A STATISTICAL METHOD OF DETERMINING THE MAXIMUM RESPONSE OF A BUILDING STRUCTURE DURING AN EARTHQUAKE

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Introduction

In recent years there have been many studies concerned with obtaining the response of buildings subjected to the ground motion of an actual earthquake by the use of digital or analog computers. It is well known that these results have given us more accurate information cocerning the dynamic behaviour of building structures during an earthquake. However, they seem still to be insufficient for obtaining a consistent knowledge of aseismic design because of the following obstacles inevitably encountered in the course of these studies: (1) the analysis methods are so complex and laborious that they cannot rapidly predict the influence of different earthquake motions as well as the different properties of vibration of a building, and (2) there is no reliable criterion of judgement, from which we can derive a design formula rationally from many response results, if they appear to be very random in character.

In order to overcome these difficulties, it is advisable to introduce a method of statistical treatment into the formulation of design criteria from detailed response results. It will be shown that such treatment facilitates analysis and even establishes a general theory of determination of earthquake load for design in an approximate but acceptable form for practical purposes. This paper is an attempt in this direction.

Previous approaches similar to the present analysis have been developed by Goodman, Rosenblueth and Newmark (Ref. 1), and again Rosenblueth (Ref. 2), making the substantial assumption that the pulse duration of an earthquake motion is much shorter than the natural period of the excited system. However, in Japan the wave duration is comparable to or larger than the building period, for there buildings have usually a fundamental period of less than 0.8 sec. It follows that we cannot ignore the "dominant period of ground motion". Indeed, Housner's velocity spectra (Ref. 3) also indicate a typical pattern characterized by an apparent transition from a linearly varying response curve for shorter periods to a constant response curve for longer periods. The period associated with the above transition will be taken as a reference period of ground motion, called here the "dominant period". For this reason, the dominant period of ground motion is used as an important factor in this work, although this period cannot always be associated with the locality.

Theory of transmission of random earthquake motions to a building

For aseismic design it is very important to determine how the vibration of a building is magnified compared with a random earthquake

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motion. It is easily recognized that such magnification is based mainly upon the frequency characteristics of the earthquake motion. A "power spectrum" serves well as a mathematical representation of the frequency characteristics. In general, the ground motion f(t) is analyzed by the Fourier integral as

$$F(i\omega)=\frac{1}{2\pi}\int_{-T}^{T} f(t)\, e^{-i\omega t} dt \quad , \tag{1}$$

where (T, -T) is the duration of that part of the earthquake motion which the accelerogram shows to have significant effects on the building. In the problem of random motion, we often take account of the mean square value of f(t) defined by

$$\overline{f^2}=\frac{1}{2T}\int_{-T}^{T} f^2(t)\, dt \quad . \tag{2}$$

However, $\overline{f^2}$ cannot be clearly defined for an earthquake motion, which is essentially nonstationary, because $\overline{f^2}$ depends on the time range (T, -T) impossible to be strictly determined. It is necessary, therefore, for the explicit definition of $\overline{f^2}$ that the earthquake motion should be so modified as to satisfy the condition of "stationary random motion", without significant loss of accuracy in the analysis. A rough modification will be made by means of adding a simulated series of motion continuously repeated to the principal motion before and after, having equal intensity and randomness to the principal motion at all points. This modification may be justified by the fact that in many cases undesirable parts of the response due to the simulated motion are damped and decay before the application of the principal motion, for a building has usually a damping ratio of 5 % or more to its critical damping. Thus the modified motion has a uniquely defined $\overline{f^2}$ in accordance with the definition of "stationary random motion". Then we can calculate $\overline{f^2}$ from $F(i\omega)$ as below, according to the Parseval theorem: if we put

$$G^2(\omega)=\lim_{T\to\infty}\frac{2\pi}{T}\left|F(i\omega)\right|^2 \quad , \tag{3}$$

we have

$$\overline{f^2}=\int_0^{\infty} G^2(\omega)\, d\omega \quad . \tag{4}$$

The function $G^2(\omega)$ is called the power spectral density and has the following property of giving the mean square of response of a linear system acted upon by the random motion f(t):

$$\overline{S^2}=\int_0^{\infty}\left|S(i\omega)\right|^2 G^2(\omega)\, d\omega \quad , \tag{5}$$

where $S(i\omega)$ is the ratio of response of the system to the applied sinusoidal motion $e^{i\omega t}$ of the frequency ω , often called the "frequency response of the system".

The foregoing discussion which has been confined to the mean behaviours of the motion is, however, not enough for our purpose of obtaining the maximum value of response. It needs, in addition, to take into account any statistical characteristics concerning the occurence of the maximum value in the time history of an earthquake motion. But they have not yet been determined from records actually observed, so let us assume the normal probability distribution, often used in the problem of random motion. According to this distribution rule, we can estimate the maximum value from the mean deviation under a definitely prescribed probability for occurence of the maximum value. Therefore, if we notice that the response of a linear system also satisfies the normal distribution rule so long as the ground motion does, we may be able to put the following relation between the maximum value f_{max} and mean value $\sqrt{\overline{f^2}}$ of the ground motion as well as s_{max} and $\sqrt{\overline{s^2}}$ of the vibratory system:

$$\lambda = \frac{s_{max}}{f_{max}} = \frac{\sqrt{\overline{s^2}}}{\sqrt{\overline{f^2}}}\,. \tag{6}$$

This ratio λ is the desired magnification and will be called the transmissibility of the system.

As regards the spectral density $G^2(\omega)$ of the ground motion of earthquake, Kanai (Ref. 4) has presented the opinion as a result of analyses of many past earthquake motions, that the spectrum observed at bedrock is characterized by a constant pattern, while that at ground surface is multiplied by the vibration property of the ground layer. This idea may be expressed by the equation in its simplest form:

$$G^2(\omega) = \frac{1 + 4h_g^2 \frac{\omega^2}{\nu_g^2}}{\left(1 - \frac{\omega^2}{\nu_g^2}\right)^2 + 4h_g^2 \frac{\omega^2}{\nu_g^2}} B\,, \qquad B = const. \tag{7}$$

Obviously, this equation posseses a "dominant frequency" ν_g for a unique peak of spectrum. h_g is a parameter, which indicates the sharpness of the peak. It is said that the dominant period T_g $(= 2\pi/\nu_g)$ depends on distance from epicenter, intensity of earthquake and especially rigidity of the ground layer, and usually lies in the range of approximately 0.2 sec to 1.0 sec. The correlation function $R(\tau)$ corresponding to Eq. 7 is written by

$$\begin{aligned} R(\tau) &= \int_0^\infty G^2(\omega)\cos\omega\tau\, d\omega \\ &= \frac{\pi}{4} B\left\{ \frac{\nu_g}{h_g}(1+4h_g^2)\, e^{-h_g\nu_g\tau}\cos(\nu_g\sqrt{1-h_g^2}\,\tau) \right. \\ &\qquad \left. + \frac{\nu_g}{\sqrt{1-h_g^2}}(1-4h_g^2)\, e^{-h_g\nu_g\tau}\sin(\nu_g\sqrt{1-h_g^2}\,\tau)\right\}. \end{aligned} \tag{8}$$

For simplicity, we will confine ourselves to Eq. 7 for the spectral density of acceleration of the ground motion in the following description. First, let us calculate the transmissibility for a single mass-spring system. This transmissiblity, when considered in more detail, may be better called the acceleration transmissibility, for the spectral density used here refers to acceleration, and the amplification thus obtained means the ratio between the two maximum accelerations. As the frequency response $S(i\omega)$ of a single mass-spring system having natural frequency ν_s and damping ratio h_s can be written by

$$S(i\omega) = \frac{1 + 2h_s \frac{i\omega}{\nu_s}}{1 - \frac{\omega^2}{\nu_s^2} + 2h_s \frac{i\omega}{\nu_s}}\,, \tag{9}$$

λ can be evaluated from the equation in accordance with Eq. 6:

$$\lambda^2 = \frac{\displaystyle\int_0^\infty \frac{1+4h_s^2\frac{\omega^2}{\nu_s^2}}{\left(1-\frac{\omega^2}{\nu_s^2}\right)^2+4h_s^2\frac{\omega^2}{\nu_s^2}} \; \frac{1+4h_g^2\frac{\omega^2}{\nu_g^2}}{\left(1-\frac{\omega^2}{\nu_g^2}\right)^2+4h_g^2\frac{\omega^2}{\nu_g^2}} \, B\,d\omega}{\displaystyle\int_0^\infty \frac{1+4h_g^2\frac{\omega^2}{\nu_g^2}}{\left(1-\frac{\omega^2}{\nu_g^2}\right)^2+4h_g^2\frac{\omega^2}{\nu_g^2}} \, B\,d\omega} . \tag{10}$$

The integration of this equation can be carried out by means of the contour-integral:

$$\left.\begin{aligned}
\lambda^2 &= \frac{1}{P_3}\left(P_1 + \frac{\alpha_1\beta_1}{\alpha_2\beta_2}\frac{\nu_s}{\nu_g}P_2\right) ,\\
P_1 &= \frac{A_1(\alpha_1\gamma_1-\beta_1\delta_1)+B_1(\alpha_1\delta_1+\beta_1\gamma_1)}{\gamma_1^2+\delta_1^2} ,\\
P_2 &= \frac{A_2(\alpha_2\gamma_2-\beta_2\delta_2)+B_2(\alpha_2\delta_2+\beta_2\gamma_2)}{\gamma_2^2+\delta_2^2} ,\\
P_3 &= (1+4h_g^2)\sqrt{1-h_g^2} ,\\
\alpha_1 &= \sqrt{1-h_g^2}, \quad \beta_1 = h_g, \quad \alpha_2 = \sqrt{1-h_s^2}, \quad \beta_2 = h_s ,\\
\gamma_1 &= \left\{1-\frac{\nu_g^2}{\nu_s^2}(1-2h_g^2)\right\}^2 - 4\frac{\nu_g^4}{\nu_s^4}h_g^2(1-h_g^2) + 4h_s^2\frac{\nu_g^2}{\nu_s^2}(1-2h_g^2) ,\\
\delta_1 &= -4\left\{1-\frac{\nu_g^2}{\nu_s^2}(1-2h_g^2)\right\}\frac{\nu_g^2}{\nu_s^2}h_g\sqrt{1-h_g^2} - 8h_s^2\frac{\nu_g^2}{\nu_s^2}h_g\sqrt{1-h_g^2} ,\\
A_1 &= 1 + \left(4h_s^2\frac{\nu_g^2}{\nu_s^2}+4h_g^2\right)(1-2h_g^2) + 4h_s^2h_g^2\frac{\nu_g^2}{\nu_s^2}\left\{(1-2h_g^2)^2-4h_g^2(1-h_g^2)\right\} ,\\
B_1 &= \left(4h_s^2\frac{\nu_g^2}{\nu_s^2}+4h_g^2\right)2h_g\sqrt{1-h_g^2} + 16\,h_s^2h_g^2\frac{\nu_g^2}{\nu_s^2}(1-2h_g^2)h_g\sqrt{1-h_g^2} .
\end{aligned}\right\} \tag{11}$$

where γ_2, δ_2, A_2, B_2 can be written by the exchange of the subscripts g and s in γ_1, δ_1, A_1, B_1, respectively. Thus we can evaluate λ as a function of the frequency ratio ν_g/ν_s or the period ratio T_s/T_g, in which $T_s = 2\pi/\nu_s$, if h_g and h_s are given. In Ref. 5, the numerical results for various combinations of these parameters are shown, though they are omitted here because of space limitations.

It is instructive to examine how the results thus obtained agree with the exactly calculated response results. For this comparison, the analyses by Housner and joint authors (Ref. 3) are avaiable, which indicate a number of plots of maximum acceleration experienced by a single mass-spring system damped or undamped against the variation of its natural period, when it is subjected to past actual earthquakes. Fortunately, the curve called there the "acceleration spectrum of response", corresponds to our curve of the transmissibility

λ versus the period ratio T_s/T_g. For convenience of inspection, let us rewrite the coordinate scales of Housner's spectrum in our dimensionless form. While the transmissibility is easily produced since the maximum acceleration of each earthquake is known, the period ratio is difficult since the dominant period is unknown. Then, the only way to estimate it, is to make the reasonable assumption that the response spectrum should have a peak at $T_g = T_S$, (Fig. 1). For this purpose, the spectrum for the damped system may be taken as appropriate rather than that for the undamped system, because in the latter case the peak can occur at the "quasi-resonance" with weaker but longer continued ground motion as well.

Thus we can compare the results obtained here with the exactly calculated results shown in Fig. 2, where by our analysis the spectral density of the ground motion is taken as Eq. 7 with a definite value $h_g = 0.3$, and by the exact results the average values of the total of the available data are plotted at the isolated period ratios of 0.5, 1.0, 2.0 and 3.0, (see Fig. 1). In Fig. 2 we find that there is reasonable agreement between the results. As a result of the examination, we can derive the following conclusion: the agreement in the effects of damping of the system proves an assurance of the assumption of stationary random motion, and also the agreement in the shape of curves shows that Eq. 7 substituted by $h_g = 0.3$ is applicable as a standard spectral density, to which we will restrict ourselves in the following description.

While Eq. 11 evaluates the acceleration transmissibility, we can obtain similarly the equation determining the velocity transmissibility λ_v which means the ratio of the maximum velocity of motion of system to that of the ground motion. In a similar manner the displacement transmissibility λ_d which implys the ratio of the maximum (absolute) displacement between them can be obtained. When the spectral density of acceleration at the ground is given as $G^2(\omega)$, the spectral density of velocity becomes $\omega^{-2}G^2(\omega)$; the density of displacement becomes $\omega^{-4}G^2(\omega)$. Hence, each transmissibility is expressed by the form:

$$\left.\begin{aligned} \lambda_v^2 &= \frac{\int_0^\infty |S(i\omega)|^2 \frac{G^2(\omega)}{\omega^2}\, d\omega}{\int_0^\infty \frac{G^2(\omega)}{\omega^2}\, d\omega}\,, \\[2ex] \lambda_d^2 &= \frac{\int_0^\infty |S(i\omega)|^2 \frac{G^2(\omega)}{\omega^4}\, d\omega}{\int_0^\infty \frac{G^2(\omega)}{\omega^4}\, d\omega}\,. \end{aligned}\right\} \quad (12)$$

Applying Eq. 11, these integrations are obtained approximately as follows:

$$\left.\begin{aligned}\lambda_v^2 &= \frac{1}{P_3}\left(P_1+\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\frac{\nu_g}{\nu_s}P_2\right),\\ \lambda_d^2 &= \frac{1}{P_3}\left\{P_1+\frac{\alpha_1\beta_1}{\alpha_2\beta_2}\left(\frac{\nu_g}{\nu_s}\right)^3 P_2\right\}.\end{aligned}\right\} \quad (13)$$

The numerical results of the above equations inserted by $h_g = 0.3$ and $h_S = 0.1$ are graphically illustrated in Fig. 3, where we see that the velocity transmissibility becomes almost constant in the range of $T_S > T_g$. This feature agrees with the Housner's velocity spectrum, though the author's is refered to the absolute velocity.

It has been shown in the preceding description that the presented method gives us results of considerable accuracy for the earthquake response problem, in spite of its simpler method. Accordingly, it is natural to proceed to the extension of this method to any system having many degrees of freedom. As an example, let us take a n-storied mass-spring system. Then, the equation of translational vibration of the system forced by the ground motion $a_o e^{i\omega t}$ can be written as

$$m_j \ddot{u}_j + \sum_{\ell=1}^{n}\left(c_{j\ell}\dot{u}_\ell + k_{j\ell}u_\ell\right) = m_j a_o \omega^2 e^{i\omega t}, \quad (j=1,2,\cdots n) \quad (14)$$

where the superscript dots indicate differentiations with respect to time, and

m_j = mass concentrated at the j-th floor level, numbered from the lowest floor,

u_j = translational displacement at the j-th floor relative to the ground,

$k_{j\ell}$ = spring constant defined by force to be applied at the j-th floor when a unit displacement is imposed at the ℓ-th floor alone,

$c_{j\ell}$ = damping constant associated with the above spring constant, which includes both internal and external damping.

It is well known that these simultaneous equations can be solved approximately by the following procedures: at first, we determine the normal function φ_p of the component φ_{jp} under the assumption of undamped system, and next, evaluate the generalized mass m_p, generalized spring constant k_p and the modal damping constant c_p by use of the expressions:

$$\left.\begin{aligned} m_p &= \sum_{j=1}^{n} m_j \varphi_{jp}^2, \quad k_p = \sum_{j=1}^{n}\sum_{\ell=1}^{n} k_{j\ell}\varphi_{jp}\varphi_{\ell p},\\ c_p &= \sum_{j=1}^{n}\sum_{\ell=1}^{n} c_{j\ell}\varphi_{jp}\varphi_{\ell p}, \quad (p=1,2,\cdots,n).\end{aligned}\right\} \quad (15)$$

Then we get the solution written in the form of the frequency response $S_j(i\omega)$ subject to the j-th mass:

$$S_j(i\omega) = \sum_{p=1}^{n} \Phi_{jp} \frac{1+2h_p\dfrac{i\omega}{\nu_p}}{1-\dfrac{\omega^2}{\nu_p^2}+2h_p\dfrac{i\omega}{\nu_p}} \quad (16)$$

where

$$h_p = \frac{c_p}{2 m_p \nu_p} \ , \qquad \nu_p^2 = \frac{k_p}{m_p} \tag{17}$$

and

$$\Phi_{jp} = \frac{\sum_{j=1}^{n} m_j \varphi_{jp}}{m_p} \varphi_{jp} \ , \tag{18}$$

which may be called the "excited normal function" for convenience. The square of absolute of S (i ω) becomes

$$S_j(i\omega)^2 = \sum_{p=1}^{n} \Phi_{jp}^2 Q_{pp}^2(\omega) + \sum_{\substack{p \\ (p\neq q)}}^{n} \sum_{q}^{n} \Phi_{jp} \Phi_{jq} Q_{pq}^2(\omega) \tag{19}$$

where

$$Q_{pp}^2(\omega) = \frac{1 + 4 h_p \frac{\omega^2}{\nu_p^2}}{\left(1 - \frac{\omega^2}{\nu_p^2}\right) + 4 h_p^2 \frac{\omega^2}{\nu_p^2}} \ ,$$

$$Q_{pq}^2(\omega) = \frac{\left(1 - \frac{\omega^2}{\nu_p^2} + 4 h_p^2 \frac{\omega^2}{\nu_p^2}\right)\left(1 - \frac{\omega^2}{\nu_q^2} + 4 h_q^2 \frac{\omega^2}{\nu_q^2}\right) + 4 h_p h_q \frac{\omega^6}{\nu_p^3 \nu_q^3}}{\left\{\left(1 - \frac{\omega^2}{\nu_p^2}\right)^2 + 4 h_p^2 \frac{\omega^2}{\nu_p^2}\right\}\left\{\left(1 - \frac{\omega^2}{\nu_q^2}\right)^2 + 4 h_q \frac{\omega^2}{\nu_q^2}\right\}} \tag{20}$$

In accordance with Eq. 10 for a single mass-spring system, the acceleration transmissibility λ_j at the j-th mass can be written as

$$\lambda_j^2 = \sum_{p=1}^{n} \sum_{q=1}^{n} \Phi_{jp} \Phi_{jq} \lambda_{pq}^2 \ , \tag{21}$$

where

$$\lambda_{pp}^2 = \frac{\int_0^\infty Q_{pp}^2(\omega) G^2(\omega)\, d\omega}{\int_0^\infty G^2(\omega)\, d\omega} \ , \qquad \lambda_{pq}^2 = \frac{\int_0^\infty Q_{pq}^2(\omega) G^2(\omega)\, d\omega}{\int_0^\infty G^2(\omega)\, d\omega} \ . \tag{22-a, -b}$$

In this expression, λ_{pp} is identical with the transmissibility for a single mass-spring system whose natural frequency is ν_p and damping ratio is h_p, and may be called the auto-transmissibility. λ_{pq} may be called the cross-transmissibility and is related to the phase difference between the p-th mode and q-th mode in a statistical sense. The reason is that, if the phase difference is denoted by Ω , the resultant transmissibility of both modes can be obtained from the expressions, as easily understood from vector diagram:

$$\lambda_j^2 = \Phi_{jp}^2 \lambda_{pp}^2 + \Phi_{jq}^2 \lambda_{qq}^2 + 2 \Phi_{jp} \Phi_{jq} \lambda_{pp} \lambda_{qq} \cos \Omega \ .$$

According to Eq. 21, the same resultant can be written in another form:

$$\lambda_j^2 = \Phi_{jp}^2 \lambda_{pp}^2 + \Phi_{jq}^2 \lambda_{qq}^2 + 2 \Phi_{jp} \Phi_{jq} \lambda_{pq}^2 \ ,$$

so that the cross-transmissibility must be

$$\lambda_{pq}^2 = \lambda_{pp} \lambda_{qq} \cos \Omega \ . \tag{23}$$

The cross-transmissibility plays a great role in a response problem of a system having many degrees of freedom, because it is true that the maximum response of any mode does not occur simultaneously with other modes, as pointed out by Clough (Ref. 6). As the integration results of Eq. 22-b becomes very lengthy, we will now write in brief, with the subscripts 1, 2, 3 in the places of p, q, g, respectively:

$$\lambda_{12}^2 = \frac{1}{(1+4h_3^2)\sqrt{1-h_3^2}}\left(\frac{b_3}{b_1}\frac{\nu_1}{\nu_3}K_1 + \frac{b_3}{b_2}\frac{\nu_2}{\nu_3}K_2 + K_3\right), \tag{24}$$

where b_1, b_2, b_3, K_1, K_2 and K_3 are given in the Appendix. The numerical results of Eq. 24 are not given here because of space limitations, but are contained in Ref. 5. Only a typical result is graphically demonstrated for the case of $h_p = h_q = 0.1$ in Fig. 4, where λ_{pq}^2 is plotted as a function of T_p/T_g for various values of parameter T_p/T_q. Though the curve should have two peaks in the neibourhoods of $T_p/T_g = 1$ and $T_q/T_g = 1$, we cannot here clearly recognize the peak at the latter, except in the case of $T_p/T_q = 1.2$. Across the peak the curve tends to zero. This means that both modes become perfectly out of phase with each other at the limit. Then we obtain the resultant transmissibility,

$$\lambda_j^2 = \sum_{p=1}^{n} \Phi_{jp}^2 \lambda_{pp}^2 , \tag{25}$$

which corresponds to the conclusion of Ref. 1.

While in the preceding description the cross-transmissibility was given for acceleration, it can be also obtained for velocity as well as displacement in the same way that we obtained Eq. 13 in the case of auto-transmissibility:

$$\lambda_{12}^2 = \frac{1}{(1+4h_3^2)\sqrt{1-h_3^2}}\left(\frac{b_3}{b_1}\frac{\nu_3}{\nu_1}K_1 + \frac{b_3}{b_2}\frac{\nu_3}{\nu_2}K_2 + K_3\right) \tag{26}$$

for the cross-transmissibility of (absolute) velocity, and

$$\lambda_{12}^2 = \frac{1}{(1+4h_3^2)\sqrt{1-h_3^2}}\left(\frac{b_3}{b_1}\frac{\nu_3^3}{\nu_1^3}K_1 + \frac{b_3}{b_2}\frac{\nu_3^3}{\nu_2^3}K_2 + K_3\right) \tag{27}$$

for the cross-transmissibility of (absolute) displacement.

It is apparent that the resultant transmissibility at any mass in a system has necessarily some phase difference against that at other masses; in other words, the individual transmissibility obtained from Eq. 21 does not occur at the same time. Therefore, it is more direct for practical design to evaluate the two values of transmissibility ${}_s\lambda_j$ of story shear and ${}_m\lambda_j$ of story moment. As for the transmissibility of story shear, if we define the excited function of story shear by

$$\chi_{jp} = \frac{\sum_{k=j}^{n} m_k \Phi_{kp}}{\sum_{k=j}^{n} m_k} \tag{28}$$

the result is given for the j-th story as

$$ {}_s\lambda_j^2 = \sum_{p=1}^{n}\sum_{q=1}^{n} \chi_{jp}\chi_{jq}\lambda_{pq}^2 \tag{29} $$

The transmissibility of story moment can be obtained in a similar manner.

Example (1)

As an illustration of how the presented method is applied, let us consider a ten mass-system, the physical properties of which are characterized by linearly varying distribution of masses as well as spring constants, as shown in Fig. 5 by the dimensionless ratios. This system is identical with that used by Tung and Newmark (Ref. 7) so that it can be compared with the exactly calculated results.

The excited normal function Φ_p of the system can be obtained as drawn in Fig. 5, where we see only the first five modes. As the remaining higher modes have less effect, they are summed algebraically and used as a 6-th mode in an approximate meaning. Let us assume all the damping ratios of the modes to be equal to 0.1. It is probable that the damping ratio of the higher mode is larger than that of the lower modes, but the influence of this difference on the response appears small. The numerical results calculated on the basis of the above quantities are graphically illustrated in Fig. 6, for the distribution of transmissibility ${}_a\lambda_j$ of acceleration as well as ${}_s\lambda_j$ of story shear with the variation of the parameter T_1/T_g. The corresponding result of Ref. 7 is limited to a specified case that T_1 is 1.32 sec and T_g is approximately 0.22 sec (Subway terminal, Los Angeles, N 39 E, Oct. 2, 1933). It follows that T_1/T_g is 6.0 and then λ_{11} for the 1-st mode is calculated to be 0.405 for h_1 = 0.1. As a final result, a plot of story shear is demonstrated in Fig. 7 together with the exact result of Ref. 7. In the examination of it, we see a rough agreement between both results, a few deviation in which are probably caused from the existence of another lower peak near 0.5 sec besides the peak at 0.22 sec in the concerned ground motion.

Example (2)

Let us consider the problem of a system, in which an earthquake induces a coupling motion of lateral translation and torsional rotation. This will be expected in a building having its center of rigidity away from the center of gravity at any story. For simplicity, we will deal with an idealized model of uniform shear-beam structure characterized by a straight axis of the center of rigidity (elastic axis) as well as a straight axis of the center of gravity, parallel and apart by the distance s. Let x, y, z be rectangular coordinates with the origin placed at the center of gravity in the base floor, the x-axis taken through the origin in the direction of the applied motion, the y-axis perpendicular to the x-axis, and the z-axis coincident with the axis of center of gravity, as shown in Fig. 8. For this configuration, we can write the following vibrational equations, when the damping is ignored:

$$\left.\begin{aligned} m\frac{\partial^2 u}{\partial t^2} &= G\,\frac{\partial^2(u-s\theta)}{\partial z^2}\ , \\ I\,\frac{\partial^2\theta}{\partial t^2} &= J\,\frac{\partial^2\theta}{\partial z^2} - Gs\,\frac{\partial^2(u-s\theta)}{\partial z^2}\ , \end{aligned}\right\} \quad (30)$$

where u = displacement in the x-direction at a point on the z-axis,
θ = twist angle around the elastic axis,
m = mass per unit height,
G = shearing rigidity per unit height,
I = moment of inertia around the z-axis per unit height,
J = torsional rigidity about the elastic axis per unit height.

For stationary vibration, these simultaneous equations can be solved by familiar expressions:

(1) The natural frequencies $\nu_n^{(1)}$ and $\nu_n^{(2)}$, where the subscript n refers to the degree of mode exhibited along the height and the superscripts (1) and (2) refer to the 1st and 2nd mode of coupling rotation, respectively, are

$$\nu_n^{(1)} = \frac{2n-1}{2}\,\frac{\pi}{\alpha_1 H}\sqrt{\frac{G}{m}}\ , \qquad \nu_n^{(2)} = \frac{2n-1}{2}\,\frac{\pi}{\alpha_2 H}\sqrt{\frac{G}{m}}\ , \qquad (31)$$

$$\alpha_{\frac{1}{2}}^2 = \frac{1}{2}\left(1+\frac{s^2}{e_o^2}+\frac{i_o^2}{e_o^2}\right) \pm \sqrt{\frac{1}{4}\left(1+\frac{s^2}{e_o^2}+\frac{i_o^2}{e_o^2}\right) - \frac{i_o^2}{e_o^2}}\ , \qquad (32)$$

where H = height of the system,
i_o = mass radius of gyration, $\sqrt{I/m}$,
e_o = elastic radius, $\sqrt{J/G}$;

(2) The distances ρ_1 and ρ_2 between the center of gravity and the two instantaneous centers of rotation are

$$\rho_{\frac{1}{2}} = \frac{\alpha_{\frac{1}{2}}^2}{\alpha_{\frac{1}{2}}^2 - 1}\, s\ , \qquad (33)$$

where the subscripts 1 and 2 correspond to the modes;

(3) The excited normal functions $\Phi_p^{(1)}(x,y)$ and $\Phi_p^{(2)}(x,y)$ are

$$\left.\begin{aligned} \Phi_p^{(1)}(x,y) &= \frac{4}{\pi}\,\frac{\rho_1 - y}{\rho_1-\rho_2}\,\frac{1}{2p-1}\,\sin\left(\frac{2p-1}{2}\,\frac{\pi x}{H}\right), \\ \Phi_p^{(2)}(x,y) &= \frac{4}{\pi}\,\frac{y-\rho_2}{\rho_1-\rho_2}\,\frac{1}{2p-1}\,\sin\left(\frac{2p-1}{2}\,\frac{\pi x}{H}\right). \end{aligned}\right\} \quad (34)$$

It is particularly informative in the present problem to examine the relation between ρ_1 , s and e_o^2/s, the last of which means the distance ρ_s between the center of gravity and the instantaneous center of rotation by the statical application of lateral force uniformly on the system. This relation is obtained from Eqs. 32 and 33, and is indicated graphically in Fig. 9 by the forms of dimensionless ratios ρ_1/i_o, s/i_o and $i_o s/e_o^2$; in addition, there are drawn the curves of period ratio $T_n^{(1)}/T_n^{(2)} = \nu_n^{(2)}/\nu_n^{(1)}$, which is equal to α_1/α_2 and hence relates to ρ_1/ρ_2. In this figure, we note that, when i_o/ρ_s keeps constant, ρ_1/i_o becomes smaller than ρ_s/i_o as s/i_o decreases. This tendency can be understood from the following. If i_o/ρ_s is unchanged, as the eccentricity s/i_o reduces so the length of e_o becomes smaller, that is, there is a greater concentration of highly rigid structural members at the center of the building. Then we can say that the mass radius of gyration, i_o, plays a determinative part in making ρ_1 smaller, while it is independent of the determination of ρ_s. As a result, it can be concluded that the building characterized by such concentration of rigidity distribution must be subjected to a more severe dynamic twist than that predicted in statics. However, there is another cause opposing the above conclusion. We find in the same figure that the smaller amounts of eccentricity produce a closer approach between the two natural periods $T_n^{(1)}$ and $T_n^{(2)}$. This tendency will have the result that the motion hardly occurs in the state of explicit resonance and hence the response decreases. These opposing results can only be resolved by the response computations.

For simplicity, let us consider only the two modes half-sinusoidal in shape along the height. We assume the damping ratios $h_1^{(1)} = 0.1$ and $h_1^{(2)} = 0.2$, and deal with the four combinations of $T_1^{(1)}/T_1^{(2)} = 1.2, 1.5$ and $\rho_1/i_o = 2.0, 4.0$. The evaluation is carried out for distribution of transmissibility of acceleration on the y-axis. The results are demonstrated in Fig. 10, where the transmissibility is plotted as a ratio C to the corresponding transmissibility for a non-eccentric system with the fundamental period $T_1^{(0)}$ so that the results are valid independently of the level of height. It is obvious that the ratio unity always holds at the instantaneous center of the 2nd mode where only the 1st mode appears. As a reference, the dotted line is added, according to the expression:

$$C_s = 1 + \frac{\rho_2 - y}{\rho_s} , \tag{35}$$

which is equivalent to the statical displacement due to uniform load. By rough examination, we can see that the distribution of Eq. 35 gives a rough agreement with the response curve, independently of $T_1^{(1)}/T_2$. This result means that the above described two contrary causes cancel each other out.

Appendix

Eq. 24 is written in detail as follows. If we use the symbols with the subscript 1 :

$$a_1 = 1 - 2h_1^2 , \quad b_1 = 2h_1\sqrt{1-h_1^2} , \quad m_1 = 1 - 4h_1^2 , \quad n_1 = \sqrt{1-h_1^2} ,$$

$$p_1 = 1 - 8h_1^2 + 8h_1^4 , \quad q_1 = 4(1-2h_1^2)h_1\sqrt{1-h_1^2} ,$$

$$r_1 = a_1^3 - 3a_1b_1^2 , \qquad s_1 = 3a_1^2 b_1 - b_1^3 ,$$

and the similar symols with the subscripts 2 and 3, we have

$$B_{11} = 1 - 2a_1a_2\left(\frac{\nu_1}{\nu_2}\right)^2 + p_1\left(\frac{\nu_1}{\nu_2}\right)^4, \qquad B_{12} = 2b_1\left\{a_1\left(\frac{\nu_1}{\nu_2}\right)^2 - a_2\right\}\left(\frac{\nu_1}{\nu_2}\right)^2,$$

$$C_{11} = 1 - 2a_1a_3\left(\frac{\nu_1}{\nu_3}\right)^2 + p_1\left(\frac{\nu_1}{\nu_3}\right)^4, \qquad C_{12} = 2b_1\left\{a_1\left(\frac{\nu_1}{\nu_3}\right)^2 - a_3\right\}\left(\frac{\nu_1}{\nu_3}\right)^2,$$

$$E_{11} = 1 - \left\{m_1 + m_2\left(\frac{\nu_1}{\nu_2}\right)^2\right\}a_1 + m_1m_2p_1\left(\frac{\nu_1}{\nu_2}\right)^2 + 4h_1h_2r_1\left(\frac{\nu_1}{\nu_2}\right)^3,$$

$$E_{12} = -\left\{m_1 + m_2\left(\frac{\nu_1}{\nu_2}\right)^2\right\}b_1 + m_1m_2q_1\left(\frac{\nu_1}{\nu_2}\right)^2 + 4h_1h_2s_1\left(\frac{\nu_1}{\nu_2}\right)^3,$$

$$F_{11} = 1 + 4a_1h_3^2\left(\frac{\nu_1}{\nu_3}\right)^2, \qquad F_{12} = 4b_1h_3^2\left(\frac{\nu_1}{\nu_3}\right)^2,$$

$$G_1 = B_{11}C_{11} - B_{12}C_{12}, \qquad H_1 = B_{12}C_{11} + B_{11}C_{12},$$

$$I_1 = E_{11}F_{11} - E_{12}F_{12}, \qquad J_1 = E_{12}F_{11} + E_{11}F_{12},$$

$$K_1 = \frac{I_1(G_1n_1 - H_1h_1) + J_1(H_1n_1 + G_1h_1)}{G_1^2 + H_1^2},$$

$$A_{21} = 1 - 2a_2a_1\left(\frac{\nu_2}{\nu_1}\right)^2 + p_2\left(\frac{\nu_2}{\nu_1}\right)^4, \qquad A_{22} = 2b_2\left\{a_2\left(\frac{\nu_2}{\nu_1}\right)^2 - a_1\right\}\left(\frac{\nu_2}{\nu_1}\right)^2,$$

$$C_{21} = 1 - 2a_2a_3\left(\frac{\nu_2}{\nu_3}\right)^2 + p_2\left(\frac{\nu_2}{\nu_3}\right)^4, \qquad C_{22} = 2b_2\left\{a_2\left(\frac{\nu_2}{\nu_3}\right)^2 - a_3\right\}\left(\frac{\nu_2}{\nu_3}\right)^2,$$

$$E_{21} = 1 - \left\{m_1\left(\frac{\nu_2}{\nu_1}\right)^2 + m_2\right\}a_2 + m_1m_2p_2\left(\frac{\nu_2}{\nu_1}\right)^2 + 4h_1h_2r_2\left(\frac{\nu_2}{\nu_1}\right)^3,$$

$$E_{22} = -\left\{m_1\left(\frac{\nu_2}{\nu_1}\right)^2 + m_2\right\}b_2 + m_1m_2q_2\left(\frac{\nu_2}{\nu_1}\right)^2 + 4h_1h_2s_2\left(\frac{\nu_2}{\nu_1}\right)^3,$$

$$F_{21} = 1 + 4a_2h_3^2\left(\frac{\nu_2}{\nu_3}\right)^2, \qquad F_{22} = 4b_2h_3^2\left(\frac{\nu_2}{\nu_3}\right)^2.$$

$$G_2 = A_{21}C_{21} - A_{22}C_{22}, \qquad H_2 = A_{22}C_{21} + A_{21}C_{22},$$

$$I_2 = E_{21}F_{21} - E_{22}F_{22}, \qquad J_2 = E_{22}F_{21} + E_{21}F_{22},$$

$$K_2 = \frac{I_2(G_2n_2 - H_2h_2) + J_2(H_2n_2 + G_2h_2)}{G_2^2 + H_2^2},$$

$$A_{31} = 1 - 2a_3a_1\left(\frac{\nu_3}{\nu_1}\right)^2 + p_3\left(\frac{\nu_3}{\nu_1}\right)^4, \qquad A_{32} = 2b_3\left\{a_3\left(\frac{\nu_3}{\nu_1}\right)^2 - a_1\right\}\left(\frac{\nu_3}{\nu_1}\right)^2,$$

$$B_{31} = 1 - 2a_3a_2\left(\frac{\nu_3}{\nu_2}\right)^2 + p_3\left(\frac{\nu_3}{\nu_2}\right)^4, \qquad B_{32} = 2b_3\left\{a_3\left(\frac{\nu_3}{\nu_2}\right)^2 - a_2\right\}\left(\frac{\nu_3}{\nu_2}\right)^2,$$

$$E_{31} = 1 - \left\{m_1\left(\frac{\nu_3}{\nu_1}\right)^2 + m_2\left(\frac{\nu_3}{\nu_2}\right)^2\right\}a_3 + m_1m_2p_3\left(\frac{\nu_3}{\nu_1}\right)^2\left(\frac{\nu_3}{\nu_2}\right)^2 + 4h_1h_2r_3\left(\frac{\nu_3}{\nu_1}\right)^3\left(\frac{\nu_3}{\nu_2}\right)^3,$$

$$E_{32} = -\left\{m_1\left(\frac{\nu_3}{\nu_1}\right)^2 + m_2\left(\frac{\nu_3}{\nu_2}\right)^2\right\}b_3 + m_1m_2q_3\left(\frac{\nu_3}{\nu_1}\right)^2\left(\frac{\nu_3}{\nu_2}\right)^2 + 4h_1h_2s_3\left(\frac{\nu_3}{\nu_1}\right)^3\left(\frac{\nu_3}{\nu_2}\right)^3,$$

$$G_3 = A_{31} B_{31} - A_{32} B_{32} , \qquad H_3 = A_{32} B_{31} - A_{31} B_{32} ,$$

$$I_3 = 4h_3^2 (1-2h_3^2) E_{31} - 8h_3^3 \sqrt{1-h_3^2}\, E_{32} ,$$

$$J_3 = 4h_3^2 (1-2h_3^2) E_{32} + 8h_3^3 \sqrt{1-h_3^2}\, E_{31} ,$$

$$K_3 = \frac{I_3 (G_3 n_3 - H_3 h_3) + J_3 (G_3 h_3 + H_3 n_3)}{G_3^2 + H_3^2} ,$$

$$\lambda_{12}^2 = \frac{1}{(1+4h_3^2)\sqrt{1-h_3^2}} \left(\frac{b_3}{b_1} \frac{\nu_1}{\nu_3} K_1 + \frac{b_3}{b_2} \frac{\nu_2}{\nu_3} K_2 + K_3 \right) .$$

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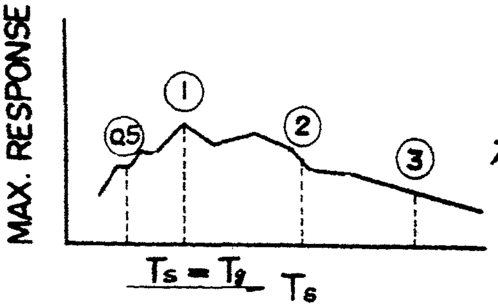

Fig. 1. Acceleration spectrum

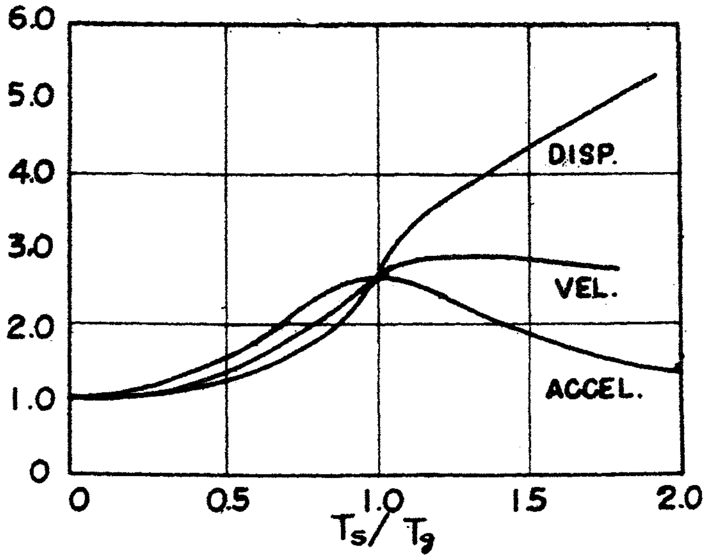

Fig. 3. Various transmissibilities versus T_s/T_g.

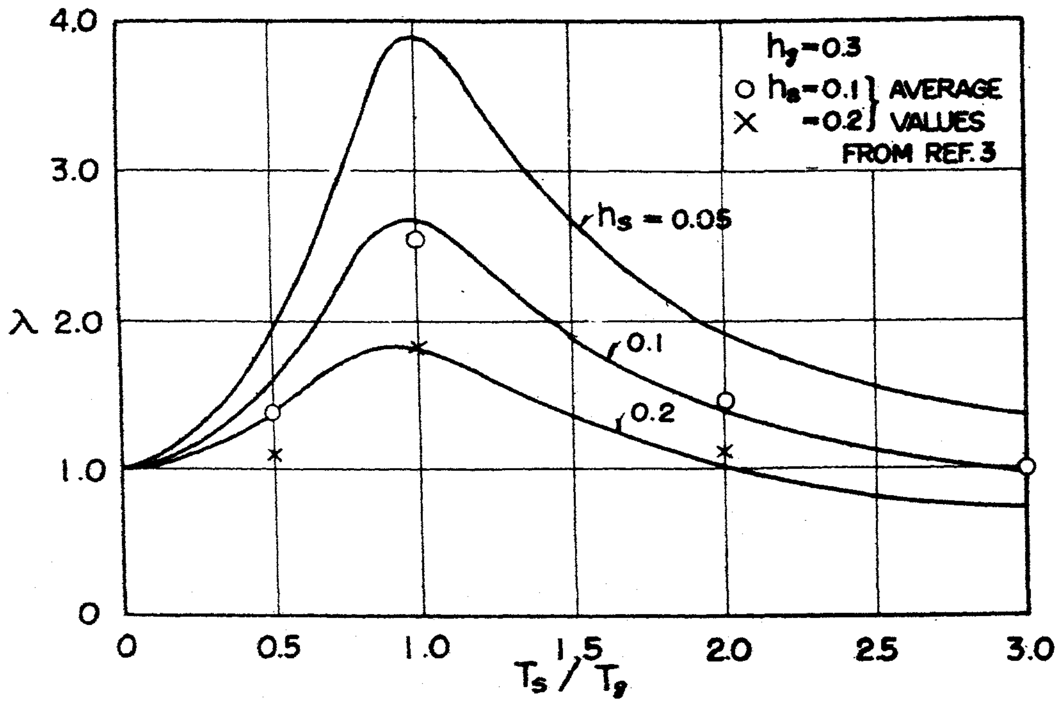

Fig. 2. Comparison of acceleration transmissibility from the presented theory and the response calculation (Ref. 3).

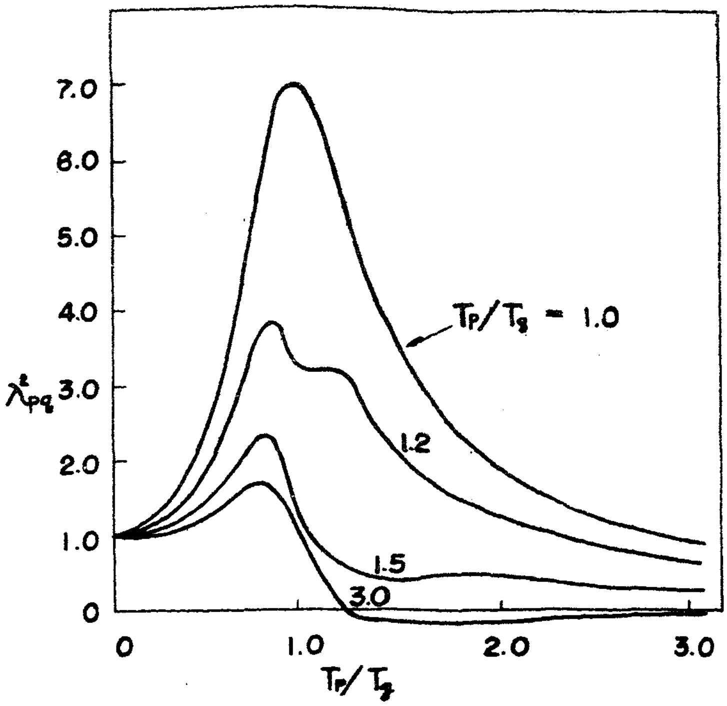

Fig. 4. Square value of cross-transmissibility of acceleration, λ^2_{pq}, from Eq. 24.

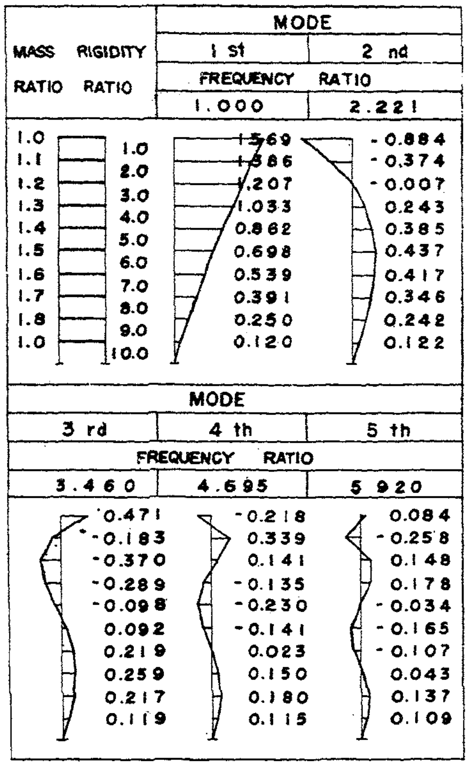

Fig. 5.

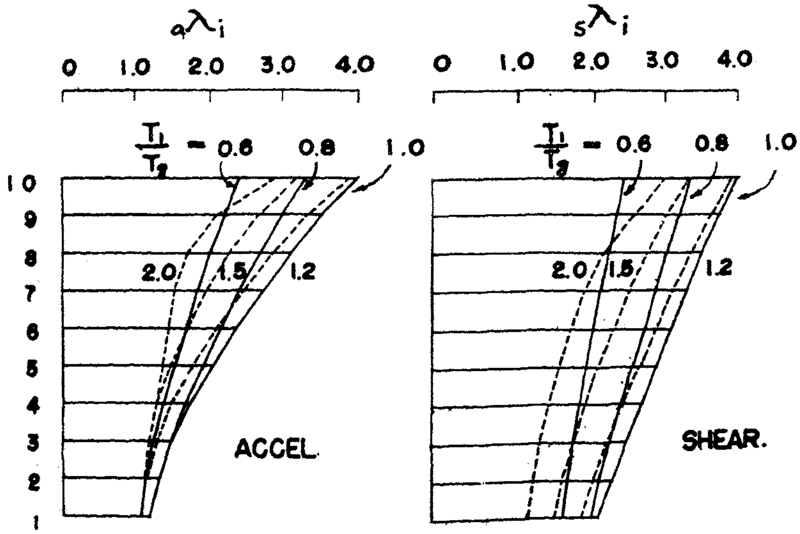

Fig. 6. Distributions of transmissibilities for the system of Fig. 5.

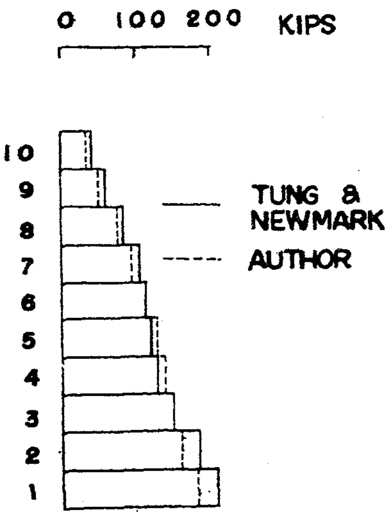

Fig. 7. Comparison of shear distribution from the presented theory and the response calculation (Ref. 7).

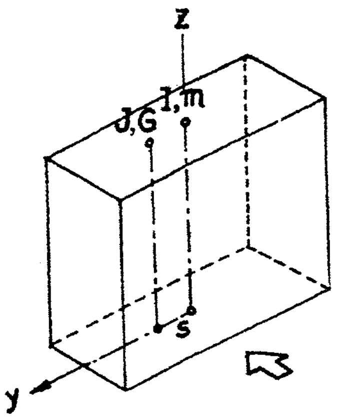

Fig. 8.

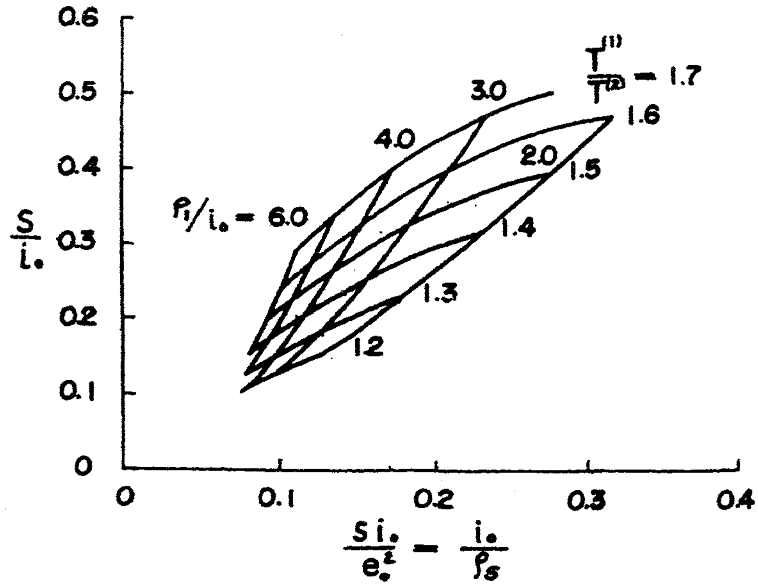

Fig. 9. Location of the instantaneous center of rotation, ρ_1/i_o, and the natural period ratio, $T_n^{(1)}/T_n^{(2)}$, for the system of

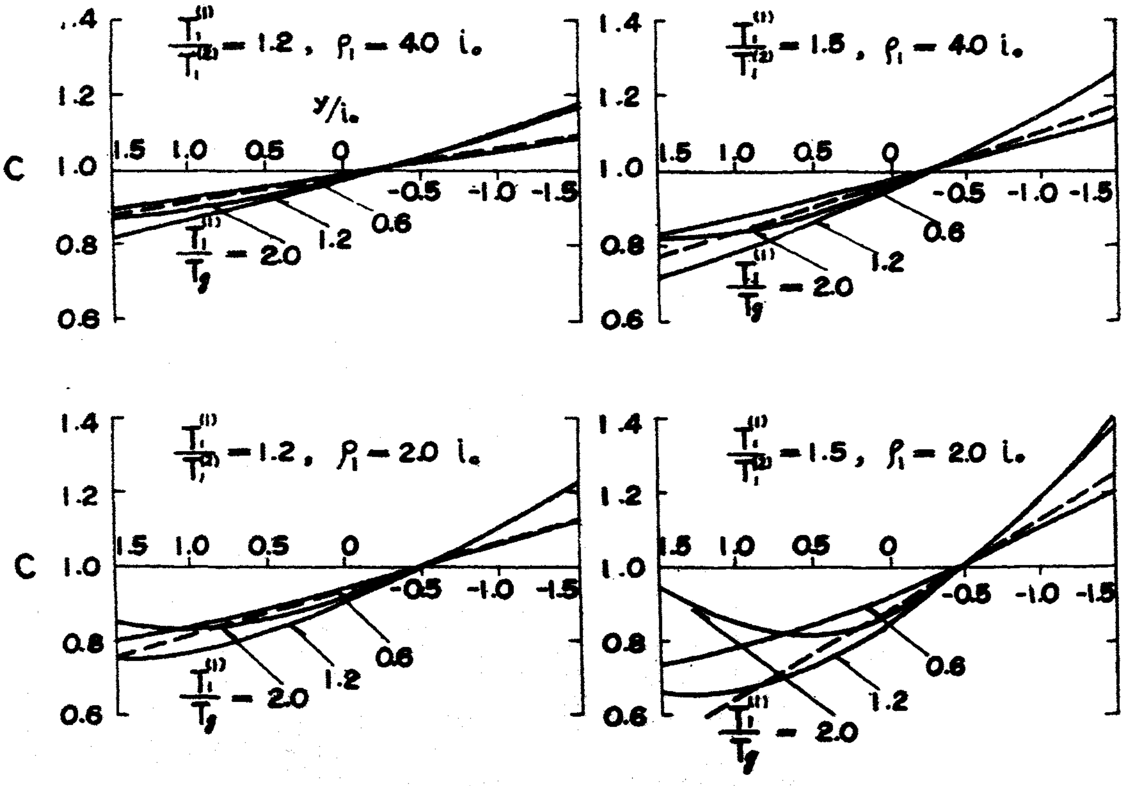

Fig. 10 Distribution of relative transmissibility of acceleration for the system of Fig. 8.

DISCUSSION

J. F. Borges, Laboratorio Nacional de Engenharia Civil, Portugal:

I much appreciate the paper presented but I wish to call the attention of the author that to compute the probability of collapse of a structure during a certain number of year, what is of principal interest from the point of view of safety, it is necessary to combine the randomness of the behaviour of the structure with the probability of occurrence of earthquakes of different intensity.

If this is done, it can be shown (1) that the mean values of the response of the structure are of more interest than extreme values.

H. Tajimi:

I appreciate your suggestive attension. I generally agree with your opinion. But, in Japan we have not yet so enough historical data or knowledge related to the probability of occurrence of earthquakes as applied to the statistical analysis. Therefore, still in this stage, I believe that extreme values of response should be used as bases of design.

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