A Survey of the Coupled Ground-Building Vibrations

By

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I. Introduction

The effect of the ground foundation properties on the dynamic behavior of buildings continues to be of interest to engineers. When a building is excited by an earthquake, what influence does the ground have on the motion of the building? Intuition indicates that stresses are relieved and natural frequencies are altered, but to what extent?

To answer these questions some investigators have assumed the building to be attached to a rigid ground by means of a torsional spring of stiffness $K\phi$. It is then possible to plot the ratio of the fundamental period of the building restrained by the torsional spring, to that of the building on a rigid ground as a function of the compliance ratio $K\phi/k\,h^2$, where k is the story stiffness in lateral shear and h the story height. For the single story building, analytical expressions are simply obtained but for the N story building a computer is generally necessary for the solution.

It has been the practice to ignore the translational stiffness K_t of the ground as being of lesser importance. However it can be shown that the effect of the translational stiffness is as important as the rotational stiffness in altering the natural frequencies of the building. It has not been clear just what the relationship is between the translational and rotational stiffness. In fact, is it justified at all to replace the ground by springs of constant modulus? Certainly the ground adjoining the foundation must move with the building, thereoy introducing a virtual mass as well as stiffness. All spring mass systems under dynamical conditions dispaly a frequency dependence and a condition of resonance.

The problem of the dynamical properties of the ground has been treated by several authors. Reissner(1) in 1936 developed solutions, based on the theory of elasticity, for the interaction of a rigid circular slab in vertical vibration on a semi-infinite ground. Sung(2) elaborated on these results for different distributions of pressure under the circular slab. Toriumi(3), in addition to verifying Reissner's results, developed the effect of horizontal translation and rocking of the circular slab about its diameter. These results given by the following three equations

(1) Horizontal displacement at the center of the disk

$$u = \frac{F_H e^{i\omega t}}{2\pi r_0 G}\left\{f_{1H}(a,b) + i\,f_{2H}(a,b)\right\} = \frac{F_H e^{i\omega t}}{K_H} \qquad (1)$$

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Vertical displacement of disk

$$w = \frac{F_v e^{i\omega t}}{\pi r_o G}\left\{ f_{1V}(a,b) + i f_{2V}(a,b) \right\} = \frac{F_V e^{i\omega t}}{K_V} \tag{2}$$

Rotation of the disk

$$\phi = \frac{4M_o e^{i\omega t}}{\pi r_o^3 G}\left\{ f_{1R}(a,b) + i f_{2R}(a,b) \right\} = \frac{M_o e^{i\omega t}}{K_R} \tag{3}$$

where $\quad a = \omega r_o \sqrt{\frac{\rho}{G}}$

$$b = \frac{G}{\lambda + 2G}$$

depend on a nondimensional quantity $\omega r_o \sqrt{\frac{\rho}{G}}$, where ω is the frequency of vibration, r_o the radius of the circular slab, and $\sqrt{\frac{G}{\rho}}$ the speed of the shear wave propagation of the soil. The quantities f_1 and f_2 for the three equations are shown in Figures 1, 2, and 3, for the sign convention adopted and indicated in each diagram.

Expressions similar to the above with an additional equation for torsion were also developed by Arnold(8) with extension to the case of a stoatum of infinite area.

The work of Reissner Toriumi and Arnold offer a basis to investigate the problem of the coupled building-ground motion. Sato and Yamaguchi(4) made use of Toriumi's results to investigate the rocking-translation motion of a solid cylindrical block which was intended to simulate a building. They also included a study of a single story frame building with shear stiffness under translation without rotation. Arnold(8) investigated the response of solid circular cylinders on an elastic semi-infinite foundation and carried out an experimental study using a foam rubber base.

II. Equations of Motion:

To study the effect of the ground properties on the dynamical behavior of buildings, it is advisable to consider an idealized structure of N equal stories with shear stiffness k per story as shown in Fig. (4). For simplicity the foundation mass will be taken equal to the story mass and circular in order to apply Toriumi's results.

The equation of motion for the n^{th} mass of the structure, where $1 \leq n \leq (N-1)$, is

$$m\ddot{y}_n - k(y_{n+1} - 2y_n + y_{n-1}) = 0 \tag{4}$$

and its amplitude relationship for harmonic oscillation $y_n = Y_n e^{i\omega t}$, is

$$Y_{n+1} - 2\left(1 - \frac{m\omega^2}{2k}\right) Y_n + Y_{n-1} = 0 \tag{4a}$$

Substituting $Y_n = e^{i\beta n}$ we obtain

$$\left(1 - \frac{m\omega^2}{2k}\right) = \cos\beta$$

$$\frac{m\omega^2}{k} = 2(1-\cos\beta) = 4\sin^2\frac{\beta}{2} \qquad (5)$$

The general solution for Y_n is then

$$Y_n = Y_0 \cos\beta n + B\sin\beta n \qquad (6)$$

The constant B is evaluated by letting n = N

$$B = \frac{Y_N - Y_0\cos\beta N}{\sin\beta N} \qquad (7)$$

which enables Y_n to be written as

$$Y_n = Y_0\frac{\sin\beta(N-n)}{\sin\beta N} + Y_N\frac{\sin\beta n}{\sin\beta N} \qquad (6a)$$

We must now satisfy the boundary conditions. There are three such equations, one for the motion of the N^{th} mass and two for the foundation mass in translation and rotation.

The boundary equation for the N^{th} mass is

$$m\ddot{y}_N + k(y_N - y_{N-1} + h\phi) = 0 \qquad (8)$$

For the foundation mass, we refer to Fig. (4). The ground is assumed to be given a displacement $Y_G = Y_G\, e^{i(\omega t+\theta)}$. and the structure will lag by the phase θ and have the displacement $y_n = Y_n\, e^{i\omega t}$. The relative displacement u of the foundation is the vector difference $y_G - y_0$ which from Eq. (1) is

$$F_H e^{i\omega t} = K_H(Y_G e^{i\theta} - Y_0)e^{i\omega t} \qquad (9)$$

where K_H is a complex number dependent on the frequency parameter. The boundary equations for the translation and rotation of the foundation mass are then

$$m\ddot{y}_0 = -\sum_{n=1}^{N} m\ddot{y}_n + K_H(y_G - y_0) \qquad (10)$$

$$(N+1)\, m r^2\ddot{\phi} = \sum_{n=1}^{N} nh(m\ddot{y}_n) - K_R\phi \qquad (11)$$

The boundary equations in terms of harmonic amplitudes are rewritten as follows

$$(k - m\omega^2)Y_N - kY_{N-1} + kh\Phi = 0 \qquad (8a)$$

$$\left(K_H - m\omega^2 \right) Y_0 - m\omega^2 \sum_{n=1}^{N} Y_n = K_H Y_G e^{i\Omega} \tag{10a}$$

$$\left[K_R - (N+1) m\omega^2 \gamma^2 \right] \Phi + m\omega^2 h \sum_{n=1}^{N} n Y_n = 0 \tag{11a}$$

These equations have (N+2) dependent variables, however since Y_n is related to Y_0, Y_N and ω by Eqs. (6a) and (5) the above equations can be reduced to three variables and ω. The procedure for this reduction is a lengthy algebraic and trigometric task which requires a systematic approach which we will first outline. The first step is the substitution of the general solution, Eq. (6a) into the three boundary equations, Eqs. (8a), (10a) and (11a). The summations $\sum Y_n$ and $\sum Y_n$ will result in a finite trigonometric series which can be summed by the equations given in the Appendix. The result will then be three equations for Y_0 Y_N and

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_N \\ h\Phi \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{K_H}{k} Y_G e^{i\Omega} \\ 0 \end{bmatrix} \tag{12}$$

where α_{ij} are complex transcendental expressions in terms of the system parameters and the frequency. It is noted here that β, K_H and K_R are functions of ω; β from Eq. (5), and the complex stiffnesses K_H and K_R from Toriumi's work.

With this outline of procedure, we leave the details of development for the Appendix and give the equations corresponding to Eq. (12).

$$-\left(\frac{\sin\beta}{\sin\beta N}\right) Y_0 + \left\{ \left(1 - \frac{m\omega^2}{k}\right) - \frac{\sin\beta(N-1)}{\sin\beta N} \right\} Y_N + h\Phi = 0 \tag{13}$$

$$\left\{ \frac{K_H}{k} - \left(\frac{m\omega^2}{k}\right) \left[1 + \frac{\sin\frac{\beta N}{2} \sin\frac{\beta}{2}(N-1)}{\sin\frac{\beta}{2} \sin\beta N} \right] \right\} Y_0 - \left\{ \left(\frac{m\omega^2}{k}\right) \frac{\sin\frac{\beta N}{2} \sin\frac{\beta}{2}(N+1)}{\sin\frac{\beta}{2} \sin\beta N} \right\} Y_N = \frac{K_H}{k} Y_G e^{i\Omega} \tag{14}$$

$$\frac{(N\sin\beta - \sin\beta N)}{\sin\beta N} Y_0 + \left\{ \frac{(N+1)\sin\beta N - N\sin\beta(N+1)}{\sin\beta N} \right\} Y_N + \left[\frac{K_R}{k h^2} - (N+1) \frac{m\omega^2}{k} \frac{\gamma^2}{h^2} \right] h\Phi = 0 \tag{15}$$

Thus the coefficients α_{ij} of Eq. (12) are identified from the above equations.

III. Forced Harmonic Vibration

The solution of the forced harmonic vibration problem is in the form

$$Y_0 = \frac{-K_H Y_G e^{i\theta}}{k D} \begin{vmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} \tag{16}$$

$$Y_N = \frac{+K_H Y_G e^{i\theta}}{k D} \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{31} & \alpha_{33} \end{vmatrix} \tag{17}$$

$$h \Phi = \frac{-K_H Y_G e^{i\theta}}{k D} \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{vmatrix} \tag{18}$$

$$D = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \tag{19}$$

Choosing a value of ω, β can be determined and the above equations computed. The shear and bending moment at the base are found from Eqs. (10a), (11a), and the above solutions (16), (17), and (18) to be

$$\text{Shear} = -m\omega^2 \sum_{n=1}^{N} Y_N = K_H Y_G e^{i\theta} \left\{ 1 + \frac{(K_H - m\omega^2)}{k D} \begin{vmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} \right\} \tag{20}$$

$$\text{Bending Moment} = m\omega^2 h \sum_{n=1}^{N} n Y_n \tag{21}$$

$$= \frac{K_H}{k} \frac{Y_G e^{i\theta}}{h D} \left[K_R - (N+1) m\omega^2 r^2 \right] \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{vmatrix}$$

IV. Free Vibration

For free vibration, the natural frequencies are found from the characteristic equation $D = 0$. Again the numerical procedure would be to assume a value for ω, determine corresponding values for β, K_H and K_R, and plot the computed value of D against ω.

At this point it is instructive to examine the first equation, Eq. (13), which after substituting for $\frac{m\omega^2}{k}$ from Eq. (5) becomes

$$2 \sin\frac{\beta}{2} \cos\beta\left(N+\frac{1}{2}\right) = \frac{Y_0}{Y_N}\sin\beta - \frac{k\Phi}{Y_N}\sin\beta N \tag{13a}$$

For the free vibration $Y_G = 0$, and $Y_0 = -U$, the elastic deformation of the ground. Thus if the ground is rigid, $U = \Phi = 0$ which results in the eigenvalues

$$\beta_i = \frac{(2i-1)\pi}{(2N+1)} \tag{22}$$

and the well known equation for the natural frequencies

$$\omega_i = 2\sqrt{\frac{k}{m}}\, \sin\frac{(2i-1)\pi}{(4N+2)} \tag{23}$$

Since the amplitude Y_N is arbitrary in free vibration, we can normalize according to the equation

$$\sum_{n=1}^{N} m Y_{ni}^2 = Nm. \tag{24}$$

which is equivalent to equating the generalized mass to the total vibrational mass. Substituting from

$$Y_{ni} = Y_{Ni}\frac{\sin\beta_i n}{\sin\beta_i N} \tag{6b}$$

we find, as shown in the Appendix, that the amplitude $\frac{Y_{Ni}}{\sin\beta_i N} = \sqrt{\frac{2N}{N+1/2}}$ and is independent of the mode number i. The mode shape is then given by

$$Y_{ni} = \sqrt{\frac{2N}{N+1/2}}\, \sin\frac{(2i-1)n\pi}{(2N+1)} \tag{6c}$$

$$n = 0, 1, 2, 3, \text{——} N$$
$$i = 1, 2, 3, \text{———} N$$

For the elastic ground Y_0/Y_N and $k\Phi/Y_N$ can be determined from Eqs. (14) and (15) and substituted into Eq. (13a). From Eq. (14) with right side equal to zero,

$$\frac{Y_0}{Y_N} = \frac{-U}{Y_N} = \frac{\left(\frac{m\omega^2}{k}\right)\dfrac{\sin\frac{\beta N}{2}\sin\frac{\beta}{2}(N+1)}{\sin\frac{\beta}{2}\sin\beta N}}{\frac{K_H}{k} - \left(\frac{m\omega^2}{k}\right)\left[1 + \dfrac{\sin\frac{\beta N}{2}\sin\frac{\beta}{2}(N-1)}{\sin\frac{\beta}{2}\sin\beta N}\right]} \tag{25}$$

From Eqs. (15)

$$\frac{h\Phi}{Y_N} = \frac{1}{\left[\frac{K_R}{k h^2} - (N+1)\left(\frac{m\omega^2}{k}\right)\left(\frac{Y}{h}\right)^2\right]} \left\{ \frac{N \sin\beta(N+1) - (N+1)\sin\beta N}{\sin\beta N} - \frac{Y_0}{Y_N}\frac{(N\sin\beta - \sin\beta N)}{\sin\beta N} \right\} \quad (26)$$

V. Ground Compliance Ratio

The ground stiffness in translation and rotation is the force or moment per unit displacement, which can be obtained from Eqs. (1), (2) and (3). These quantities are complex numbers depending on the frequency parameter $\omega\, r_0\sqrt{\frac{\rho}{G}}$, and therefore are not constant. From Eqs. (1), (2) and (3), we can write

$$\frac{K_H}{k} = \frac{2\pi r_0 G}{k} \frac{1}{(f_{1H} + i f_{2H})} \quad (27)$$

$$\frac{K_R}{k h^2} = \frac{\pi r_0^3 G}{4 k h^2} \frac{1}{(f_{1R} + i f_{2R})} \quad (28)$$

$$K_V = \pi r_0 G \frac{1}{(f_{1V} + i f_{2V})} \quad (29)$$

It is at this point that we must consider acceptable design values of r_0, k, and h, depending on the number of stories N and the soil properties of bearing strength and elasticity.

The foundation area A is established from the total load mg(N + 1) and the allowable bearing pressure P_b

$$A = \frac{mg(N+1)}{P_b} \quad (30)$$

The fundamental period of actual buildings is very nearly proportional to the number of stories, and the equation $T_1 = 0.1\ N$ seems to be applicable to many buildings. From Eq. (23) the design value of k/m must then conform closely to the equation.

$$\frac{k}{m} = \frac{\pi^2}{T_1^2 \sin^2\left(\frac{\pi}{4N+2}\right)} \quad (31)$$

where for tall buildings $\sin^2\left(\frac{\pi}{4N+2}\right) \cong \frac{\pi^2}{(4N+2)^2}$, and $\frac{k}{m} \cong \left(\frac{4N+2}{0.1N}\right)^2$

We now write Eq. (28) with r_0G replaced from Eq. (29)

$$\frac{K_R}{k h^2} = \frac{A K_V (f_{1V} + i f_{2V})}{4\pi k h^2 (f_{1R} + i f_{2R})} \tag{32}$$

Substituting from Eqs. (30) and (31) the compliance ratio reduces to

$$\frac{K_R}{k h^2} = \frac{K_V}{A P_b} \frac{(f_{1V} + i f_{2V})}{(f_{1R} + i f_{2R})} \left(\frac{r^2}{4h^2}\right) \frac{g(N+1)\tau_1^2 \sin^2\left(\frac{\pi}{4N+2}\right)}{\pi^2} \tag{33}$$

We can compare this equation with the compliance ratio determined on a static basis which is

$$\frac{K_R}{k h^2} = \frac{P_V}{P_b} \left(\frac{\gamma}{h}\right)^2 \frac{g(N+1)\tau_1^2 \sin^2\left(\frac{\pi}{4N+2}\right)}{\pi^2} \tag{34}$$

where $r^2 = I/A$ = radius of gyration of the foundation area and P_V is the vertical pressure per unit deflection. This equation holds regardless of the shape of the foundation, the ratio $(\gamma^2/h)^2$ for a rectangular foundation of the shape shown in Fig. 5 being

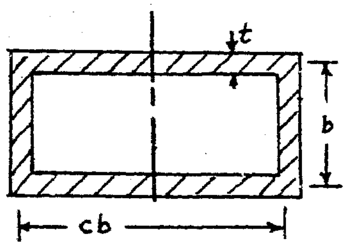

Fig. 5

$$\left(\frac{\gamma}{h}\right)^2 \cong \frac{b^2}{12h^2}\left(\frac{1+3c}{1+c}\right) \tag{35}$$

By comparison of Eqs. (33) and (34) it is evident that the term $\frac{K_V}{A P_b} \frac{(f_{1V} + i f_{2V})}{(f_{1R} + i f_{2R})}$ represents a sort of a dynamical value of P_V/P_b for any frequency parameter $\omega r_0 \sqrt{\frac{\rho}{G}}$. Although Eq. (33) is applicable only for the circular foundation area of radius r_0 which has a radius of gyration of $\gamma^2 = r_0^2/4$, intuition indicates that for the more general

* See Ref. (5).

shape, the dynamical compliance ratio must have the form

$$\frac{K_R}{k h^2} = F_R\left(\omega r_0 \sqrt{\frac{\rho}{G}}\right)\left(\frac{r}{h}\right)^2 \frac{g(N+1)\,\tau_1^2 \sin^2\left(\frac{\pi}{4N+2}\right)}{\pi^2} \tag{33a}$$

Housner has defined p_V to be the elastic rebound slope of the load settlement curve, and gives a value of 5.2×10^5 lb/ft^3 for medium clay. He also gives the allowable bearing strength of the same soil as 4000 lb/ft^2. For very poor soil p_V may be as low as 35,000 lb/ft^3 with bearing strength ranging near 500 lb/ft^2. Since p_V/P_b, or $F(\omega r_0 \sqrt{\frac{\rho}{G}})$ in the general case, is the ratio of two physical properties of the soil, and the frequency effect as indicated by $\left(\frac{f_{1V} + i f_{2V}}{f_{1R} + i f_{2R}}\right)$ is nearly a constant over the low frequency range, $F(\omega r_0 \sqrt{\frac{\rho}{G}})$ must be nearly a constant for any soil.

Lorenz (6) has shown that the stiffness property p_V of soil can be determined graphically from the frequency response curve of a vibration test using three different eccentric mass loads. The stiffness curve obtained is the equivalent static stiffness curve from which p_V can be computed.

Returning to the horizontal compliance ratio, the same procedure as before leads to the equation

$$\frac{K_H}{k} = \frac{K_V}{A P_b}\left(\frac{f_{1V} + i f_{2V}}{f_{1H} + i f_{2H}}\right) 2g \frac{(N+1)\,\tau_1^2 \sin^2\left(\frac{\pi}{4N+2}\right)}{\pi^2} \tag{36}$$

Since for small ω, f_{2V}, f_{2H} and f_{2R} are all small compared to f_{1V}, f_{1H} and f_{1R} we might write the two compliance ratios for large N as

$$\frac{K_H}{k} \cong \left(\frac{.845 K_V}{A P_b}\right) g (N+1) \left(\frac{0.1N}{4N+2}\right)^2 \tag{36a}$$

$$\frac{K_R}{k h^2} \cong \left(\frac{1.16 K_V}{A P_b}\right)\left(\frac{r_0}{h}\right)^2 g (N+1) \left(\frac{0.1N}{4N+2}\right)^2 \tag{33b}$$

These equations indicate that the compliance ratio in rotation is approximately $(r_0/h)^2$ times the compliance ratio in horizontal translation, and that knowing the soil properties these values are dependent on the number of stories N, and increasing with N. Thus for very tall buildings the ground appears stiffer and the rigid ground assumption, as pointed out by Housner, is justified.

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Conclusions and Recommendations:

(1) The equations of motion for the N-story building coupled with the ground are expressed by the three equations (13), (14) and (15).

(2) For N greater than one, the computational labor required is the same for any N.

(3) The shear and bending moment at the base of the building, given by Eqs. (20) and (21), should be compared to those for the rigid ground for the same ground motion.

(4) The compliance ratio in rotation for any shape of the foundation must have the form indicated by Eq. (33a), where $F_R(\omega r_0\sqrt{\frac{\rho}{G}})$ should be available by an experimental test.

(5) The compliance ratio in horizontal translation for any shape of the foundation must have a similar form to Eq. (36) with the first factor replaced by $F_H(\omega r_0\sqrt{\frac{\rho}{G}})$.

(6) For tall buildings the compliance ratios become large thereby approaching a rigid ground condition. For poor soil and low buildings the effect of the ground becomes important.

Appendix

A number of formulas for the finite sum of trigonometric terms were derived and checked against Reference 7. These are given as follows:

$$\sum_{n=1}^{N} \sin n\beta = \frac{\sin\frac{\beta N}{2}}{\sin\frac{\beta}{2}} \sin\frac{\beta(N+1)}{2} \tag{A1}$$

$$\sum_{n=1}^{N} \cos n\beta = \frac{\sin\frac{\beta N}{2}}{\sin\frac{\beta}{2}} \cos\frac{\beta(N+1)}{2} \tag{A2}$$

$$\sum_{n=1}^{N} n\sin n\beta = \frac{\sin\beta N}{4\sin^2\frac{\beta}{2}} - \frac{N}{2}\frac{\cos\beta(N+\frac{1}{2})}{\sin\frac{\beta}{2}} \tag{A3}^*$$

$$\sum_{n=1}^{N} n\cos n\beta = \frac{N}{2}\frac{\sin\beta(N+\frac{1}{2})}{\sin\frac{\beta}{2}} - \frac{\sin^2\frac{\beta N}{2}}{2\sin^2\frac{\beta}{2}} \tag{A4}$$

$$\sum_{n=1}^{N} \sin^2 n\beta = \frac{1}{2}\left\{N - \frac{\sin\beta N}{\sin\beta}\cos\beta(N+1)\right\} \tag{A5}$$

These formulas were used to express in closed form the summations appearing in the paper as follows:

(A)
$$\sum_{n=1}^{N} Y_n = y_0\sum_{n=1}^{N}\cos n\beta + B\sum_{n=1}^{N}\sin n\beta$$
$$= y_0\frac{\sin\frac{\beta N}{2}}{\sin\frac{\beta}{2}}\cos\frac{\beta(N+1)}{2} + B\frac{\sin\frac{\beta N}{2}}{\sin\frac{\beta}{2}}\sin\frac{\beta(N+1)}{2}$$

Eliminating B from Eq. (7) and simplifying

$$\sum_{n=1}^{N} Y_n = \frac{\sin\frac{\beta N}{2}}{\sin\frac{\beta}{2}\sin\beta N}\left\{y_0\sin\frac{\beta}{2}(N-1) + y_N\sin\frac{\beta}{2}(N+1)\right\}$$

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*Jolley's Eq. 374, P. 110 (Ref.7) is in error.

(B)
$$\sum_{n=1}^{N} nY_n = y_0 \sum_{n=1}^{N} n\cos n\beta + B\sum_{n=1}^{N} n\sin n\beta$$
$$= y_0\left\{\frac{N}{2}\frac{\sin\beta(N+\frac{1}{2})}{\sin\frac{\beta}{2}} - \frac{\sin^2\frac{\beta N}{2}}{2\sin^2\frac{\beta}{2}}\right\} - B\left\{\frac{N}{2}\frac{\cos\beta(N+\frac{1}{2})}{\sin\frac{\beta}{2}} - \frac{\sin\beta N}{4\sin^2\frac{\beta}{2}}\right\}$$

Again substituting for B from Eq. (7) and simplifying

$$\sum_{n=1}^{N} nY_n = \frac{1}{4\sin^2\frac{\beta}{2}\sin\beta N}\left\{y_0(N\sin\beta - \sin\beta N) + y_N\left[(N+1)\sin\beta N - N\sin\beta(N+1)\right]\right\}$$

(C) Eq. (24) requires the evaluation of $\sum_{n=1}^{N} Y_{ni}^2$. Substituting Y_{ni} from Eq. (6b) and using Eq. (A5) we obtain

$$\sum_{n=1}^{N} Y_n^2 = \frac{Y_{ni}^2}{2\sin^2\beta_i N}\left\{N - \frac{\sin\beta N}{\sin\beta}\cos\beta(N+1)\right\} = N.$$

The quantity $\sin\beta N\cos\beta(N+1)$ can be changed to

$$\sin\beta N\cos\beta(N+1) = \frac{1}{2}\left[\sin\beta(2N+1) - \sin\beta\right]$$

and since $\cos\beta_i(N+\frac{1}{2}) = 0$ from Eq. (13a), $\sin\beta_i(2N+1) = 0$
The result is the relationship

$$\frac{Y_{Ni}}{\sin\beta_i N} = \sqrt{\frac{2N}{N+\frac{1}{2}}}$$

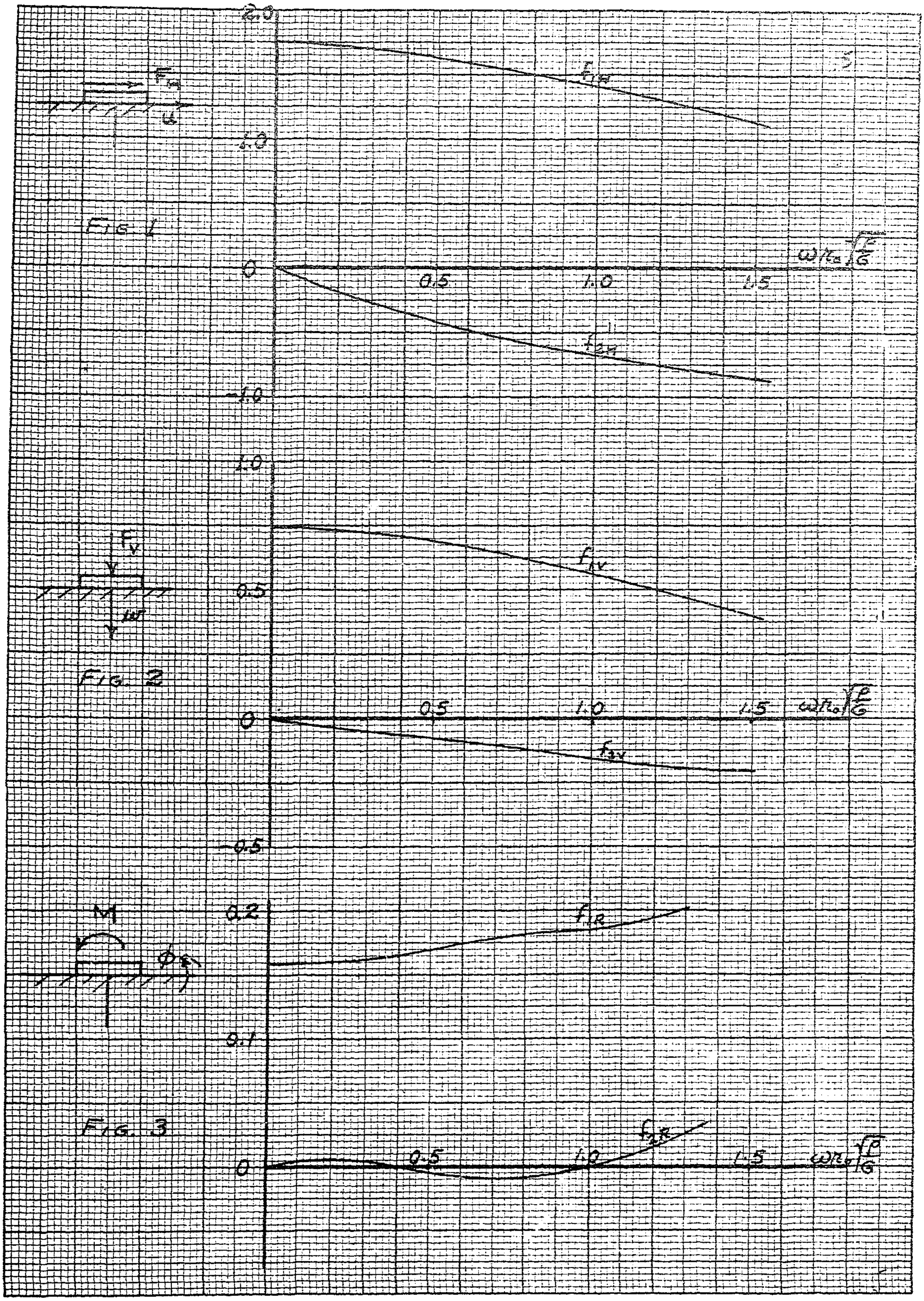
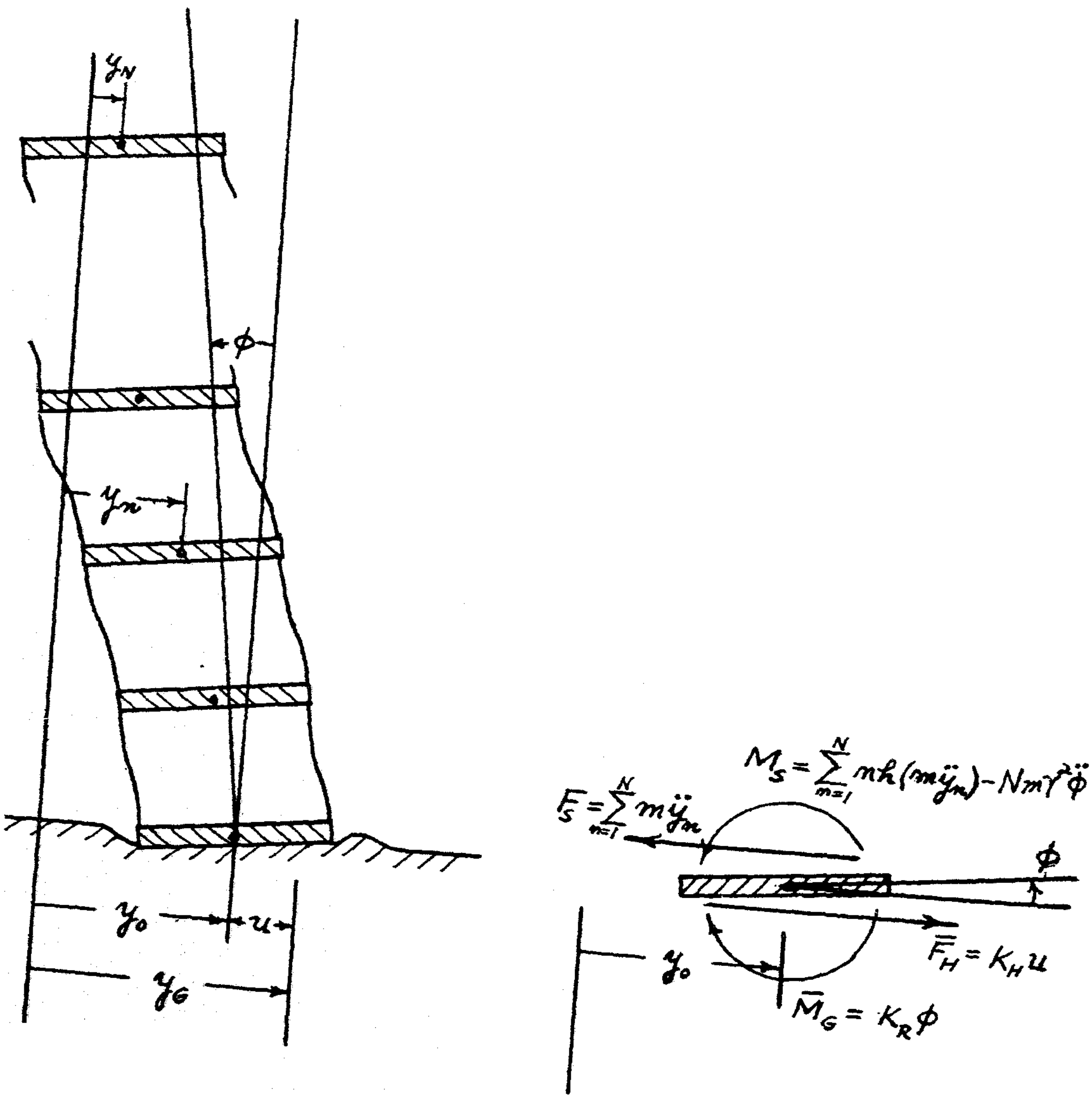


Fig. 1
2.0
1.0
0
-1.0
0.5
1.0
1.5
$\omega r_0\sqrt{\frac{\rho}{G}}$
F_H
u
f_{1H}
f_{2H}
Fig. 2
1.0
0.5
0
-0.5
F_V
w
f_{1V}
f_{2V}
Fig. 3
0.2
0.1
0
M
ϕ
f_{1R}
f_{2R}



y_N
ϕ
y_n
y_0
u
y_G
$M_S = \sum_{n=1}^{N} nh(m\ddot{y}_n) - Nm\gamma^2\ddot{\phi}$
$F_S = \sum_{n=1}^{N} m\ddot{y}_n$
ϕ
y_0
$\bar{F}_H = K_H u$
$\bar{M}_G = K_R \phi$

FIG. 4

DISCUSSION

L. S. Jacobsen, Stanford University, U. S. A.:

Professors Thomson, Sato and Yamaguchi have in reality written their paper together, and I want to compliment them on a most significant contribution to our knowledge of the ground's influence on building motion. Their discovery that the two spring factors are not constant, and Professor Thomson's lucid explanation of the phenomenon constitute, in my estimation, an extremely important step forward.

I think that it is very modest of Professor Thomson to describe his paper as being only a survey, it is more than that, it is in fact a significant, real contribution to our understanding of a hitherto obscure situation. Again I express my sincere compliments.

W. T. Thomson:

I wish to thank Professor Jacobsen for his generous compliments. There is much work that can be done in this field. After a more complete numerical study of the N-story problem as outlined in this paper, an attempt should be made towards the development of the dynamical stiffness of the ground for foundations conforming more closely to those of actual buildings. Due to the mathematical difficulties of the half space problem, the analytical expressions available today are limited to a circular disk resting on the ground surface. Dynamical stiffness of the ground for foundations of other shapes and particularly those extending into the ground is needed. When such expressions are available, an approach as given by this paper will enable a more realistic dynamical analysis of the combined ground-building system. Response of such systems to arbitrary and random inputs would normally follow.