Seism. Soc. Amer., 32(1942), 163.
A. E. Jones, "Empirical Studies of Some of the Seismic Phenomena of Hawaii", ditto, 28(1938), 313.

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corresponding period for a given epicentral distance.

The following empirical formula, which has been previously reported[2], describes the relation of maximum displacement to corresponding period of seismic waves at depth:

$$A_{ms} = 53\ T_m^{2.56} \tag{2}$$

in which A_{ms} in microns and T_m in seconds represent, respectively, the displacement and period coordinates of the peak of the displacement spectrum for a station at an epicentral distance of 100 km..

Except for periods much shorter or much longer than the above peak period, the spectrum of seismic waves at depth can be expressed as

$$\frac{\bar{u}}{T}\ \left(\equiv \frac{\text{velocity}}{2\pi}\right) = \text{constant} \tag{3}$$

in which $\bar{u}$ and T represent, respectively, the displacements and periods of individual points on the spectrum. This relation implies that seismic waves at depth are compatible over a considerable range of period with the concept of equipartition of energy among wave components. (Fig. 1 shows schematically the relations mentioned above.)

AMPLITUDE, PERIOD, MAGNITUDE AND EPICENTRAL DISTANCE

Many empirical formulae have been advocated to represent the relationship among magnitude, epicentral distance, and largest amplitude[3]. In the present case the following formula[4] is adopted:

$$\log A_m = M - 1.73 \log \Delta + 0.83 \tag{4}$$

where M is the Richter magnitude, Δ the epicentral distance in km., and A_m is the largest displacement amplitude in microns. In general, for long period waves such as are usually involved in (4), the largest amplitude at bed rock is approximately the same as that at the ground surface.

Considering a special case of (4) where $\Delta = 100$ km., and combining (4) with (2), there is obtained

$$\log T_m = 0.39\ M - 1.70 \tag{5}$$

From (4) and (5) we obtain the relation

$$\log \frac{A_m}{T_m} = 0.61\ M - 1.73 \log \Delta + 2.53 \tag{6}$$

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2) K. Kanai and S. Yoshizawa, loc. cit., 1).

3) B. Gutenberg and C. F. Richter, loc. cit., 1).

4) C. Tsuboi, "Determination of the Gutenberg-Richter Magnitude of Earthquakes occurring in and near Japan", Jour. Seism. Soc., Japan, (ii) 7, No. 3(1954), 185.

If the amplitude-period relation (3) is considered to be applicable to (6), we obtain

$$\bar{u} = T \times 10^{0.61\,M - 1.73 \log \Delta - 1.47} \tag{7}$$

which expresses the displacement amplitude $\bar{u}$ in cm. as a function of period of waves in seconds in bed rock.

It follows directly from (7) that

$$\bar{v} = 2\pi \times 10^{0.61\,M - 1.73 \log \Delta - 1.47} \tag{8}$$

$$\bar{a} = \frac{(2\pi)^2}{T} \times 10^{0.61\,M - 1.73 \log \Delta - 1.47} \tag{9}$$

where $\bar{v}$ is spectral velocity in cm./sec. and $\bar{a}$ is spectral acceleration in cm./sec.^{2}.

VIBRATIONAL CHARACTERISTICS OF GROUND

Understanding of the seismic characteristics of the ground surface has been becoming more clear as a result of observational investigations of earthquakes and microtremors, statistical analyses of earthquake damage, and theoretical studies of seismic waves. Based on earthquake and microtremor observations, and adding theoretical considerations, there has been obtained$^{5)}$ a semi-empirical formula for the seismic characteristics of ground consisting of a layer overlying a semi-infinite medium:

$$G(T,q) = 1 + \left\{ \left[\frac{1+c}{1-c} \left\{ 1 - \left(\frac{T}{T_0} \right)^2 \right\} \right]^2 + \left\{ \frac{0.3}{\sqrt{T_0}} \left(\frac{T}{T_0} \right) \right\}^2 \right\}^{-\frac{1}{2}} \tag{10}$$

in which T and T_0 represent, respectively, in seconds, the period of earthquake waves and the predominant period of the ground,

$$c = \sqrt{\frac{\rho_1 \mu_1}{\rho_2 \mu_2}},$$

and ρ_1, μ_1 and ρ_2, μ_2 are, respectively, the densities and elastic constants of the layer and the subjacent medium. (The expression(10) has been modified from that reported previously$^{5)}$ in order to make it more closely analogous to the expression for motion of a pendulum relative to its support.)

The following special cases are presented as illustrations of (10).

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5) K. Kanai, "Semi-empirical Formula for the Seismic Characteristics of the Ground", Bull. Earthq. Res. Inst., 35(1957), 309.

K. Kanai, R. Takahasi and H. Kawasumi, "Seismic Characteristics of Ground", Proc. World Conf. Earthquake Engrg., Berkeley (1956), 31.

(i) The case of no surface layer and very long wave periods:

$$T_0 \longrightarrow 0 \ (T \longrightarrow \infty) \text{ and } G(T,q) \longrightarrow 1.$$

(ii) The case of a thick surface layer and very short wave periods:

$$T_0 \longrightarrow \infty \ (T \longrightarrow 0) \text{ and } G(T,q) \longrightarrow \frac{2}{1+c} .$$

(iii) The case of resonance:

$$T = T_0, \text{ and } G(T,q) = 1 + \frac{\sqrt{T_0}}{0.3}$$

In cases (i) and (ii) the ground magnifies the wave motion only very slightly. In case (iii) the values from (10) correspond very closely with those obtained from the previous version[6].

COMPARISONS OF COMPUTED AND OBSERVED SPECTRAL VALUES

Combining with equations (7), (8), (9) and (10) yields the following empirical formulae for surface ground motion spectra:

$$u = 0.034 \times T \times 10^{0.61\,M - 1.73 \log \Delta}$$

$$\times \left[1 + \left(\left[\frac{1+c}{1-c} \left\{ 1 - \left(\frac{T}{T_0} \right)^2 \right\} \right]^2 + \left\{ \frac{0.3}{\sqrt{T_0}} \left(\frac{T}{T_0} \right) \right\}^2 \right)^{-\frac{1}{2}} \right] \qquad (11)$$

$$v = 0.213 \times 10^{0.61\,M - 1.73 \log \Delta}$$

$$\times \left[1 + \left(\left[\frac{1+c}{1-c} \left\{ 1 - \left(\frac{T}{T_0} \right)^2 \right\} \right]^2 + \left\{ \frac{0.3}{\sqrt{T_0}} \left(\frac{T}{T_0} \right) \right\}^2 \right)^{-\frac{1}{2}} \right] \qquad (12)$$

$$a = \frac{1.34}{T} \times 10^{0.61\,M - 1.73 \log \Delta}$$

$$\times \left[1 + \left(\left[\frac{1+c}{1-c} \left\{ 1 - \left(\frac{T}{T_0} \right)^2 \right\} \right]^2 + \left\{ \frac{0.3}{\sqrt{T_0}} \left(\frac{T}{T_0} \right) \right\}^2 \right)^{-\frac{1}{2}} \right] \qquad (13)$$

where u, v and a are respectively, the spectral displacement(cm.), velocity(cm./sec.) and acceleration(cm./sec.^{2}) at the ground surface.

The displacement, velocity and acceleration spectra, in the cases where $M = 7$ and 8 and $\Delta = 50$ km., 100 km. and 150 km. besides the condition that $c = 1/5$ obtained by using equations (11)-(13) are shown in Fig.3.

At Tokyo in the 1923 Kwanto earthquake there were obtained $T_0 = 1.35$

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[6] K. Kanai, loc. cit., 5).

sec., $T_0 = 0.3$ sec., $M = 7.9$, and $\Delta = 100$ km.. The 1.35 sec. and 0.3 sec. periods were noted as corresponding to the predominant waves in the uptown area by Imamura[7] and Ishimoto[8], respectively. Applying (11), (12) and (13) the resonance values are as follows:

$$\left.\begin{array}{l} T_0 = 1.35 \text{ sec.}; \; u = 5.1 \text{ cm.}, \; v = 24 \text{ cm./sec.}, \; a = 110 \text{ cm./sec.}^2 \\ T_0 = 0.3 \text{ sec.}; \; u = 0.66 \text{ cm.}, \; v = 14 \text{ cm./sec.}, \; a = 290 \text{ cm./sec.}^2 \end{array}\right\} \qquad (14)$$

For comparison, the following values are estimated from the seismograms obtained by Imamura and Ishimoto:

$$\left.\begin{array}{l} T_0 = 1.35 \text{ sec.}; \; u = 4.43 \text{ cm.}, \; a = 100 \text{ cm./sec.}^2 \\ T_0 = 0.3 \text{ sec.}; \; a > 250 \text{ cm./sec.}^2 \end{array}\right\} \qquad (15)$$

The agreement between calculated and observed values is rather good.

The instrumental records and spectra[9] of eleven United States earthquakes recorded by the United States Coast and Geodetic Survey in the period 1933 to 1949 have been examined. The seismograph constants and earthquake data appear in Tables I and II. The resonance values of acceleration have been computed from (13), using as T_0 the periods of the maximum trace accelerations from Table II. The calculated values are compared with values of maximum recorded accelerations in Table III and Fig. 2. (In Fig. 2 the two circles of different radii representing each observation station correspond with the two orientations of measurement of horizontal acceleration.) Again the agreement of calculated and observed values is good.

CONCLUSIONS

In determining earthquake ground motion characteristics for purposes of structural design, equations (11), (12) and (13) seem to be applicable. The magnitudes and epicentral distances of anticipated earthquakes may be estimated by the application of engineering judgment to statistical analyses of seismicity, and there can be thus obtained an approximate spectrum of seismic waves in bed rock. If this spectrum is combined with the vibration characteristics of the ground at the structure, the lateral force coefficient can be determined.

The author's many thanks are due to Professor C. M. Duke, University of California, Los Angeles, who has assisted him in preparing this paper.

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[7] A. Imamura, "Report of the Investigation of the Kwanto Earthquake", Rep. Imper. Earthquake Inv. Com., No. 100A(1925), 27 (in Japanese).

[8] M. Ishimoto, Kokin-Shoin, Tokyo (1925), 113 (in Japanese).

[9] J. L. Alford, G. W. Housner and R. R. Martel, "Spectrum Analysis of Strong-Motion Earthquakes", 1st Tech. Rep. Office Nav. Res., Contr. No. 6onr-244, Task Order 25, Project Design, NR-081-091-(1951).

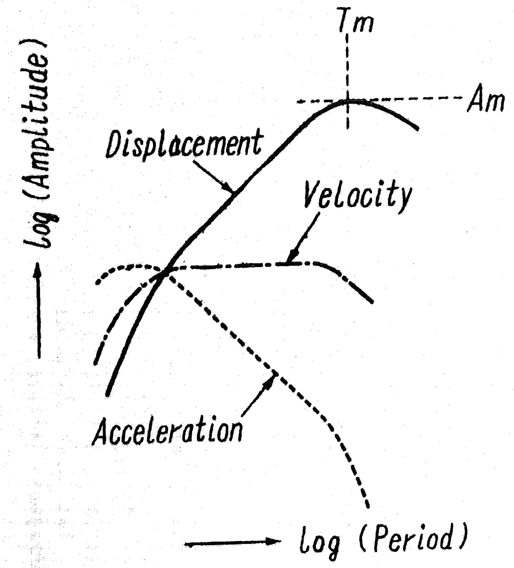

Fig. 1.
Schematic figures of the displacement, velocity and acceleration spectra of seismic waves at bed rock.

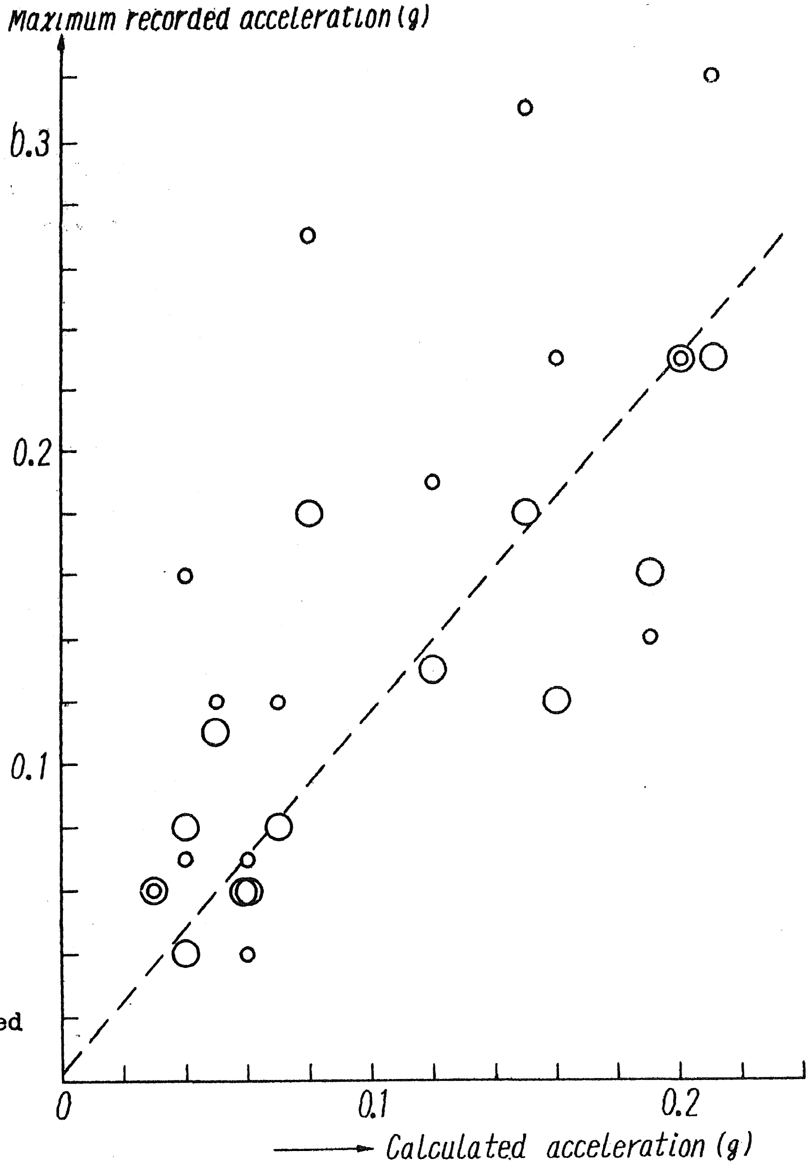

Fig. 2.
Relation between the maximum recorded accelerations and the values of calculated accelerations.

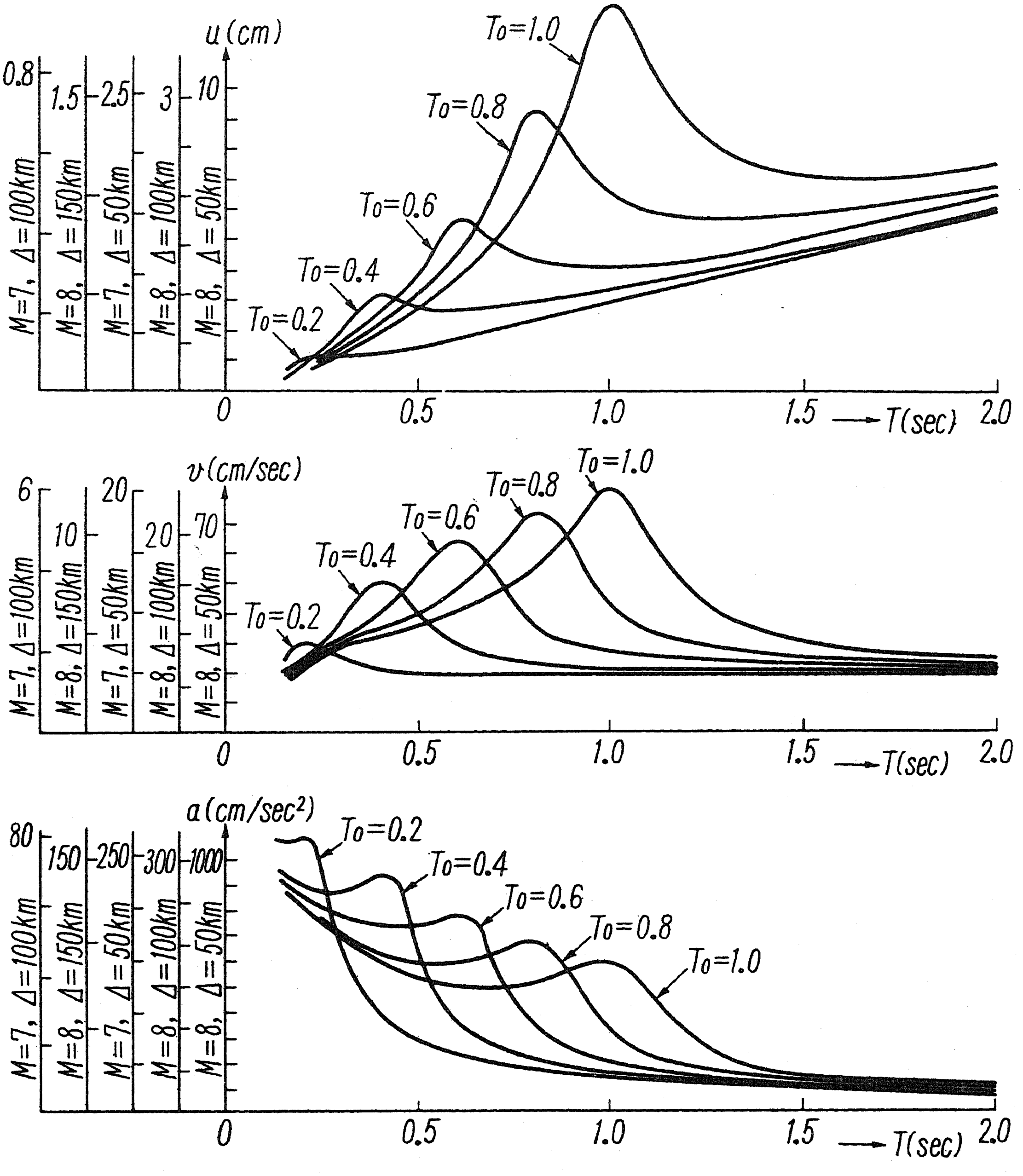

Fig. 3. The displacement(u), velocity(v) and acceleration(a) spectra obtained by using equations (11)-(13). T; period of waves, M; Richter magnitude, Δ ; epicentral distance, T_0; predominant period of ground.

TABLE I

Instrumental Data

Station			Instrument		
No.	Location	No. of stories	Location	Orientation	Period (sec.)
I	Vernon, Calif. Centr. Manuf. Dist. Term.	6	Basement	S82E	0.065
				N08E	0.065
II	Los Angeles, Calif. Subway Terminal Bldg.	13	Subbasement	N51W	0.065
				N39E	0.065
III	El Centro, Calif. Irrig. Dist. Sub-station	1	Ground level, slab	NS	0.065
				EW	0.065
IV	Helena, Montana Carroll College	5	Basement	NS	0.080
				EW	0.080
V	Ferndale, Calif. City Hall	2	Slab	N45E	0.066
				S45E	0.065
VI	Santa Barbara, Calif. Court House	2-4	Basement	N42E	0.064
				S48E	0.065
VII	Hollister, Calif. City Library	1	Basement	S01W	0.066
				N89W	0.066
VIII	Olympia, Washington Hiway Test Lab.	1	Slab	S10E	0.081
				S80W	0.080
IX	Seattle, Washington Army Deport	1	Shed	N88W	0.085
				S02W	0.085

TABLE II

The Values of Periods

Earthquake			Station		Period (sec.)	
No.	Date	M	No.	Δ (km.)	Max. accel. recorded	Max. accel. of spectrum
1	Mar. 10, '33	6.3	I	45	0.3	0.3
			II	53	0.65	0.65, 0.3
2	Oct. 2, '33	5.3	I	27	0.35	0.3
			II	35	0.65	0.65, 0.3
3	Dec. 30, '34	6.5	III	56	0.5	0.5, 0.2
4	Oct. 31, '35	6.0	IV	24	0.4	0.4, 0.25
5	Sept. 11, '38	5.5	V	56	0.2	0.2, 0.4
6	May 18, '40	7.0	III	48	0.5	0.5, 0.2
7	Feb. 9, '41	6.6	V	121	0.3	0.3, 0.6
8	June 30, '41	5.9	VI	24	0.3	0.3, 0.6
9	Oct. 3, '41	6.4	V	80	0.3	0.3, 0.6
10	Mar. 9, '49	5.3	VII	16	0.35	0.35
11	Apr. 13, '49	7.1	VIII	72	0.35	-
			IX	88	0.9	0.9, 0.3

TABLE III

The Values of Accelerations

Earthquake Number (Table II)	Station Number (Table I)	Orien-tation	Acceleration (g)	
			Observed	Calculated
1	I	S82E	0.19	0.12
		N08E	0.13	
	II	N51W	0.06	0.06
		N39E	0.04	
2	I	S82E	0.12	0.07
		N08E	0.08	
	II	N51W	0.06	0.03
		N39E	0.06	
3	III	NS	0.27	0.08
		EW	0.18	
4	IV	NS	0.14	0.19
		EW	0.16	
5	V	N45E	0.08	0.04
		S45E	0.16	
6	III	NS	0.32	0.21
		EW	0.23	
7	V	N45E	0.07	0.04
		S45E	0.04	
8	VI	N42E	0.23	0.20
		S48E	0.23	
9	V	N45E	0.11	0.05
		S45E	0.12	
10	VII	S01W	0.23	0.16
		N89W	0.12	
11	VIII	S10E	0.18	0.15
		S80W	0.31	
	IX	N88W	0.07	0.06
		S02W	0.06	

DISCUSSION

C. Tsuboi, University of Tokyo, Japan:

You have made use of my formula

$$\log A = M - 1.73 \log \Delta + 0.83$$

in your study. This formula was originally designed to give M of earthquakes in and near Japan from A and Δ data for these earthquakes observed in Japan. I wish to know if you have found that this formula works for earthquakes in America also. For instance, have you found agreements or disagreements between the Pasadena magnitudes of American earthquakes, and those determined by this formula?

K. Kanai:

From the result of the present investigation, it may be said that the agreements between the Pasadena magnitudes of American destructive earthquakes and those determined by your formula are rather good.