DESIGN CRITERIA FOR SHEAR AND OVERTURNING MOMENT

by John E. Rinne*

Introduction

Criteria for earthquake-resistant design in California have varied considerably over the fifty-four years since the San Francisco Earthquake of April 18, 1906. In the early post-quake era, earthquake resistance was achieved through design for increased wind forces. While some structural engineers designed for accelerations applied to gravity loads, it was not until 1933 that earthquake design was written into the State laws. The Riley Act provides for minimum lateral earthquake forces of 2 per cent of the vertical design loads. The Field Act, also passed in 1933, following the Long Beach Earthquake, is applicable to schools and other institutional buildings under the jurisdiction of the State Division of Architecture. The rules established under this Act are much more stringent than the Riley Act. The latter is generally applicable statewide where more stringent requirements are not prescribed by local ordinances. Most of the cities and counties do impose much more involved design criteria in their building codes.

As this is being written, the Los Angeles and Uniform Building Codes are considering adopting the recommendations of the Seismology Committee of the Structural Engineers Association of California. These recommendations are the subject of another paper at this Conference. All of the modern codes have recognized the need for variable coefficients for seismic shears, with higher coefficients for low, rigid buildings; lower coefficients for high, flexible buildings. In the Los Angeles and Uniform Building Codes, these coefficients relate to the number of stories above the story under consideration and are applied to the dead load generally to arrive at the story earthquake shears. Overturning criteria correspond to 80 per cent of the story shear multiplied by the height to the center of gravity of the loads above the story under consideration.

Following the report of the Joint Committee on Lateral Forces (San Francisco Section, American Society of Civil Engineers, and Structural Engineers Association of Northern California) known as "Separate 66", San Francisco adopted a code where, for the first time in California at least, earthquake forces and shears were directly related to the fundamental period of vibration of the structure. The overturning moments were recognized not to be cumulative without limit. An arbitrary limit on design overturning moment was given:

> "Provision for overturning moment shall be made for the specified earthquake forces in the top ten stories of buildings or the top 120 feet of other structures, and the moments shall be assumed to remain constant from these levels into the foundations."

The "overturning moment" at any horizontal plane is the moment on the structure as a whole resulting from the dynamic earthquake forces above the plane, giving due regard to signs of the modal forces.

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The Joint Committee was well aware that this was an interim provision and recommended that further research was needed for "establishment of a design criterion for the overturning effects in earthquakes, based upon dynamic considerations."

The author undertook a study of shears and overturning moments of stack-like structures and more recently, as part of the work of the Seismology Committee of the Structural Engineers Association of California, extended this work to include shear deflection type structures. From these studies, criteria have evolved for design overturning moment. It is the purpose here to present the reasoning leading to these criteria.

Steps in the Analysis

To correlate the ensuing discussion, the following outline of the steps in the development may be helpful toward an understanding of the significance of these steps and of the assumptions made.

1. Determine equivalent spring-mass systems representative of the modes of vibration for uniform cantilevers which deflect (a) due to bending moments exclusively, and (b) due to shear exclusively.

2. Establish a modal response spectrum which indicates the response of the simple spring-mass systems to earthquake ground motion in terms of relative lateral force, or base shear, on these structures as a function of the period of vibration.

3. Determine the combined modal response of both moment and shear type structures, and for both base shear and base moment. Compare the base moment response with the base moment corresponding to the SEAOC code provisions for base shear and overturning corresponding to that base shear.

4. Indicate a criterion for reducing the base moment to reflect the ratio of the base moment response to the base moment corresponding to the design base shear.

5. Recommend a basis for reducing the design overturning moment from the base to the top of the structure.

6. Recommend a basis for distributing the design overturning moment to the vertical resisting elements of the structure.

Equivalent Spring-Mass Systems

The response of a complex structure to earthquake ground motion theoretically can be determined at any instant during an earthquake by the superposition of the responses in the different modes in which the structure can respond or vibrate. Each of these modes can be represented by a spring-mass system of period equal to that of the mode under consideration, and a mass which is a certain proportion of the total mass of the structure.

For cantilever structures of uniform distribution of mass or weight, Mr. Milton Ludwig has determined these equivalent spring-mass systems for moment deflecting structures and for shear deflecting structures. With his permission, Mr. Ludwig's development is given in detail in the Appendix. The resulting equivalent masses of the modal spring-mass systems, together with the ratios of the higher mode periods to the fundamental period, are given in TABLE I. Graphically these are shown in Figures 1 and 2 which indicate heights of the spring-mass systems such that the modal shears multiplied by these heights give the respective modal base overturning moments in the actual structure.

It is seen by the summation of W_e/W columns of TABLE I that most of the mass of either type structure, but particularly the shear type, is assigned to the first three modes. For purposes of relating design shear to design overturning moment, a simplifying assumption can be made that all of the mass is assigned or prorated to the first three modes. The higher modes contribute little to overturning and even the third mode is not very significant in this regard. The result is shown in TABLE II and in Figures 1 and 2.

Response Spectrum

The spectral or maximum responses of spring-mass systems to a number of recorded ground motions have been determined for both undamped and damped systems, using analog computers. It bears repeating that the spectral response is the maximum response during the entire earthquake. As units of measurement, the spectra may be in terms of displacement, velocity, acceleration, or functions related to these units. For this analysis, the acceleration spectrum is particularly useful since this spectrum is directly related to the lateral force, or the base shear, on the spring-mass system.

Reviewing a number of undamped spectra, which qualitatively can indicate the response of structures within the elastic range, the striking similarity between these is interesting. A response spectrum which seems to envelop quite adequately the spectra from a number of earthquakes of varying intensity is shown in FIGURE 3.

An assumption will be made in this analysis which is admittedly not entirely accurate. It is that spectral responses can be added to get the combined modal response. The maximum responses in the second and third modes do not necessarily occur at the same instant as the spectral fundamental response. But it is probable that the higher modes are making a significant contribution at that instant. The accuracy of this assumption is probably well within the accuracy of several other assumptions inherent to this kind of an analysis.

Combined Modal Responses

The relationship between base moment and base shear in the first three modes is shown in TABLE III. The calculation of the modal base moments and the ratio of these to the moments corresponding to the design base shear is given in TABLE IV for moment structures, and TABLE V for shear structures. These calculations have been made for certain periods selected either arbitrarily or to create the maximum response in the second or third modes. For example, T_1 = 1.56 seconds was selected to give a second mode period, T_2 = 0.25 for the moment type structure, at which period the second mode response is a maximum as seen from FIGURE 3.

Moment Reduction Criterion

The results of TABLES IV and V are plotted in FIGURE 4. Also shown thereon is the proposed base moment reduction factor, J.
FIGURE 2 indicates that if the base shear for design is given by a formula:

$$V = C\ W, \quad \text{where } C = k/\ T^{1/3}$$

and if this base shear is resolved into forces F_x by the formula:

$$F_x = V \frac{w_x\ h_x}{\sum w\ h}$$

and if the modal response to earthquake ground motion is given by FIGURE 1:

$$S = 0.25/\ T \quad \text{for T greater than or equal to 0.25}$$

$$\text{and} \quad S = 0.30 + 2.80\ T \quad \text{for T between 0 and 0.25}$$

and if it is a reasonable approximation to assume that the maximum response of the structure is the sum of the spectral modal responses, then the design base moment M can be given by:

$$M = J \sum F_x\ h_x$$

$$\text{where} \quad J = J_b \text{ for buildings} = 0.5/\ T_1^{2/3} \quad \text{with } J_{b\ max.} = 1.00$$

$$\text{and } J_{b\ min.} = 0.33$$

$$\text{and} \quad J = J_m \quad \text{for moment deflection structures, like stacks}$$

$$J_m = 0.6/\ T_1^{1/2} \quad \text{with } J_{m\ max.} = 1.00$$

$$J_{m\ min.} = 0.40$$

The J_m factor is in the same form as the factor k suggested in the author's earlier paper, but is somewhat lower.

$$k = 0.7/\ T_1^{1/2} \quad \text{with } k_{max} = 1.00$$

$$k_{min} = 0.50$$

The difference is attributable to the difference in the criteria for the base shear coefficient C. Here the formula advocated by the Seismology Committee of the Structural Engineers Association of California has been used. Earlier, the formula for $C = 0.08\ /\ T_1$.

It would have been possible to fit empirical curves much closer to the response curves in FIGURE 2. A degree of conservatism has been introduced to account for the possibility that the modal response might peak at a period somewhat higher than 0.25 seconds. Most of the spectra peak in the range of 0.20 to 0.30 seconds. The recommended formula for J recognizes this contingency.

Base Shear

The SEAOC Seismology Committee's recommendation for base shear is:

$$V = K C W \quad \text{where} \quad C = 0.05/T^{1/3}$$

$$\text{and} \quad K = 0.67 \text{ to } 1.33 \text{ for buildings, depending upon the type of construction}$$

$K = 1.50$ for structures other than buildings.

While the recommended code places no lower limit on the value of C as given by the formula above, it does place a ceiling or maximum value of C as that corresponding to $T = 0.10$ seconds. This is $C = 0.108$. One and two story buildings have a fixed $C = 0.10$. A $C = 0.10$ maximum for all structures is a simpler and more consistent upper limit. Depending upon the type of structure and construction, the combined K C factor then would vary from 0.067 to 0.15. For the base shears so determined, the base moment M can be defined by the formulas given above.

The Moment M_x

The variation of the overturning moment on the structure as a whole over the height of the structure depends upon the relative importance of the fundamental and higher modes of vibration. This could lead to a quite involved relationship which hardly seems justified. For a moment deflection structure of the uniform cantilever type it has been shown that a straight-line diminishing moment from the bottom to the top of the structure provides a reasonable design criterion. For more rigid structures, where the fundamental mode predominates, this straight-line variation is conservative. However, recognizing that overturning considerations do not usually impose difficulties to the design of low structures, and in the interest of simplification, it is suggested that the moment M_x at height h_x be given by:

$$M_x = M\left(1 - \frac{h_x}{H}\right)$$

The code further recommends or stipulates that the total moment at any level x shall be assigned to the vertical resisting elements in the same proportion as the shears. Further amplification of this is to be found in the manual which explains the code in more detail.

Conclusion

It is believed that the SEAOC criteria for shear and overturning moment are more refined and more realistic than like criteria used heretofore. However, it must be acknowledged that experience in applying these criteria to date have been very limited. Hence, it might be expected that further modifications in the code will be forthcoming with further experience. For now, the criteria appear to be a sound step forward in the resolute progress of earthquake-resistant design.

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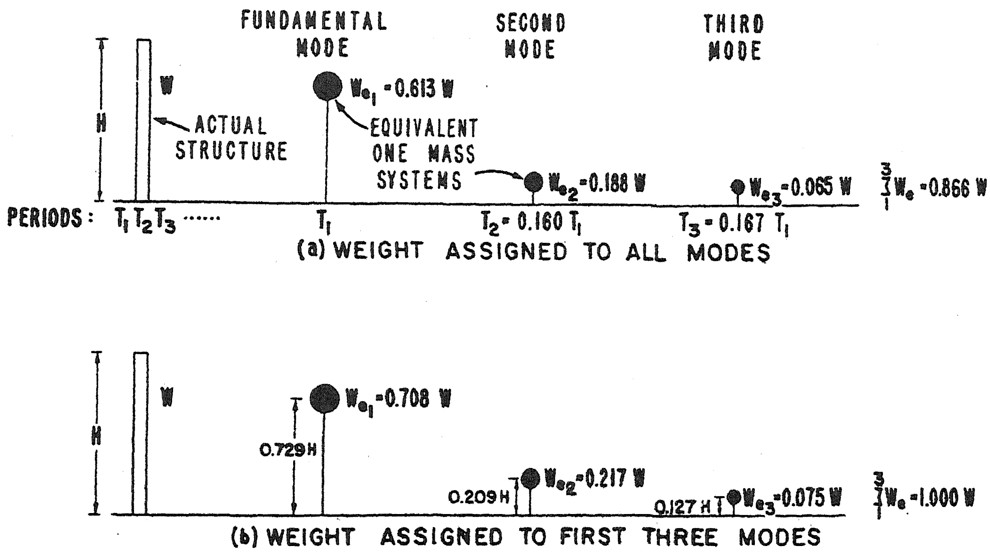
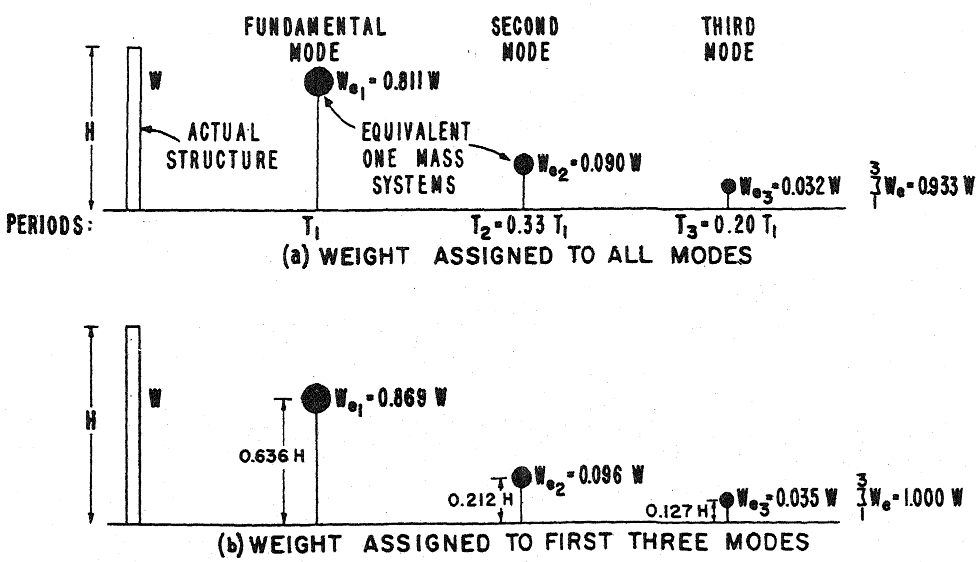
Nomenclature

C	Base shear coefficient in $V = K\ C\ W$
F_x	Earthquake lateral force at level x
h_x	Height to level x from the base of the structure
H	Total height of the structure from the base
J	Base moment reduction coefficient
J_b	J as applied specifically to buildings and structures other than those structures which deflect primarily due to bending, like stacks.
J_m	J as applied to moment deflection type structures
K	Coefficient for type of structure and type of construction in determining the base shear, $V = K\ C\ W$
M	Base overturning moment for design; also the base moment response in Tables IV and V
M_d	Base moment corresponding to $\sum F_x\ h_x$
M_x	Overturning moment for design at level x
S	Relative spectral response of spring-mass systems
T_n	Period of vibration in mode n
V	Base shear; also the base shear response in Tables IV and V
V_d	Base shear corresponding to $C = k/\ T^{1/3}$ in Tables IV and V
w_x	That part of the total weight of the structure which is at or is assigned to level x for purposes of lateral force design
W	Total weight of the structure
W_{en}	Proportion of the total weight W of a uniform cantilever assigned as an equivalent spring-mass system for mode n.



EQUIVALENT SPRING — MASS SYSTEMS FOR MOMENT DEFLECTION STRUCTURES

FIGURE 1



EQUIVALENT SPRING—MASS SYSTEMS FOR SHEAR DEFLECTION STRUCTURES

FIGURE 2

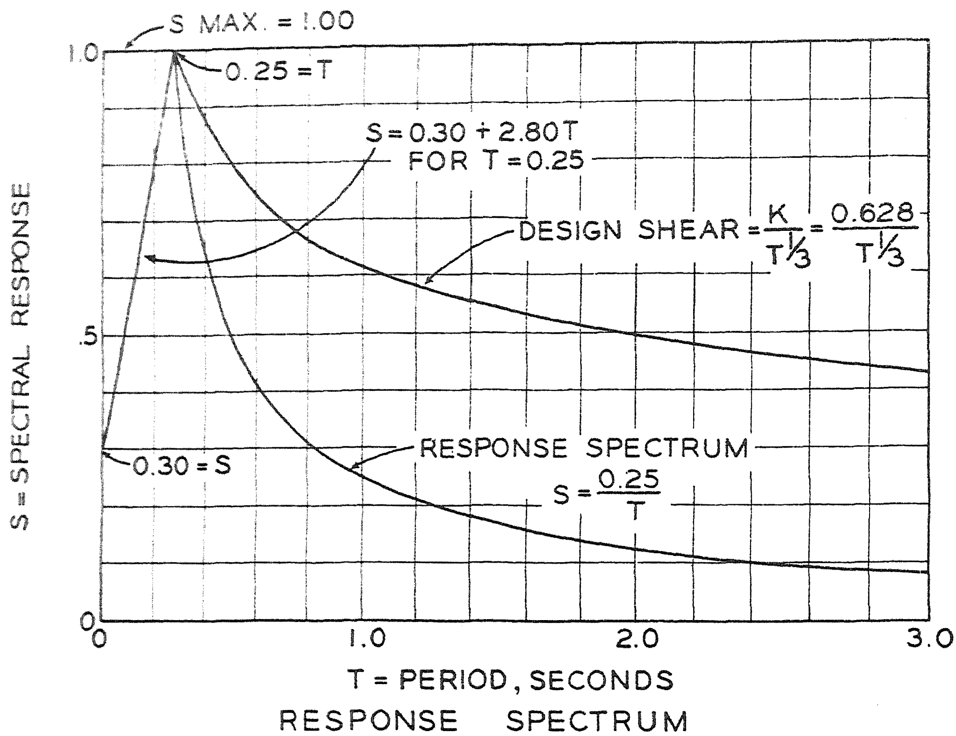


FIGURE 3

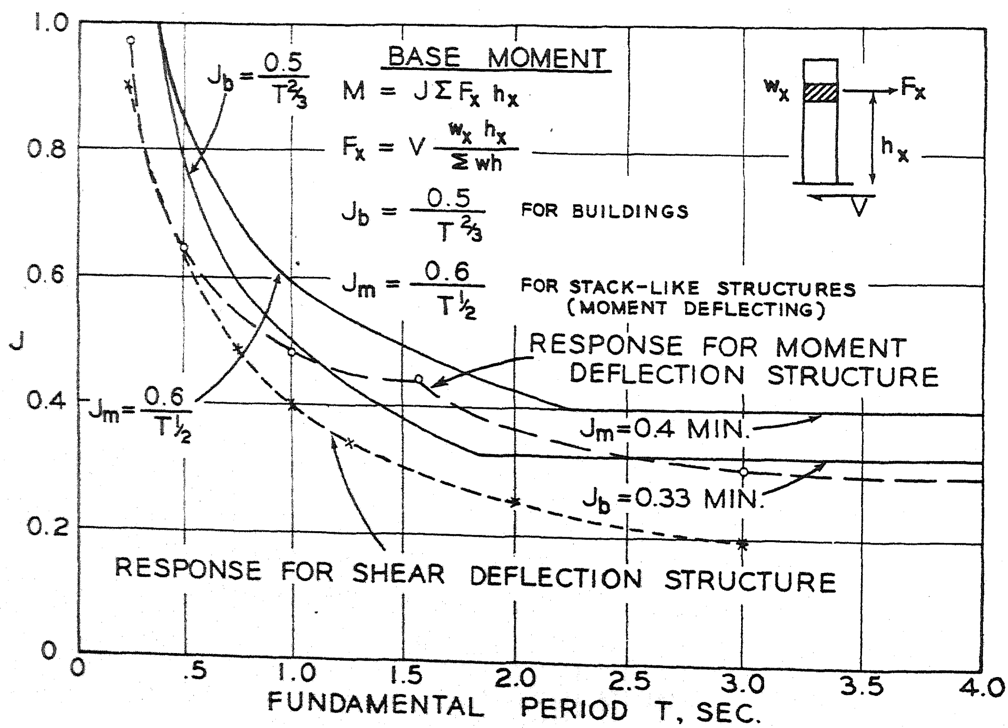


FIGURE 4

TABLE I
EQUIVALENT MODAL SPRING-MASS SYSTEMS*

Mode	Moment Deflection Structures			Shear Deflection Structures		
n	W_e/W	$\Sigma W_e/W$	T_n/T_1	W_e/W	$\Sigma W_e/W$	T_n/T_1
1	0.613	0.613	1.00	0.811	0.811	1.00
2	0.188	0.801	0.160	0.090	0.901	0.33
3	0.065	0.866	0.057	0.0324	0.933	0.20
4	0.033	0.899	0.029	0.0166	0.950	0.142
5	0.020	0.919	0.0176	0.0100	0.960	0.111
6	0.013	0.932	0.0117	0.0067	0.967	0.091

*For cantilevers of uniformly distributed weight.
W_e/W = proportion of total weight assignable to the indicated mode's equivalent spring-mass system.
T_n/T_1 = ratio of the modal period to the fundamental period.

TABLE II
PROPORTIONATE ASSIGNMENT OF ALL WEIGHT TO MODES 1, 2, 3

Mode	Moment Deflection Structures			Shear Deflection Structures		
n	W_e/W	$\Sigma W_e/W$	T_n/T_1	W_e/W	$\Sigma W_e/W$	T_n/T_1
1	0.708	0.708	1.00	0.869	0.869	1.00
2	0.217	0.925	0.160	0.096	0.965	0.33
3	0.075	1.000	0.057	0.035	1.000	0.20
	1.000			1.000		

TABLE III
BASE MOMENT - BASE SHEAR RELATIONSHIPS

Mode	Moment Structures M/VH*	Shear Structures M/VH
1	0.729	0.636
2	0.209	0.212
3	0.127	0.127

*When H = 1, M/VH = M/V

TABLE IV
MOMENT RESPONSE FOR MOMENT DEFLECTION STRUCTURES

Item	Description	Fundamental Period, T_1					
		0.25	0.50	1.00	1.56	3.00	4.38
S_1	Response	1.00	0.50	0.25	0.16	0.085	0.057
W_{e1}	Equiv. mass	0.708	0.708	0.708	0.708	0.708	0.708
$V_{1'}$	$S_1 W_{e1}$	0.708	0.354	0.177	0.113	0.060	0.0403
T_2	$0.160\ T_1$	0.040	0.080	0.160	0.250	0.480	0.703
S_2	Response	0.42	0.523	0.746	1.000	0.52	0.355
W_{e2}	Equiv. mass	0.217	0.217	0.217	0.217	0.217	0.217
$V_{2'}$	$S_2 W_{e2}$	0.091	0.114	0.162	0.217	0.114	0.0771
T_3	$0.057\ T_1$	0.014	0.0285	0.057	0.089	0.171	0.250
S_3	Response	0.33	0.38	0.46	0.55	0.78	1.00
W_{e3}	Equiv. mass	0.075	0.075	0.075	0.075	0.075	0.075
$V_{3'}$	$S_3 W_{e3}$	0.025	0.0285	0.0345	0.0412	0.0586	0.075
V'	$V_{1'} + V_{2'} + V_{3'}$	0.824	0.4965	0.3735	0.3712	0.2326	0.1924
V_1	$V_{1'}/\ 0.824$	0.860	0.430	0.215	0.1375	0.0731	0.0490
V_2	$V_{2'}/\ 0.824$	0.111	0.138	0.196	0.263	0.139	0.0932
V_3	$V_{3'}/\ 0.824$	0.029	0.0343	0.042	0.050	0.0715	0.0910
V	$V_1 + V_2 + V_3$	1.000	0.6023	0.453	0.450	0.2386	0.2332
M_1	$0.729\ V_1$	0.6270	0.3140	0.1570	0.1010	0.0533	0.0358
M_2	$0.209\ V_2$	0.0232	0.0289	0.0410	0.0550	0.0291	0.0195
M_3	$0.127\ V_3$	0.0038	0.0043	0.0053	0.0063	0.0091	0.0115
M	$M_1 + M_2 + M_3$	0.6540	0.3472	0.2033	0.1623	0.0915	0.0668
V_d*	Design shear	1.000	0.794	0.628	0.541	0.436	0.386
M_d	$0.667\ V_d$	0.667	0.530	0.420	0.361	0.291	0.258
J	$M/\ M_d$	0.980	0.655	0.484	0.449	0.314	0.259

$$* \ V_d = \frac{K}{T_1^{1/3}} = \frac{0.628}{T_1^{1/3}}$$

TABLE V
MOMENT RESPONSE FOR SHEAR DEFLECTION STRUCTURES

Item	Description	Fundamental Period, T_1					
		0.25	0.75	1.00	1.25	2.00	3.00
S_1	Response	1.00	0.33	0.25	0.20	0.125	0.083
W_{e1}	Equiv. mass	0.869	0.869	0.869	0.869	0.869	0.869
$V_{1'}$	$S_1 W_{e1}$	0.869	0.290	0.217	0.174	0.1085	0.0721
T_2	$0.333\ T_1$	0.083	0.250	0.333	0.417	0.667	1.00
S_2	Response	0.532	1.00	0.757	0.600	0.375	0.250
W_{e2}	Equiv. mass	0.096	0.096	0.096	0.096	0.096	0.096
$V_{2'}$	$S_2 W_{e2}$	0.0510	0.096	0.0728	0.0576	0.0360	0.0240
T_3	$0.200\ T_1$	0.050	0.150	0.200	0.250	0.400	0.600
S_3	Response	0.44	0.72	0.86	1.00	0.625	0.416
W_{e3}	Equiv. mass	0.035	0.035	0.035	0.035	0.035	0.035
$V_{3'}$	$S_3 W_{e3}$	0.0154	0.0253	0.0301	0.0350	0.0219	0.0146
V'	$V_{1'} + V_{2'} + V_{3'}$	0.9354	0.4112	0.3199	0.2666	0.1664	0.1107
V_1	$V_{1'}/0.9354$	0.929	0.310	0.232	0.186	0.116	0.077
V_2	$V_{2'}/0.9354$	0.055	0.102	0.0775	0.062	0.0386	0.0256
V_3	$V_{3'}/0.9354$	0.016	0.0268	0.0321	0.0374	0.0232	0.0155
V	$V_1 + V_2 + V_3$	1.000	0.4388	0.3416	0.2854	0.1778	0.1181
M_1	$0.636\ V_1$	0.5910	0.197	0.148	0.118	0.074	0.049
M_2	$0.212\ V_2$	0.0117	0.0216	0.0164	0.0131	0.0082	0.0054
M_3	$0.127\ V_3$	0.0020	0.0034	0.0041	0.0047	0.0029	0.0020
M	$M_1 + M_2 + M_3$	0.6047	0.2220	0.1685	0.1358	0.0851	0.0564
V_d *	Design shear	1.000	0.692	0.628	0.584	0.498	0.436
M_d	$0.667\ V_d$	0.667	0.462	0.420	0.390	0.333	0.291
J	M/M_d	0.907	0.480	0.401	0.348	0.255	0.194

$$* \ V_d = \frac{0.628}{T_1^{1/3}}$$

APPENDIX 1

Development of Equivalent Spring-Mass Systems for Uniform Cantilever Beams of the Moment Deflection Type

(From notes prepared by Mr. Milton Ludwig, Engineering Department, Standard Oil Company of California, San Francisco, California)

The modal periods of vibration of a uniform cantilever beam deflecting due to bending moment is well documented as:

$$T_n = \frac{2\pi}{\theta_n^2}\sqrt{\frac{W\,H^3}{E\,I\cdot g}} \qquad \text{.........(1)}$$

where θ_n is defined by:

$$\cos\theta_n \cosh\theta_n + 1 = 0 \qquad \text{.........(2)}$$

The spring constant and mass for the equivalent spring-mass systems may be evaluated in several ways. One method is to determine the effect on frequency for a slight lateral flexibility at the base. An easier and more accurate method is to work from the known response to a sudden change of lateral velocity at the base. The solution to this problem has been given* and with some changes in notation, the equation for the modal shear is:

$$V_n = B_n\,v\sqrt{\frac{E\,I\cdot W}{H^3\,g}}\ \sin\omega_n t \qquad \text{.........(3)}$$

where
- V_n = base shear in the nth mode
- B_n = numerical constant for the nth mode
- v = base velocity
- W = total mass of the beam
- ω_n = frequency of nth mode, radians per sec.
- t = time, sec.
- H = total height of cantilever
- EI = Young's modulus x moment of inertia

From equation (1) for small values of t:

$$\sin\omega_n t = \omega_n t = \theta_n^2\sqrt{\frac{E\,I\,g}{W\,H^3}}\ t \qquad \text{.........(4)}$$

which substituted into equation (3) yields:

$$V_n = B_n\,v\,\theta_n^2\,\frac{E\,I}{H^3}\,t \qquad \text{.........(5)}$$

The spring constant is defined as the force-deflection ratio, so:

$$k_n = \frac{V_n}{v\,t} = B_n \theta_n^2 \frac{E\,I}{H^3} \qquad \text{........(6)}$$

The basic equation for the resonant frequency of a spring-mass system may be used to give the equivalent mass for each mode, as follows:

$$W_{en} = \frac{k_n g}{\omega_n^2} = W \frac{B_n}{\theta_n^2} \qquad \text{........(7)}$$

in which W_{en} = mass associated with the nth mode

The report* that provides the solution for V_n (Equation 3) also gives the equation for the moment developed at the base of the cantilever for a sudden change of velocity:

$$M_n = A_n v H \sqrt{\frac{E\,I\,W}{H^3 g}} \qquad \text{........(8)}$$

Then the ratio of the moment arm to the total height of the beam is:

$$\frac{M_n}{H V_n} = \frac{A_n}{B_n} \qquad \text{........(9)}$$

Values of A_n and B_n and calculated equivalent masses and moment factors are as follows:

Mode, n	θ_n	A_n	B_n	$\frac{W_{en}}{W} = \frac{B_n}{\theta_n^2}$	$\frac{M_n}{H V_n} = \frac{A_n}{B_n}$
1	1.875	1.566	2.146	0.610	0.729
2	4.694	0.868	4.149	0.188	0.209
3	7.855	0.509	3.994	0.065	0.127
4	10.996	0.364	4.000	0.033	0.091
5	14.137	0.283	4.000	0.020	0.071
6	17.279	0.231	4.000	0.013	0.058

These calculated values are perfectly general and are not limited to the case for a sudden change of velocity. The values of $M_n/H V_n$ can be calculated from the moment and shear equations for harmonic vibration of a cantilever beam, and equivalent masses can be checked by other methods. It was merely convenient to use the constants from the NACA report which seem to have been worked out carefully and accurately.

*Report No. 828 National Advisory Committee for Aeronautics (NACA) "Bending and Shear Stresses Developed by the Instantaneous Arrest of the Root of a Moving Cantilever Beam." by Elbridge Z. Stowell, Edward B. Schwartz, and John C. Houbolt. 1945. Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. Price 15 cents.

APPENDIX 2

Development of Equivalent Spring-Mass Systems for Uniform Cantilever Beams of the Shear Deflection Type

(From notes prepared by Mr. Milton Ludwig, Engineering Department, Standard Oil Company of California, San Francisco, California)

In this development:

M = total mass of the shear deflection type structure

M_1, M_2, etc. = masses of the equivalent spring-mass systems, corresponding to the fundamental, second, mode etc.

K = total spring constant for the structure $= \dfrac{\text{force at top}}{\text{deflection at top}}$

k_1, k_2, etc., = spring constants for the equivalent spring-mass systems

ω = angular velocity of an applied simple harmonic base motion

ω_1, ω_2, etc., = angular velocity corresponding to the fundamental, second mode, etc.

$v = v_{max} \sin \omega t$ = applied lateral velocity at the base

V = periodic base shear resulting from the applied velocity

$Z = V/v$ = input impedance of the system

$j = \sqrt{-1}$, which merely indicates the phase relationship between the base shear V and the velocity v.

L = inductance of an equivalent electrical system or analog.

C = capacitance of the electrical analog.

It can be demonstrated that the structure involved here is equivalent to a long short-circuited transmission line with no resistance.

$\dfrac{M}{g} = L$ = equivalent total inductance of the structure

$\dfrac{1}{K} = C$ = equivalent total capacitance of the structure

for which $Z = j\sqrt{\dfrac{L}{C}}\ \tan \omega\sqrt{LC} = j\sqrt{\dfrac{MK}{g}}\ \tan \omega\sqrt{\dfrac{M}{Kg}}$

For the equivalent spring-mass systems:

$$Z = j\,\frac{\omega M_1/g}{1 - \omega^2 \dfrac{M_1}{k_1 g}} + j\,\frac{\omega M_2/g}{1 - \omega^2 \dfrac{M_2}{k_2 g}} + \cdots\cdots\cdots$$

The impedance Z is infinite when $\frac{M}{K\,g} = \pi/2, \quad 3\pi/2, \quad 5\pi/2,$ etc.

or when $\omega\sqrt{\frac{M_1}{k_1\,g}} = 1, \; \omega\sqrt{\frac{M_2}{k_2\,g}} = 1,$ etc.

from which, $\frac{M_1}{k_1} = \left(\frac{2}{\pi}\right)^2 \frac{M}{K}, \quad \frac{M_2}{k_2} = \left(\frac{2}{3\pi}\right)^2 \frac{M}{K}$, etc.

Now, the response at a frequency very close to a resonant frequency is very high so that the response of the other masses of the equivalent system can be neglected. So to determine M_1:

$$\tan \omega\sqrt{\frac{M}{K\,g}} = \frac{1}{\tan \frac{\pi}{2}\left(1 - \frac{2}{\pi}\,\omega\sqrt{\frac{M}{K\,g}}\right)} = \frac{1}{\tan \frac{\pi}{2}\left(1 - \frac{\omega}{\omega_1}\right)}$$

in which $\omega_1 = \frac{\pi}{2}\sqrt{\frac{K\,g}{M}}$ = first resonant frequency

As $\frac{\omega}{\omega_1}$ approaches 1, $\tan \omega\sqrt{\frac{M}{K\,g}}$ approaches $\frac{1}{\frac{\pi}{2}\left(1 - \frac{\omega}{\omega_1}\right)}$

For the shear deflection structure: $Z = \frac{j\sqrt{\frac{M\,K}{g}}}{\frac{\pi}{2}\left(1 - \frac{\omega}{\omega_1}\right)}$

For the first mass of the equivalent spring-mass system:

$$Z = \frac{j\,\omega\,\frac{M_1}{g}}{1 - \frac{\omega^2 M_1}{k_1\,g}} = \frac{j\,\omega\,\frac{M_1}{g}}{1 - \left(\frac{\omega}{\omega_1}\right)^2} = \frac{j\,\frac{\omega}{\omega_1}\;\frac{M_1}{g}\;\frac{\pi}{2}\;\frac{K\,g}{M}}{\left(1 + \frac{\omega}{\omega_1}\right)\left(1 - \frac{\omega}{\omega_1}\right)}$$

Then equating values of Z:

$$M_1 = \frac{4}{\pi^2}\;\frac{1 + \frac{\omega}{\omega_1}}{\frac{\omega}{\omega_1}}\;M = \frac{8}{\pi^2}\;M = 0.811\;M$$

$$k_1 = \frac{\pi^2}{4}\;\frac{M_1}{M}\;K = 2\,K$$

as $\frac{\omega}{\omega_1}$ approaches 1

By an exactly analogous development:

$$M_2 = \frac{1}{9}\;\frac{8}{\pi^2}\;M = 0.090\;M$$

$$M_3 = \frac{1}{25}\;\frac{8}{\pi^2}\;M = 0.0324\;M$$

$$M_4 = \frac{1}{49}\;\frac{8}{\pi^2}\;M = 0.0166\;M$$