EARTHQUAKE RESPONSE OF BENDING STRUCTURES DERIVED FROM A MIXED MECHANICAL-ELECTRICAL ANALOGUE

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1. INTRODUCTION

If the rational design of earthquake-resistant structures is to be used extensively, throughout seismic regions, convenient methods must be developed for the assessment of earthquake-generated forces and motions. A measure of the seismic forces to which a structure will be subjected can be obtained by calculating or measuring the forces which would have been generated in the structure by a number of the large earthquakes already recorded.

As a contribution towards simplified rational design this paper describes the measurement of the dynamic response (shear force, bending moment, and displacement) at any floor, of various types of building, to some of the largest recorded earthquakes. The measuring techniques used reduce the data to a form suitable for presentation in a design handbook. Such a handbook, covering the earthquake response of a range of building types (shear, bending, mixed shear and bending, flexing floors) is being prepared. The handbook sections for shear buildings are described in a companion paper presented at this conference - "Paper "B [1] Skinner, Adams, Brown. The building data which the structural engineer must obtain in order to use the handbook are the effective floor masses (and rotary inertias), the inter-storey stiffnesses, and an estimate of the damping factors of the normal modes. The building stiffnesses are taken as independent of deflection, and velocity damping assumed, so that linear analysis may be used throughout.

The mixed mechanical-electrical analogue used to obtain the earthquake responses of buildings, which deflect in bending, and with flexing floors, avoids the mechanical difficulties encountered in applying an earthquake to a model, and avoids the complexity [2] or the stringent demands on electrical components [3] of suggested electrical analogues. Limitations of the mechanical model as used are the difficulty of setting up mixed bending and shear, and the impracticability of including yield.

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The earthquake responses of any particular building are obtained in two steps. The properties of the building normal modes (at any floor) are measured on the most convenient building model or analogue. Then the measured normal modes are set up on a simple electrical analogue and an electrical signal, corresponding to a particular earthquake motion, is applied to it. The earthquake response, at a particular building floor, is then obtained as a voltage proportional to the building shear force, bending moment, or displacement. This normal mode approach has a number of advantages over the direct application of the earthquake motion to a building model or analogue.

Attempts to excite a mechanical model with earthquake ground motions lead to difficult problems of drive, controlled normal mode damping, and measurement of the responses. However, the normal mode properties of a mechanical model can be measured with ease and precision. The model damping should be small but need not be known. A simple sinusoidal drive, of variable period, is applied to the model at any convenient point and adjusted to one of its natural periods. The only quantities which need be measured are the ratios of the displacements; from these the modal properties are calculated. The displacements are obtained as voltages by allowing the model floors to shutter lights which fall on photo-multipliers. The voltage ratios are measured on a resistance bridge which has a phase-sensitive detector to discriminate against unwanted normal modes. When a convenient electrical analogue exists, as for shear buildings, a sinusoidal voltage is applied to it and the voltage ratios measured in the same way.

A conceptual advantage of the normal-mode approach is the clear presentation of the effects of building masses and stiffnesses upon the normal mode properties, and hence upon the earthquake responses. Interpolation and extrapolation from the normal mode properties of already known buildings can be made readily and this greatly extends the range of buildings which can be covered in a handbook. The normal mode approach also reduces the number of significant building parameters, since only a few normal modes are significant in the earthquake response of any particular building.

The electrical analogue of a set of normal modes, to which the earthquake is applied as an electrical signal, is very simple. Each normal mode is represented by an inductor and a capacitor, with a resistor to give the velocity damping. For greater precision and convenience a conventional adding circuit is used to give the normal modes the required relative weights, and to provide for negative responses. A section of the handbook gives the responses, to a number of large earthquakes, of normal modes of various periods, dampings, and relative weights. As further large earthquakes are recorded they can be applied to these normal modes and the earthquake responses issued as a supplement to this section of the handbook. It is hoped that eventually a sufficient number of earthquakes will be obtained to define the modal responses as smooth probability curves which will allow reliable interpolation. Any knowledge of local earthquake probabilities and types should be applied to the interpretation of these normal mode responses.

2. THEORY OF NORMAL MODE PROPERTIES

The theory of the normal modes of systems with a number of degrees of freedom is treated thoroughly by Rayleigh [4] (1877). Those features which affect the measurement of the modal properties of a building, and the use of these modal properties to obtain its earthquake responses, are outlined here.

2.1 Idealized Building

The following theory applies exactly to idealized buildings with the masses concentrated at the floors, Fig. 1. Deflection may occur as mixed bending and shear together with flexure of the floors. Initially the effects of angular momentum will be neglected. The base moves inexorably without rotation.

2.2 Normal Modes Without Damping

It is shown by Rayleigh [4] that the motion of such a system of N masses may be expressed as the sum of N independent normal modes. Each normal mode moves with all the masses in phase or antiphase and with fixed ratios of the N displacements.

The properties of a normal mode are completely defined by these N-1 displacement ratios, together with the system masses and stiffnesses.

In principle a constraining mechanism could be applied to the N masses of an idealized building to hold constant the N-1 displacement ratios at the values corresponding to a particular normal mode as shown in Fig. 1(b). It is then evident that the building has only one degree of freedom and will respond to base movements as a simple resonator. The complete motion of the N-mass building can be resolved into that of its N normal modes by setting up N buildings, each with a mechanical constraint which allows one mode only to operate, Fig. 1(c). Now, if an earthquake motion is applied to the base of Fig. 1(c), the displacement, shear force, and bending moment at any floor, and in any normal mode, occur at the appropriate floor of the constrained buildings. The response at any floor is then the sum of the modal responses. The method of analysis described below uses this approach adapted to permit convenient and precise measurements to be made.

A simple mechanical resonator can be set up, Fig. 1(d), whose shear force response to a base motion is numerically equal to the building shear force, bending moment, or displacement, at a given floor, and in a given normal mode. Before setting up equations defining the equivalence, we will examine the effect of velocity damping on these idealized buildings.

2.3 Idealized Building - Damping and Drives

The effects of damping in the building must be taken into account and also the effect of the unavoidable damping in any model set up to measure the normal mode shapes.

Consider the system of Fig. 1(a) oscillating in a single normal mode. Without damping or driving forces it oscillates steadily in this mode with no constraints.

The following damping forces are now added.

Absolute velocity damping,

$$\lambda_r = -A m_r(\dot{x}_r + \dot{x}_b) \tag{1}$$

Relative velocity damping; the damping forces must satisfy the relationship,

(Energy loss, storey r) = B(maximum potential energy, storey r) (2)

It is shown by Rayleigh [4] (1877) that for such damping the normal modes retain their undamped properties of independence and equi-phase mass movements.

If we substitute the velocities of mode m in eqns (1) and (2) and negative values of A and B, we obtain a set of driving forces which excite normal mode m only. Other distributions of driving force generally excite all modes. In particular, a base acceleration A_b may be translated, by d'Alembert's Principle, into forces $A_b m_r$ which are not of the above form and therefore excite all normal modes.

Damping forces with other than the above distributions cause the normal modes to interact with consequent mode distortion and change of natural period.

2.4 Normal Mode Equivalent Resonators

Apply velocity damping, as given by eqn (2), to the idealized building and let this give a damping factor n_m in the normal mode m. Now apply a base acceleration A_b of period T to the systems of Fig. 1(c) and let the natural period of mode m be T_m. Since the normal modes behave as simple resonators, the displacement, shear force, and bending moment responses, at floor r, in mode m, are given (for small damping) by

$$\left\{x_{r,m}, S_{r,m}, M_{r,m}\right\} = \frac{A_b}{g}\left\{x'_{r,m}, S'_{r,m}, M'_{r,m}\right\} \frac{T^2}{T^2 - T_m^2 + j2n_m T_m T} \quad (3)$$

where $x'_{r,m}$, $S'_{r,m}$ and $M'_{r,m}$ are the responses, at floor r in mode m, to a static acceleration of amplitude g . It should be noted that these dashed quantities are the actual static displacements, shear forces and bending moments that occur in the building with the hypothetical mechanical constraints of Fig. 1(c). Further, the sum of these modal static responses is equal to the actual static displacement, shear force, or bending moment of the building. This follows since removal of the constraint allows the static responses of all normal modes to occur simultaneously.

We may compare eqn. (3) with the static response of a simple resonator of mass $\mathcal{M}_{r,m}$, period T_m, and damping factor n_m as illustrated in Fig. 1(d).

$$\text{Shear force} = A_b \frac{\mathcal{M}_{r,m}}{g} \frac{T^2}{T^2 - T_m^2 + jn_m T_m T} \quad \text{lb} \quad (4)$$

The shear force response of the simple resonator is therefore numerically equal to the modal response of eqn. (3) when

$$\left\{\mathcal{M}_{r,m}\right\} = \left\{x'_{r,m}, S'_{r,m}, M'_{r,m}\right\} \quad (5)$$

Hence, if an earthquake motion is applied to the base of Fig. 1(d) the shear force responses give either the displacement, shear force, or bending moment responses of the building normal modes at floor r .

The two steps by which the earthquake response of a building are obtained may be described now in terms of Figs. 1(c) and 1(d). The static normal mode responses, $x'_{r,m}$, $S'_{r,m}$ and $M'_{r,m}$, are calculated from the modal displacement ratios of the building $x_{r,m}/x_{N,m}$, which are measured on a lightly damped mechanical model. The next step is to set up the modal equivalent resonators for any floor r , as defined by eqn. (5) and Fig. 1(d), as an electrical analogue. An electrical current, corresponding to the ground motion of a particular earthquake is applied to the analogue. With appropriate calibration the earthquake response of the building at floor r is obtained as the sum of the voltages across the electrical analogue resonators.

3. MEASUREMENT OF NORMAL MODE DISPLACEMENT RATIOS - BENDING STRUCTURES

A mechanical model is set up with the masses and stiffnesses proportional to those of the building under investigation. The radius of gyration of each floor mass must be scaled by the same factor as the inter-storey heights. The damping of the model is kept as low as possible and a sinusoidal driving force is applied to the lowest mass. A moving mass acts as a variable shutter for a uniform illuminated strip which is viewed with a photo-multiplier, giving an electrical signal proportional to the mass displacement.

The period of the driving force is varied until the force is in phase quadrature with the modal displacements. It follows from eqn. (4) that the model is now being driven at the natural period of one of its normal modes. The motion of the other normal modes, which are non-resonant, will be much smaller. A resistance bridge is set up which measures voltage ratios equal to the model displacement ratios. The bridge has a phase-sensitive detector which discriminates against the phase quadrature motion (as given by eqn. (4)) of the other normal modes. This feature is particularly useful when investigating the shape of a normal mode near one of its nodes. Since scaling the model leaves the shapes of the normal modes unaltered, the measured displacement ratios are equal to those of the building itself.

The natural period T_m of the building may be calculated from the measured displacement ratios $x_{r,m}/x_{r,N}$, together with the building masses and stiffnesses, or it may be obtained conveniently by scaling from the period of the model.

4. CALCULATION OF NORMAL MODE STATIC RESPONSES

It is now necessary to calculate the modal static responses $x'_{r,m}$. $S'_{r,m}$ and $M'_{r,m}$ from the known building masses m_r , stiffnesses k_r, K_r, and normal mode dampings, and from the measured modal displacement ratios $x_{r,m}/x_{N,m}$, and periods T_m. The modal responses, at their natural periods, are calculated first. The static modal responses are then obtained by dividing by the resonant rise of amplitude, $1/2n_m$. (Eqn. 3)

4.1.1 Modal Displacement at Resonance - $x_{r,m}$.

The actual displacements of the building normal modes are now calculated from the measured displacement ratios by using a power relationship which defines the modal damping factor. This relationship may be considered as written for the system constrained to move in mode m , Fig. 1(c). When the base has an acceleration A_b of period T_m ,

$$\text{maximum kinetic energy} = \frac{1}{2n_m}\left(\frac{T_m}{2\pi}\right) \text{ input power} \qquad (6)$$

We apply d'Alembert's principle to convert the base acceleration A_b to a driving force $A_b\, m_r/g$ at floor r, and obtain a fixed base. Eqns. (3) and (4) show that these driving forces are in phase with the building mass velocities for T equals T_m. This gives the input power as

$$\sum_{i=1}^{N} \frac{1}{2} A_b \frac{m_i}{g} \dot{x}_{i,m}$$

Eqn. (6) becomes

$$\sum_{i=1}^{N} \frac{1}{2} \frac{m_i}{g} (\dot{x}_{i,m})^2 = \frac{1}{2 n_m} \left(\frac{T_m}{2\pi}\right) \sum_{i=1}^{N} \frac{1}{2} A_b \frac{m_i}{g} \dot{x}_{i,m}$$

Hence
$$x_{r,m} = \frac{A_b}{2 n_m} \left(\frac{T_m}{2\pi}\right)^2 \frac{\sum_{i=1}^{N} m_i \frac{x_{i,m}}{x_{N,m}}}{\sum_{i=1}^{N} m_i \left(\frac{x_{i,m}}{x_{N,m}}\right)^2} \left(\frac{x_{r,N}}{x_{N,m}}\right) \quad \text{ft} \qquad (7)$$

This gives the actual resonant displacement of the building at floor r, in normal mode m, in terms of the measured displacement ratios.

4.1.2 Modal Shear Force at Resonance - $S_{r,m}$.

The shear force just below floor r may be obtained by summing the inertia forces above that point.

$$S_{r,m} = -\sum_{i=r}^{N} \frac{m_i}{g} \ddot{x}_{i,m}$$

Hence
$$S_{r,m} = \left(\frac{2\pi}{T_m}\right)^2 \sum_{i=r}^{N} \frac{m_i}{g} x_{i,m} \quad - \text{lb} \qquad (8)$$

4.1.3 Modal Bending Moment at Resonance - $M_{r,m}$.

The bending moment at floor r, in normal mode m, may be obtained by summing the moments due to all the inertia forces above floor r. For a uniform inter-storey height h,

$$M_{r,m} = -\sum_{i=r+1}^{N} h(i-r)\,\frac{m_i}{g}\,\ddot{x}_{i,m}$$

Now $-\frac{m_i}{g}\ddot{x}_{i,m} = S_{i,m} - S_{i+1,m}$

Hence $M_{r,m} = h\sum_{i=r+1}^{N} S_{i,m}$ - lb-ft (9)

4.2.1 Responses of Normal Modes to Static Acceleration - g $X'_{r,m}$ $S'_{r,m}$ and $M'_{r,m}$

When eqns. (7), (8) and (9) are compared with eqn. (3), it is seen that the resonant modal responses have an increase of $1/2n_m$ over the static responses to a uniform acceleration. Therefore if the modal responses at resonance are divided by $1/2n_m$, and A_b made equal to g, the modal responses to a static acceleration g are obtained. The displacement is given as $X'_{r,m}$ inches

$$X'_{r,m} = 12g\left(\frac{T_m}{2\pi}\right)^2 \rho_m \frac{x_{r,m}}{x_{N,m}} \quad \text{- in} \qquad (10)$$

$$S'_{r,m} = \rho_m \sum_{i=r}^{N} m_i \frac{x_{i,m}}{x_{N,m}} \quad \text{- lb} \qquad (11)$$

$$M'_{r,m} = h\sum_{i=r+1}^{N} S'_{i,m} \quad \text{- lb-ft} \qquad (12)$$

Where
$$\rho_m = \frac{\sum_{i=1}^{N} m_i \frac{x_{i,m}}{x_{N,m}}}{\sum_{i=1}^{N} m_i \left(\frac{x_{i,m}}{x_{N,m}}\right)^2}$$

An overall check on the accuracy of the measurements and calculations is provided by the fact that the sum of the modal static responses of a building is equal to its static response.

$$\text{Hence} \quad \sum_{i=1}^{N} X'_{r,i} = X'_r \tag{13}$$

$$\sum_{i=1}^{N} S'_{r,i} = S'_r \tag{14}$$

$$\sum_{i=1}^{N} M'_{r,i} = M'_r \tag{15}$$

where X'_r , S'_r and M'_r are the static responses of the building, at floor r , to an acceleration g .

4.2.2 Static Responses of Normal Modes with Rotary Inertia

Let the mass of floor r have a moment of inertia I_r about a horizontal axis through its centre of gravity, and let θ_r be the angular displacement of floor r . The building now has 2 N degrees of freedom and the shape of normal mode m is defined by the 2 N-1 ratios

$$\frac{x_{r,m}}{x_{N,m}} \quad \text{and} \quad \frac{\theta_{r,m}}{x_{N,m}}$$

The argument follows closely that of sections 4.1 and 4.2.1 giving the responses to a static acceleration g as follows,

$$\left\{ X'_{r,m} \text{ or } \theta'_{r,m} \right\} = g\left(\frac{T_m}{2\pi}\right)^2 Q_m \left\{ 12 \frac{x_{r,m}}{x_{N,m}} \text{ or } \frac{\theta_{r,m}}{x_{N,m}} \right\} \text{ inches or radians} \tag{10a}$$

$$S'_{r,m} = Q_m \sum_{i=r}^{N} m_i \frac{x_{i,m}}{x_{N,m}} \quad \text{- lb} \tag{11a}$$

$$M'_{r,m} = h \sum_{i=r+1}^{N} S'_{i,m} + Q_m \sum_{i=r}^{N} I_i \frac{\theta_{i,m}}{x_{N,m}} \quad \text{- lb-ft} \tag{12a}$$

Where

$$Q_m = \frac{\sum_{i=1}^{N} m_i \frac{x_{i,m}}{x_{N,m}}}{\sum_{i=1}^{N}\left[m_i \left(\frac{x_{i,m}}{x_{N,m}}\right)^2 + I_i \left(\frac{\theta_{i,m}}{x_{N,m}}\right)^2 \right]}$$

5. DETERMINATION OF EARTHQUAKE RESPONSES OF NORMAL MODES

The normal mode equivalent resonators of eqn. (5) and Fig. 1(d) are now completely defined by the modal static responses of eqns. (10), (11) and (12) (section 4.2), the measured modal periods, and the estimated modal damping factors.

Let R_m be the earthquake-generated shear force in a simple resonator of one pound mass, natural period T_m, and damping factor n_m. Then the earthquake response of a normal mode equivalent resonator of mass $\mathcal{M}_{r,m}$ is given by

$$\mathcal{R}_{r,m} = R_{r,m}\mathcal{M}_{r,m} \qquad (16)$$

The maximum value of the total earthquake response at floor r, due to all the normal modes, is obtained by summation. Let N equal the number of normal modes.

$$(\mathcal{R}_r)_{max} = \left(\sum_{i=1}^{N} \mathcal{R}_{r,i}\right)_{max} = \mathcal{M}_{r,1}\, E\left(1, \frac{\mathcal{M}_{r,2}}{\mathcal{M}_{r,1}}, \cdots, \frac{\mathcal{M}_{r,N}}{\mathcal{M}_{r,1}}\right) \qquad (17)$$

Where

$$E\left(1, \frac{\mathcal{M}_{r,2}}{\mathcal{M}_{r,1}}, \ldots, \frac{\mathcal{M}_{r,N}}{\mathcal{M}_{r,1}}\right) = \left(R_1 + R_2\frac{\mathcal{M}_{r,2}}{\mathcal{M}_{r,1}} + \cdots + R_N\frac{\mathcal{M}_{r,N}}{\mathcal{M}_{r,1}}\right)_{max}$$

The magnitude of E is measured, for particular earthquakes, by an electric analogue which is set up with resonant circuits for each of the modal equivalent resonators of Fig. 1(d). This analogue, Fig. 3, is the classical one in which inductance corresponds to mass, reciprocal capacitance to stiffness, current to velocity, and voltage to force. The earthquake ground velocity is obtained as an electric current from substantially the same equipment as that described by Murphy et al.[5] A convenient time scale is chosen for the electrical circuits and a corresponding time scale given to the earthquake electrical current. The earthquake-velocity current is passed through each of the resonant circuits to give voltages across them proportional to the modal responses. The voltages across each of the electrical circuits are added (with the appropriate sign) in a conventional electronic adder to give a voltage proportional to the earthquake-generated building displacement, shear force, or bending moment at floor r. The maximum voltage is therefore proportional to $(\mathcal{R}_r)_{max}$ of eqn. 17. The next step is to obtain a voltage which is related to $\mathcal{M}_{r,1}$, of eqn. 17, by the same factor of proportionality. Since $\mathcal{M}_{r,1}$ is the static response of normal mode 1, a current corresponding to an acceleration g is applied to the circuit for mode 1 with C_1 removed to give the static response. The voltage obtained is proportional to $\mathcal{M}_{r,1}$, and the ratio of the earthquake-voltage across all the resonant circuits to this voltage is E, in accordance with eqn. 17.

Values of E (1, 0.06, 0.05) for $T_1 : T_2 : T_3 = (0.72):(0.143):(0.054)$ and for $n_1 = n_2 = n_3 = 0.05$ are given for the earthquake recorded at Taft, California, (N.21.E - 21/7/1952). Values of $E(1, 0, 0)$ from the response of mode 1 only, are also given for comparison.

In a practical circuit layout a single current proportional to the earthquake ground velocity is passed through all the resonant circuits in series. The inductors have ferrite pot-core magnetic circuits and any convenient inductance. The periods and dampings are adjusted by varying the condensers and the resistors respectively. The relative weights of the various modes are then obtained by setting up appropriate values of the adder scaling factors.

Typical tall buildings are found to give almost all their earthquake response in the first three modes. The amount added to the maximum earthquake response at a given floor by any one normal mode is generally considerably less than the maximum response of the individual normal mode since the maxima of two added responses seldom coincide in time. Since its individual maximum response sets an upper limit to the contribution of a given normal mode, this gives a convenient means of recognizing those which are insignificant and gives a rough indication of the relative contributions of the significant normal modes.

6. RESPONSE OF A PARTICULAR BUILDING TO THE TAFT EARTHQUAKE

As an illustrative example the foregoing methods are applied to find the forces generated in a particular building by the Taft earthquake.

A five-storey bending building has the floor masses, rotary inertias, and inter-storey bending stiffnesses given in columns (2), (3) and (4) of the table at the end of the paper. The damping factor of the normal modes is estimated as 0.05. The shear stiffness is assumed to be infinite.

A mechanical model is set up as described in section 3, with the masses and stiffnesses proportional to those tabulated in columns (2) and (4). The rotary inertias of the model masses are obtained by scaling the radii of gyration by the same factor as the inter-storey heights. The modal displacement ratios are measured by the methods described in section 3, and the values for the first three normal modes listed in columns (5), (6) and (7). The modal periods are scaled from the model and included in the table.

The building modal displacements $X'_{r,m}$ for a static acceleration of g, are calculated from eqn. 10(a) and listed in columns (8), (9) and (10). (These include a small contribution from the rotary inertias.)

The static modal responses are summed for each floor in column (11) and the static deflection of the building (for a static acceleration g) is given in column (12). The static modal shear forces $S'_{r,m}$ and bending moments $M'_{r,m}$ are calculated from eqns. (11a) and (12a) and listed, together with the building static shear forces and bending moments in columns (13) to (22). If the measurements of building normal mode shapes and periods are exact, and the sums in columns (11), (16) and (21) are taken over all ten of the building normal modes, then the values obtained would be exactly equal to the building static deflections, shear forces, and bending moments in columns (12), (17) and (22). The static responses of the first three normal modes of the building have now been tabulated. When substituted in eqn. (5) these modal static responses specify the equivalent modal resonators of Fig. 1(d).

The modal bending moments at the base of the building $M'_{0,1} = 40.4.10^7$, $M'_{0,2} = 2.4.10^7$, and $M'_{0,3} = 1.77.10^7$ lb-ft, are set up on the electrical analogue of Fig. 3. The periods of the electrical resonators are adjusted to give the natural period ratios of (0.720):(0.143):(0.054), and the damping factor of each resonant circuit adjusted to the estimated value of 0.05. The adder scaling factors are set to give the resonant circuit static weights the same ratios as the modal static moments at the base; (40.4):(2.4):(1.77). This is equivalent to using resonant circuits with inductances in these ratios and using equal values for all the adder scaling factors.

The analogue is calibrated by applying a current corresponding to an acceleration of g to the circuit of mode 1, made to correspond to a mass rigidly attached to the base by removing C_1. This gives a voltage proportiona_ to $M'_{0,1} = 40.4.10^7$ lb-ft. The earthquake-velocity current is now applied to the analogue and the maximum output voltage is proportional to the maximum bending moment at the base of the building, with the same factor of proportionality as for the calibration voltage. The ratio of this voltage to the calibration voltage gives E (1, 0.06, 0.046).

For the Taft earthquake, the analogue measurement gives $E(1,\ 0.06,\ 0.046) = 0.415$.

The response of the three bending modes at the base of the building is given by eqn. 17 as

$$\left(\sum_{i=1}^{3} \mathcal{R}_{0,i}\right)_{max} = M'_{0,1} \cdot E\,(1,\ 0.06,\ 0.046)$$
$$= (40.4)(0.415).10^7$$
$$= 1.7.10^8 \quad \text{lb-ft}$$

The electrical analogue, as set up for measuring the bending moment at the base, can be altered to measure the earthquake-generated bending moment, shear force or displacement, at any building floor r, simply by changing the adder scale factors to appropriate values.

To present a picture of the effect of varying the building stiffness, we have given E for a range of values of fundamental period, T_1, Fig. (4), together with the response of mode 1 only, E (1, 0, 0).

From eqns. (13) and (14) the response of an individual mode, at any floor, may be obtained as

$$\mathcal{R}_{r,m} = \mathcal{M}_{r,m} \, E\,(1,\ 0,\ 0) \tag{18}$$

where $E\,(1,\ 0,\ 0)$ is the response for the modal period T_m. For example the shear force at floor r in mode 2 ($T_2 = 0{,}143$ sec) is given by

$$\mathcal{R}_{3,2} = S'_{3,2} \; E\;(1,\ 0,\ 0)$$

$$= -0.08.10^6\,(0.3\ ) = 0.024.10^6\text{-lb}$$

from the table, column (14) and Fig. (4). To obtain the true sum of two or more modal responses they must be added throughout the duration of the earthquake since the maximum values do not in general occur at the same instant and may even occur in opposite directions.

DISCUSSION

Although the responses of a particular set of normal modes to a given earthquake can be measured easily and quickly with the electrical analogue*, this does not immediately give the probabilities of various earthquake-generated forces occurring during the life of the building, and it is on such information that a rational design must be based. However, when sufficient earthquake records are obtained and their statistical properties assessed, it should be possible to express the modal responses directly as probabilities of the occurrence of various forces and motions. (* See Fig. 3)

There are two factors in the dynamic response of a building on which more information is required before they can be defined adequately. The values to be taken for the modal damping factors of a particular building are difficult to assess. The effect of the building foundations on its dynamic response is also difficult to assess. The foundations of some buildings probably make an important contribution to the normal mode damping.

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NOMENCLATURE

m_r - mass of floor r - lb

I_r - moment of inertia of floor r - lb-ft^2

k_r - shear stiffness between floor $r-1$ and r - lb/in

K_r - bending stiffness between floor $r-1$ and r - lb-ft/rad.

N - number of floors

$x_r, \dot{x}_r, \ddot{x}_r$ - displacement, velocity, acceleration, of floor r, relative to the base - ft-sec. units

$x_{r,m}, \dot{x}_{r,m}, \ddot{x}_{r,m}$ - displacement, velocity, acceleration, of normal mode m at floor r, relative to the base - ft-sec. units

$X_r, X_{r,m}$ - displacement of floor r - in.

$\theta_{r,m}, \dot{\theta}_{r,m}$ - angular displacement, angular velocity, of floor r, in normal mode m, relative to the base - rad., sec. units.

$x_b, \dot{x}_b, A_b$ - displacement, velocity, acceleration, of base relative to rest position - ft., sec. units.

S_r - shear force just below floor r - lb

M_r - bending moment just below floor r - lb-ft

$S_{r,m}$, $M_{r,m}$ - shear force, bending moment, just below floor r, in normal mode m - lb., ft units.

$\mathcal{M}_{r,m}$ - mass of equivalent resonator - lb

T_m - natural period (corresponding to normal mode m) - sec

n_m - damping factor of normal mode m, fraction of critical

$x'_{r,m}$, $S'_{r,m}$, $M'_{r,m}$ - displacement, shear force, bending moment, due to a static acceleration g of base - ft., lb units

λ_r, $\mathcal{C}_r$ - absolute, relative, velocity damping force - lb

$R_{r,m}$ - ratio of the earthquake response of a simple resonator to its response to a static acceleration g.

$E(1, a, b, \cdots)$ - ratio of the earthquake response of a set of weighted resonators to the response of a single resonator to a static acceleration g. (maximum value)

$\mathcal{R}_r$ - response at floor r to earthquake base motion.

$\mathcal{R}_{r,m}$ - response of mode m at floor r to earthquake base motion.

NORMAL MODE RESPONSE OF BENDING BUILDING TO A STATIC ACCELERATION g

BENDING BUILDING WITH STATIC ACCELERATION g

(1) Storey No. r	(2) Floor Mass	(3) Rotary Inertia I_r lb-ft^2	(4) Inter-storey Stiffness lb/ft	(5) Mode 1 $X_{r,1}/X_{5,1}$	(6) Mode 2 $X_{r,2}/X_{5,2}$	(7) Mode 3 $X_{r,3}/X_{5,3}$
5	$1.29.10^6$	$6.32.10^6$	10^9	1.000	- 1.000	1.000
4	1.78	12.30	10^9	0.548	- 0.048	- 0.086
3	5.00	13.60	10^9	0.226	0.122	- 0.356
2	5.00	13.60	10^9	0.060	0.276	0.173
1	5.00	12.30	10^9	0.006	0.065	0.503

DISPLACEMENTS OF NORMAL MODES-INCHES

(1) Storey No. r	(8) $X'_{r,1}$	(9) $X'_{r,2}$	(10) $X'_{r,3}$	(11) $\sum_{m=1}^{3} X'_{r,m}$	(12) X'_r
5	8.98	-0.207	0.014	8.78	8.90
4	6.68	-0.045	-0.004	6.63	
3	4.28	0.073	-0.008	4.22	
2	2.22	0.109	0.006	2.34	
1	0.67	0.053	0.010	0.73	

SHEAR FORCES OF NORMAL MODES - lb

(1) Storey No. r	(13) $S'_{r,1}$	(14) $S'_{r,2}$	(15) $S'_{r,3}$	(16) $\sum_{m=1}^{3} S'_{r,m}$	(17) S'_{r}
5	$2.27.10^6$	$-1.34.10^6$	$0.63.10^6$	$1.56.10^6$	$1.29.10^6$
4	4.59	-1.74	0.38	3.23	3.07
3	8.78	0.08	-1.08	7.78	8.07
2	10.95	2.80	-0.07	13.68	13.07
1	11.60	4.12	1.67	17.39	18.07

BENDING MOMENTS OF NORMAL MODES - lb-ft

(1) Storey No. r	(18) $M'_{r,1}$	(19) $M'_{r,2}$	(20) $M'_{r,3}$	(21) $\sum_{m=1}^{3} M'_{r,m}$	(22) M'_{r}
5	$0.25.10^7$	$-0.52.10^7$	$0.48.10^7$	$0.21.10^7$	$0.00.10^7$
4	3.08	-2.76	1.63	1.96	1.29
3	8.29	-5.15	1.60	4.74	4.36
2	17.55	-5.08	-0.01	12.47	12.45
1	28.76	-1.73	0.13	27.16	25.50
0	40.36	2.40	1.80	44.56	43.57