VIBRATIONS IN FOUNDATION, STRUCTURE AND IN ITS VICINITY ON THE ELASTIC GROUND

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Part I (Toriumi)

Introduction

Vibrations in foundations may be due to either one of two causes, viz. they may be caused by a train of waves passing through the ground, or they may be set up by fluctuating load on the foundation itself. These problems are similar essentially irrespective of the sign in propagation of waves, and of the sort of foundation for machine or building.

We treat these problems theoretically as a propagation of tremors over the surface of a semi-infinite elastic ground. Thus the vibrations in foundation of machine, vibrations of elastic structure on the elastic ground and the distribution of vibration in vicinity of these foundations and moreover the interference in vibration between two foundations etc. can be solved in series.

1. Fundamental

For the elastic solid in cylindrical coordinates, the displacements are given by

$$(1)\quad U=\left[-A_m\frac{1}{k^2}\frac{\partial H_m^{(2)}(kr)}{\partial r}e^{-\alpha z}+B_m\frac{m}{k^2}\frac{H_m^{(2)}(kr)}{r}e^{-\beta z}+C_m\frac{\beta}{mj^2}\frac{\partial H_m^{(2)}(kr)}{\partial r}e^{-\beta z}\right]e^{ipt}\cdot\cos m\theta \quad \text{etc.}$$

where A_m, B_m and C_m are unknown coefficients. The stresses are given by

$$(2)\quad \sigma_z=\mu\left[A_m\frac{-2k^2+\gamma^2}{k^2}e^{-\alpha z}+2C_m\frac{\beta k^2}{mj^2}e^{-\beta z}\right]H_m^{(2)}(kr)\cdot e^{ipt}\cdot\cos m\theta \quad \text{etc.}$$

Horizontal force transmitted to the ground Q_H ($=\pi r_0^2 q_0$) acts on the circular area with radius r_0 on the surface of the ground. (Fig. 1)

B.C.
1) $z=0$: $\sigma_z=0$
2) $z=0$, $r>r_0$: $\tau_{zr}=\tau_{z\theta}=0$
3) $z=0$, $r\leqq r_0$: $\tau_{zr}=-q_0\cdot\cos\theta\cdot e^{ipt}$, $\tau_{z\theta}=q_0\cdot\sin\theta\cdot e^{ipt}$

A_m, B_m and C_m in eq. 1 are decided by the B.C. adapting Fourier-Bessel integral. Thus horizontal displacement at the center of circular area ($r=0$) is given by

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$$(3) \qquad U = \frac{r_0 q_0}{2\mu} e^{ipt} \int_0^\infty \frac{F(k) - \beta^2 \gamma^2}{\beta F(k)} J_1(k r_0)\, dk$$
$$= \frac{Q_H}{2\mu\pi r_0}(f_1 \cos pt - f_2 \sin pt) \qquad \text{(Real part)}$$

The real-part f_1 and the imaginary part f_2 of the above integral evaluated in the case $\lambda = \mu$ is shown in Fig.2 in which abscissa denotes frequency parameter $a_0 (= p r_0 \sqrt{\rho/\mu})$.

We get similar results for vertical and rocking vibrations (Fig.3 and 4).

$$(4) \qquad W_{z, r=0} = \frac{Q_V}{\mu\pi r_0}(f_1 \cos pt - f_2 \sin pt)$$

$$(5) \qquad \Upsilon = \frac{W_{r=r_0}}{r_0} = \frac{4 \bar{M}_{z=0}}{\pi r_0^3 \mu}(f_1 \cos pt - f_2 \sin pt)$$

2. <u>Vibrations in foundation</u>[1]

When additional mass is set on the circular area and unbalanced force P_H (independent of circular frequency p) acts upon it, equation of horizontal vibration of this foundation is given by

$$(6) \qquad Q_H \cos pt + m_0 \frac{d^2 U}{dt^2} = P_H \cos(pt + \varepsilon)$$

where ε is phase angle of P_H to Q_H.

Substituting eq.3 to eq.6, we obtain

$$(7) \qquad Q_H = P_H \left[\left(1 - \frac{m_0 p^2}{2\mu\pi r_0} f_1\right)^2 + \left(\frac{m_0 p^2}{2\mu\pi r_0} f_2\right)^2 \right]^{-\frac{1}{2}}$$

Substituting eq.7 again to eq.3, we obtain the amplitude A_H of displacement of the foundation as under

$$(8) \qquad A_H = \frac{P_H}{2\mu\pi r_0} \left[\frac{f_1^2 + f_2^2}{\left(1 - \frac{m_0 p^2}{2\mu\pi r_0} f_1\right)^2 + \left(\frac{m_0 p^2}{2\mu\pi r_0} f_2\right)^2} \right]^{\frac{1}{2}}$$

The phase angle of displacement of the foundation to unbalanced force P_H is given by

$$(9) \qquad \varphi_H = \tan^{-1} \frac{-f_2}{f_1 - \frac{m_0 p^2}{2\mu\pi r_0}(f_1^2 + f_2^2)}$$

The behaviours of A_H and φ_H to frequency parameter a_0 are shown in Fig. 5 and 6 at some values of mass parameter $b (= m_0/\rho r_0^3)$. We get similar results for vertical and rocking vibrations. From these curves we get generalized results as under--

The semi-infinite elastic solid itself has no natural frequency, however if an additional mass is set at the finite area of the surface, resonant curves appear. But by the dispersion to the infinite solid, damping acts to resonance.

3. Vibrations in vicinity of foundation [2,3]

Distributions of vibrations in vicinity of foundations of machine are peculiar. Distribution of statical displacements are featureless (Fig.7), however it turns to complex type in the case of vibration. This problem is in domain where the surface wave is formed, and this phenomenon seems to have relation essentially to the problem of sound field near the radiator although the generated waves are more complex.

From the equation of vibration in semi-infinite elastic solid, horizontal components U , V of displacement at any point ($z=0; r, \theta$) on the surface and under ground (z-axis) are given by

$$(10)\quad \begin{cases} U_{z=0}(r,\theta) = \dfrac{r_0 q_0}{2\mu} e^{ipt}\left[\displaystyle\int_0^\infty \frac{F(k)-\beta^2 j^2}{\beta F(k)} J_0(kr)\cdot J_1(kr_0)\,dk - (2\sin^2\theta-1)\int_0^\infty \frac{F(k)+\beta^2 j^2}{\beta F(k)} J_1(kr_0)\cdot J_2(kr)\,dk\right] \\ V_{z=0}(r,\theta) = \dfrac{r_0 q_0}{2\mu} e^{ipt}\sin 2\theta \times \displaystyle\int_0^\infty \frac{F(k)+\beta^2 j^2}{\beta F(k)} J_1(kr_0)\cdot J_2(kr)\,dk \\ U_{r=0}(z) = \dfrac{r_0 q_0}{2\mu} e^{ipt} \displaystyle\int_0^\infty \left\{-2k^2\beta\left(e^{-\alpha z} - e^{-\beta z}\right) + \frac{F(k)-\beta^2 j^2}{\beta} e^{-\beta z}\right\} \frac{J_1(kr_0)}{F(k)}\cdot dk \end{cases}$$

The distribution diagrams of U and V in four lines (three on the surface and one under ground) are shown in Fig. 8. From the phase angle at some points $\varphi \sim r/r_0$ curves are drawn and the wave velocity is decided. (Fig. 9)

Results :

1) On line 1-1 S-wave velocity appears and the distribution of amplitude is simple.

2) On line 2-2 from origine to $r/r_0 = 8 \sim 12$ P-wave velocity appears and then changes to Rayleigh wave velocity at a distance. This changing point of wave velocity is comparatively distinct and it is nearly situated in one length of P-wave from the origine.

3) On line 3-3 standing for lines ($0^\circ \sim 90^\circ$), U and V components viz. u and v components reduced to cylindrical coordinates coexist at some amplitude ratio. Besides velocities of these u and v components are the same as on line 2-2 and 1-1 respectively. So the locus at any point becomes to loop.

4) On line 4-4 it shows S-wave velocity and the distribution of amplitude is simple and nearly equal to statical case.

4. Interference in vibration between two foundations [4]

By combining the steady vibration in foundation itself (division 2) with the distribution of vibration in vicinity of foundation (division 4), the interferences in vibration between two foundations are solved.

Though there are many cases in combinations , here two cases A and B as under are treated. (Fig. 10)

1) Two foundations a and b are the same.

2) Relations to direction of wave propagation and direction of wave vibration are indicated in Fig. 10.

These are simple cases in many combinations, because the wave of interference is only S-wave in these cases and line 1-1 in Fig. 8 is of simple character in many radial lines from the origin.

Case-A :

It is symmetrical and the behaviours of two foundations a and b are the same. The velocity of propagation of tremors U_1 is optional.

Original ground motion; $U_1 = A_1 e^{ipt}$

Term of compensation in foundation a to U_1 ; ${}_1U_2 = -A_1 E (f_1^2 + f_2^2)^{1/2} e^{i(pt-\varepsilon-\varphi)}$

Propagation of term of compensation in vicinity of foundation a ; ${}_1U_2' = -KA_1 E (f_1^2 + f_2^2)^{1/2} e^{i(pt-\varepsilon-\varphi-\theta)}$

Term of compensation in foundation b to ${}_1U_2'$; ${}_2U_2 = KA_1 E^2 (f_1^2 + f_2^2) e^{i(pt-2\varepsilon-2\varphi-\theta)}$

....

General term of compensation ; ${}_nU_2 = (-)^n A_1 K^{(n-1)} E^n (f_1^2 + f_2^2)^{n/2} e^{i(pt-n\varepsilon-n\varphi-(n-1)\theta)}$

General term of propagation ; ${}_nU_2' = (-)^n A_1 K^n E^n (f_1^2 + f_2^2)^{n/2} e^{i(pt-n\varepsilon-n\varphi-n\theta)}$

where $\varphi = \tan^{-1} f_2/f_1$, $\theta = pd/v_s$. Thus horizontal displacements in foundation a & b are given by

$$U = U_1 + ({}_1U_2 + {}_1U_2') + ({}_2U_2 + {}_2U_2') + \cdots + ({}_nU_2 + {}_nU_2') + \cdots$$

$$(11) \quad = A_1 e^{ipt}\left[1 - Ge^{-i(\varepsilon+\varphi)}(1+Ke^{-i\theta})\{1 + (-KGe^{-i(\varepsilon+\varphi+\theta)}) + (-KGe^{-i(\varepsilon+\varphi+\theta)})^2 + \cdots + (-KGe^{-i(\varepsilon+\varphi+\theta)})^n + \cdots\}\right]$$

$$= A_1 e^{ipt}\left[1 - Ge^{-i(\varepsilon+\varphi)} \frac{(1+Ke^{-i\theta})}{(1+KGe^{-i(\varepsilon+\varphi+\theta)})}\right]$$

where $G = E(f_1^2 + f_2^2)^{1/2}$, K : Amplitude at distance d from origin on line 1-1 (Fig. 8)

The series in eq.11 converges in domain $|KG| < 1$. As horizontal displacement in single foundation is given by $U = U_1 + {}_1U_2$, so the ratio of terms of compensation is shown by

$$(12) \quad {}_A\gamma_{a,b} = \frac{\text{Term of compensation in case-A}}{\text{Term of compensation in single}} = \frac{1+Ke^{-i\theta}}{1+KGe^{-i(\varepsilon+\varphi+\theta)}} = a_1 + ib_1$$

Evaluating a_1 & b_1 in complex number of eq. 12, the ratio of amplitude of case-A to single is shown by $(a_1^2+b_1^2)^{1/2}$ and phase-lag is shown by $\tan^{-1}\text{-}b_1/a_1$.

Case-B :

In this case, foundations a & b are different in phase to receive the original wave, so the effects of interference in a & b are not the same. On the contrary to case-A, case-B is not independent of the velocity of propagation of tremors. It seems to be suitable here to adopt the velocity of S-wave. Further it is supposed that there are no damping in amplitude of original wave in the distance d between two foundations as original tremors is very strong.

By the similar process to case-A, ratio of term of compensation in foundation b to single is given by

$$(13) \qquad {}_B\gamma_b = \frac{(1+K)-KGe^{-i(\xi+\eta)}-K^2Ge^{-i(\xi+\eta+2\theta)}}{1-K^2G^2e^{-2i(\xi+\eta+\theta)}}$$

and for foundation a

$$(14) \qquad {}_B\gamma_a = \frac{1+Ke^{-2i\theta}-(1+K)KGe^{-i(\xi+\eta+2\theta)}}{1-K^2G^2e^{-2i(\xi+\eta+\theta)}}$$

In Figs.11 & 12, the curves of γ and phase-lag of eq. 12, 13 & 14 evaluated for frequency parameter $a_0 = 0{,}8$ between $d/r_0 = 2\sim20$ are shown.

Results :

1) The curves decrease demping oscillationally to the distance of two foundations in both cases A and B as expected.

2) But they differ in pitch about two times, that is the pitch in case-A agrees with about one wave length of S-wave and the one in case-B agrees with a half wave length of S-wave.

3) In case-B foundation a which receives farther the ground motion is affected relatively more notably than foundation b , and foundation b is forward in phase always to compare with single foundation.

Part II (Satô and Yamaguchi)

Shear vibration of a continuous circular column, excited by ground motion, was discussed considering the coupling effect of the vibrating body and the elastic foundation, the calculation being based on the theory of Toriumi.

Maximum values of the response curves of rocking motion[5] are given taking various values of the ratio of the density of structures and that of the foundation.

Empirical formulas are also presented which hold between the parameters giving maximum response.

5. <u>Shear vibration of continuous circular columns</u>

In the present study structures are assumed to be continuous, and purely horizontal disturbance with amplitude A_H and frequency p travel vertically upward and is reflected at the surface.

At the free surface the displacement should be $U_1 = 2A_H \exp(ipt)$, but the extra displacement $U_D = U\exp(ipt)$ must be added because of the existence of a structure upon the ground. Displacement of the structure at height x relative to the base is expressed by $y = Y\exp(ipt)$.

The equation of motion connecting above quantities is

$$\frac{\partial^2}{\partial t^2}(U_1 + U_D + y) = \frac{\mu_0}{\rho_0}\frac{\partial^2 y}{\partial x^2} \tag{15}$$

Boundary conditions are :

At the base of the structure $x = 0$: $Y = 0$ and $E_0\mu_0 \frac{dY}{dx} = Q_H$

At the top of the structure $x = 2l_0$: $E_0\mu_0 \frac{dY}{dx} = 0$

Combining these expressions we have the following equation, which gives the responce at the base of structure

$$\begin{aligned} \psi_{HB}\exp(-i\vartheta_H) &= (2A_H + U)/2A_H \\ &= 1/\left\{1 - \frac{1}{2}\frac{\rho_0 V_{s0}}{\rho V_s} a_0 (f_1 + if_2)\tan\left(2\frac{V_s l_0}{V_{s0} r_0} a_0\right)\right\} \end{aligned} \tag{16}$$

and at the top

$$\begin{aligned} \psi_{HT}\exp(-i\vartheta_H) &= (2A_H + U + Y)_{x=2l_0}/2A_H \\ &= \psi_{HB}\exp(-i\vartheta_H)/\cos\left(2\frac{V_s l_0}{V_{s0} r_0} a_0\right) \end{aligned} \tag{17}$$

Results of numerical computation of these quantities are given in Figs. 13-18.

6. Empirical formulas

There is a certain relation between the maximum amplitude of the base movement and the corresponding frequency. (See Fig. 19) This relation is easily obtained as follows with a good accuracy. From (16)

(18) $$\mathcal{V}_{HB} \exp(-i\vartheta_H) = 1/\left\{1 - \frac{1}{2}\frac{\rho_0 V_{S0}}{\rho V_S} a_0 (f_1 + i f_2) \tan\left(2\frac{V_S l_0}{V_{S0} r_0} a_0\right)\right\}$$
$$= 1/\{1 - C_0(f_1 + i f_2)\}$$
$$= \exp(-i\vartheta_H)/\sqrt{(1 - C_0 f_1)^2 + C_0^2 f_2^2}$$

If f_1 does not change too rapidly, $\mathcal{V}_{HBMAX}$ is obtained putting simply $1 - C_0 f_1 = 0$ in the above expression. Then we have

(19) $$\mathcal{V}_{HBMAX} = \frac{1}{C_0 f_2} = \frac{f_1}{f_2} \doteqdot 2.45/a_0$$

For the top of the structure we have the next relation from Fig. 20,

(20) $$\mathcal{V}_{HTMAX} \doteqdot 11.5\,(V_S/V_{S0})/a_0^2$$

(21) $$\mathcal{V}_{HTMAX} \doteqdot 8.5/\{(l_0/r_0)\,a_0^3\}$$

from which the next formula is obtained.

(22) $$a_{0MAX} \doteqdot 0.74/\{(V_S/V_{S0})(l_0/r_0)\}$$

Therefore the period which gives the maximum $\mathcal{V}_{HT}$ is

(23) $$T_{MAX} \doteqdot 8.5\,(l_0/V_{S0})$$

Similar formula was obtained before by K. Suyehiro[6]. He did not consider the coupling effect of the structure and foundation, but only assumed a definite displacement at the base of the structure. His formula has the same form with (23), but the coefficient was not 8.5 but 8.0.

Simple formulas were not obtained for the case other than $\rho_0/\rho = 1/4$, but the effect of ρ_0/ρ can be seen in Figs. 21 and 22.

7. Maximum response for the rocking motion

The calculation of rocking motion was carried out assuming parameters covering wider range than before[5]. However, simple relations like our previous study were not deduced. Only the formula for T_{MAX} is obtained as follows.

(24) $$T_{MAX} \doteqdot 12\sqrt{\rho_0/\rho}\,(l_0/V_S)$$

Results are given in the Figs. 23-25.

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Nomenclature

a_0	$= pr_0\sqrt{(\rho/\mu)} = pr_0/V_S = 2\pi r_0/$(Wavelength of S-waves)
E_0	Cross section of the structure
f_1, f_2	Cf. eq.3 etc.
l_0	Height of the center of gravity of the structure
Q_H	Horizontal force propagated to the foundation
r_0	Radius of the structure
V_S, V_{S0}	Velocity of S-waves propagated in the foundation and the structure
μ, μ_0	Rigidity of the foundation and the structure respectively
ρ, ρ_0	Density of the foundation and the structure respectively
ϑ_H	Phase difference between the incident waves and the oscillation of structure
Ω_{HB}	Amplitude of the base of structure
Ω_{HT}	Amplitude of the top of structure

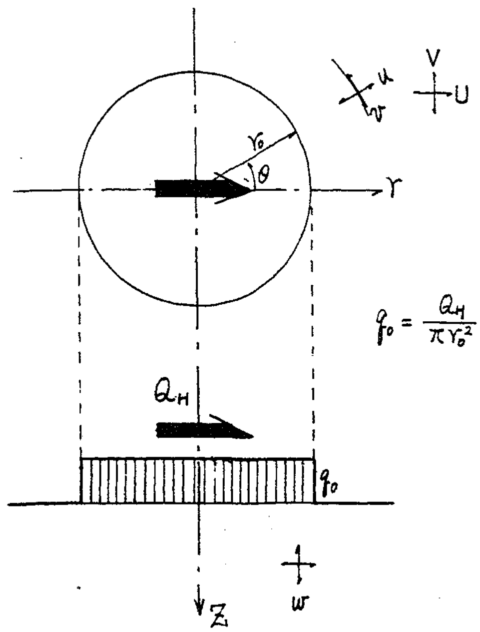

Fig.1

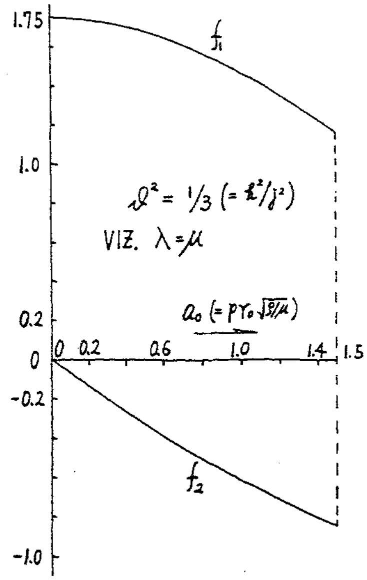

Fig.2 f_1, f_2 curves for horizontal vibration

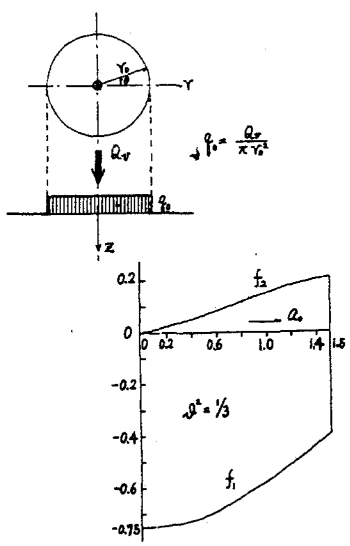

Fig.3 f_1, f_2 curves for vertical vibration

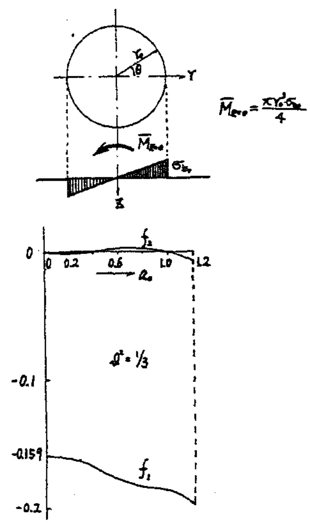

Fig.4 f_1, f_2 curves for rocking vibration

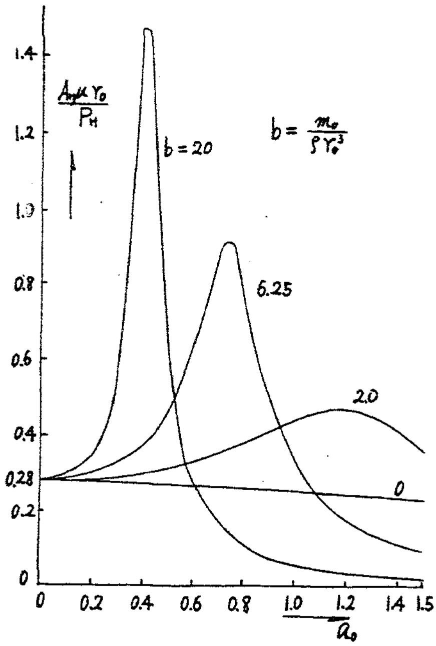

Fig.5 Amplitude curves for horizontal vibration

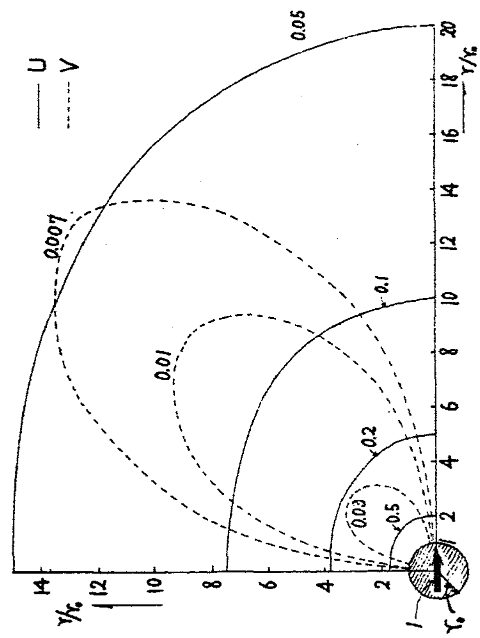

Fig.7 Contour line for horizontal displacement (statical)

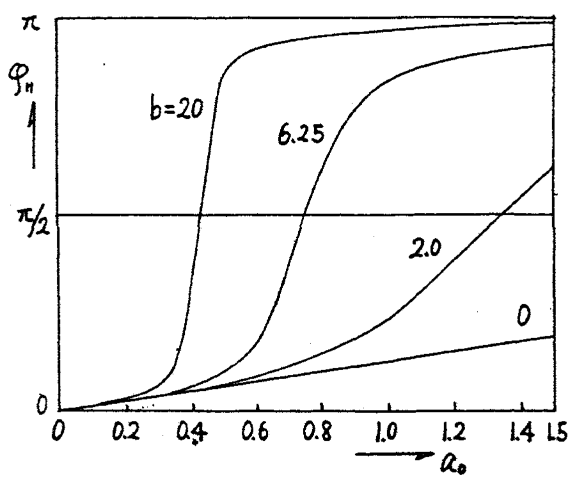

Fig. 6 Phase angle curves for horizontal vibration

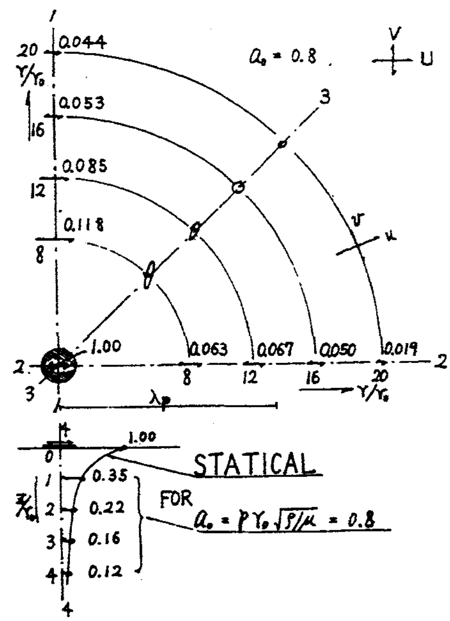

Fig.8 Amplitude distribution in vicinity of foundation

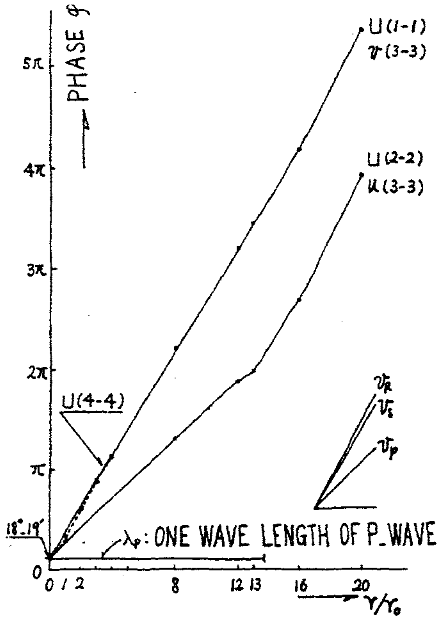

Fig.9 Wave velocity in vicinity of foundation for horizontal vibration

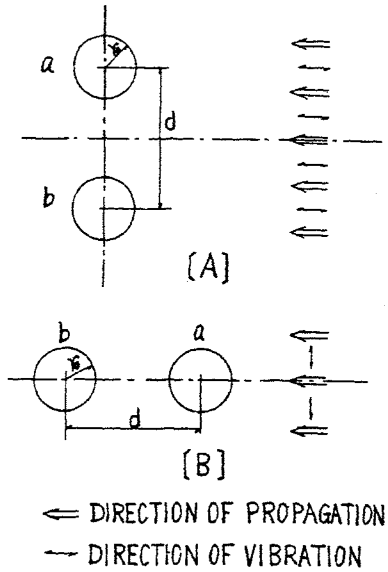

Fig. 10

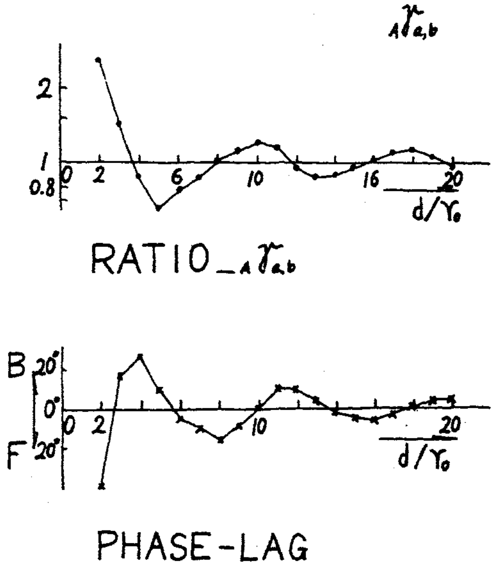

Fig. 11 γ & phase-lag curves for case-A in Fig.10

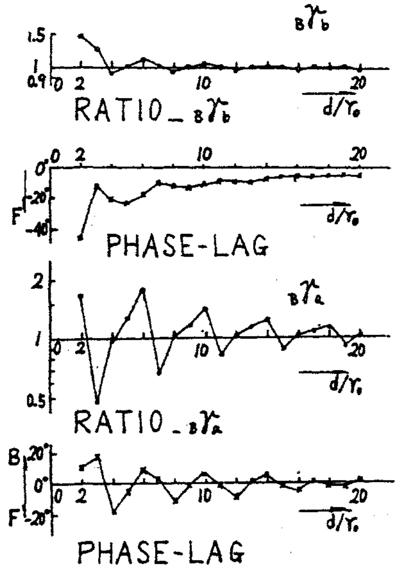

Fig. 12 γ & phase-lag curves for case-B in Fig. 10.

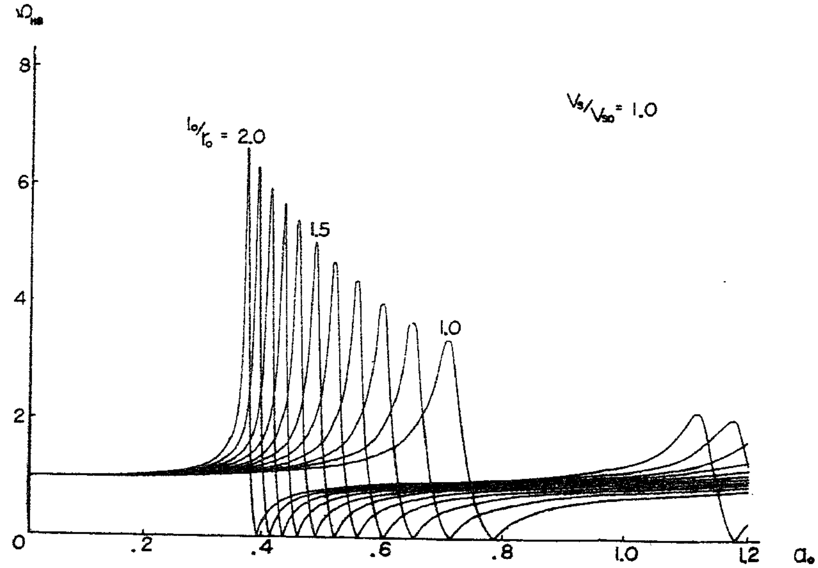

Fig.13 Amplitude of the base of structure for the case $V_s/V_{so}=1.0$

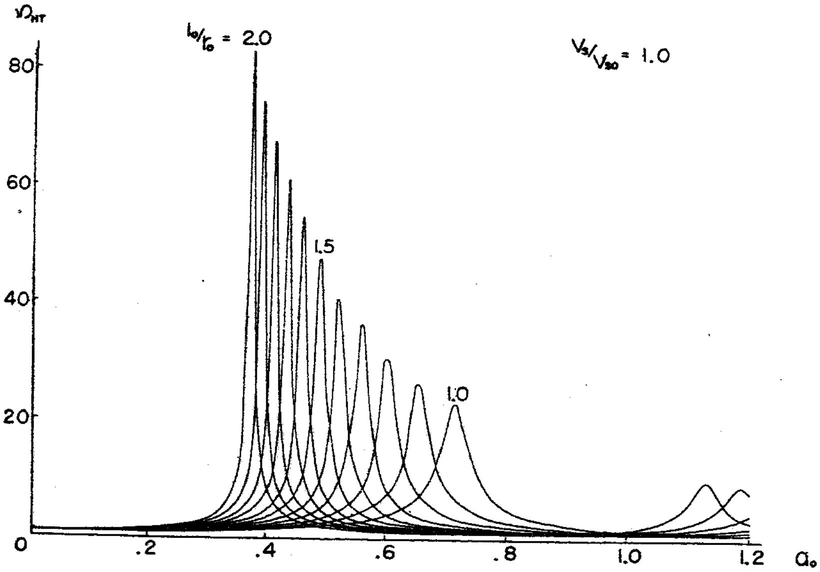

Fig.14 Amplitude of the top of structure for the case $V_s/V_{so}=1.0$

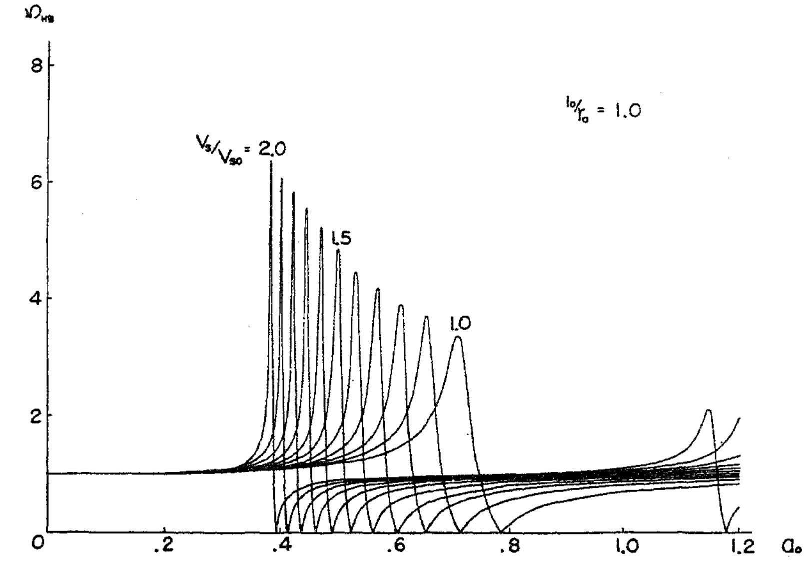

Fig.15 Amplitude of the base of structure for the case $l_o/r_o = 1.0$

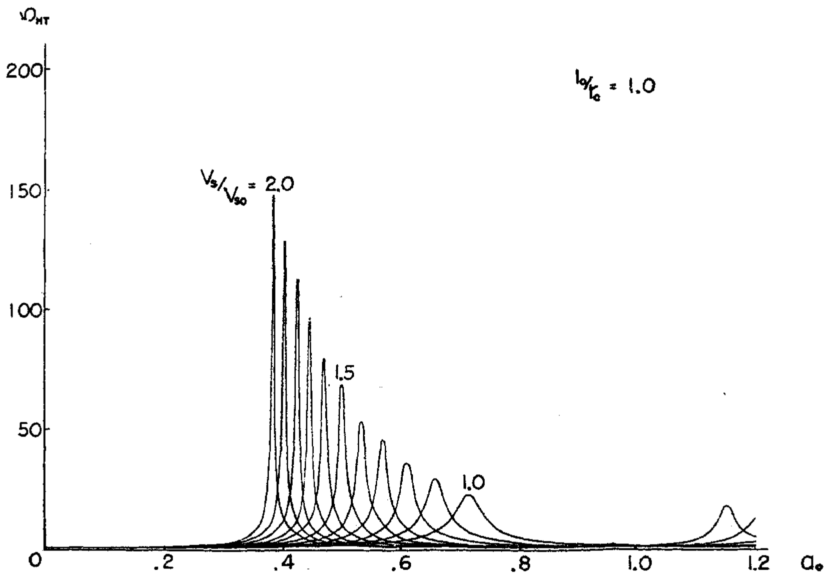

Fig.16 Amplitude of the top of structure for the case $l_o/r_o = 1.0$

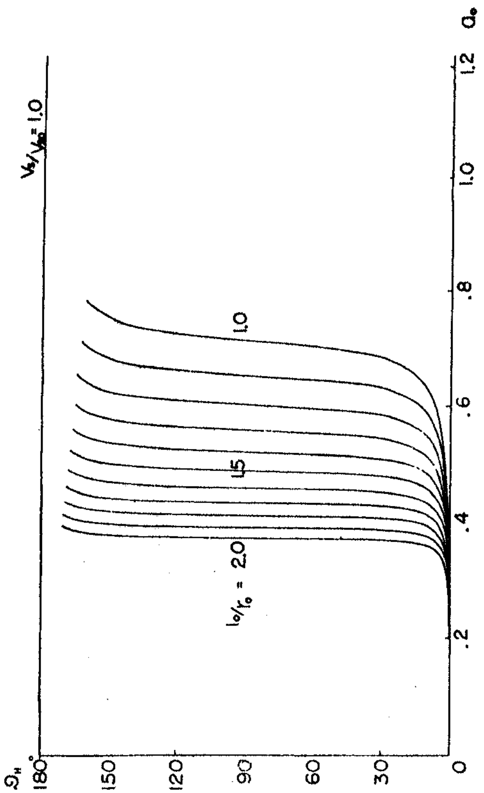

Fig.17 Phase difference between the incident waves and the oscillation of structure in the case of $V_s/V_{s0} = 1.0$

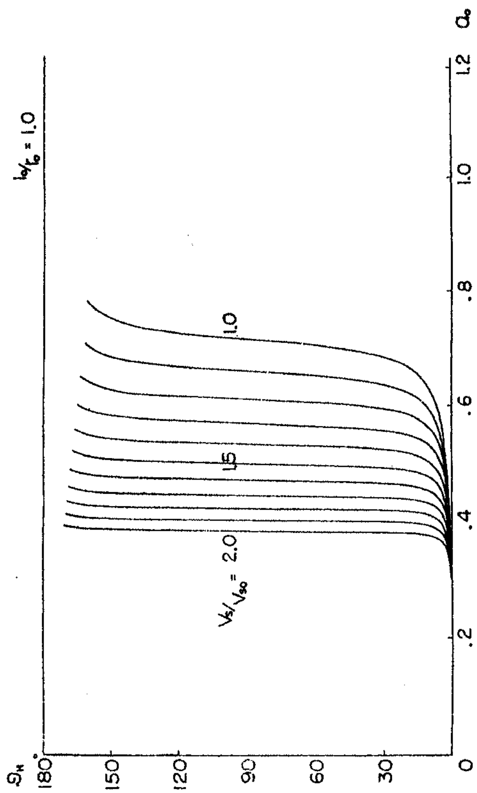

Fig.18 Phase difference between the incident waves and the oscillation of structure in the case of $l_o/r_o = 1.0$

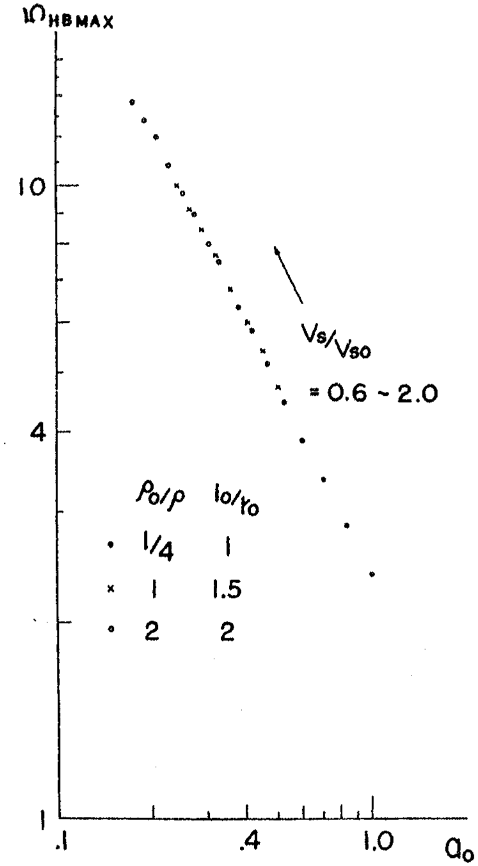

Fig. 19 Frequency which give the maximum amplitude of the base of structure

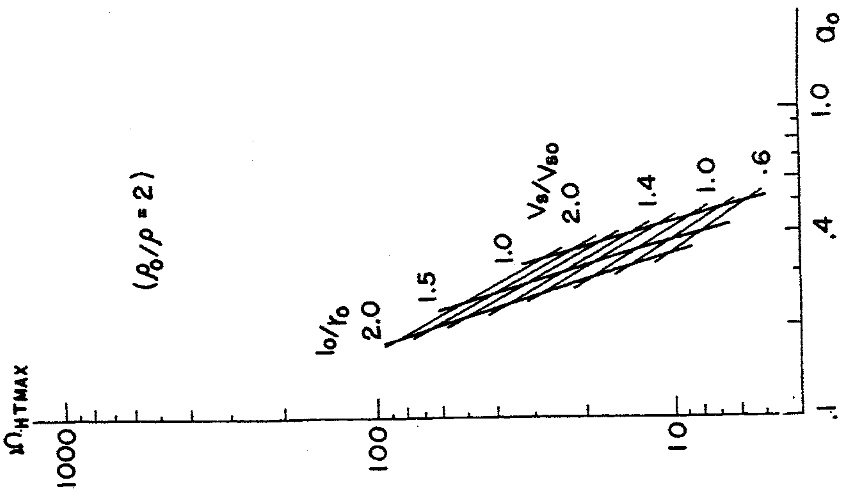

Fig. 22

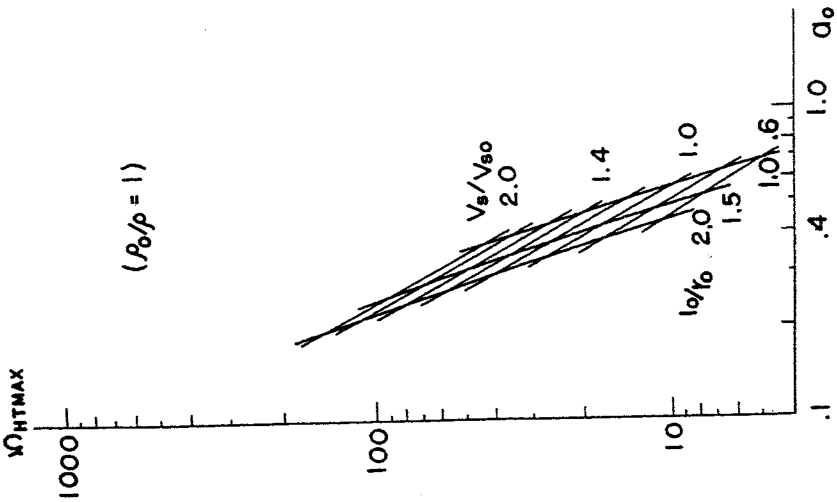

Fig. 21

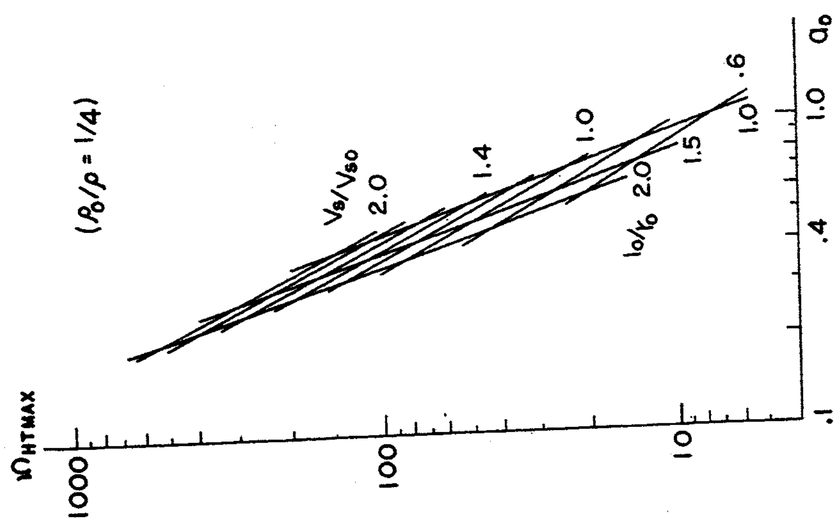

Fig. 20

Maximum amplitudes of the top of structure for various density ratio

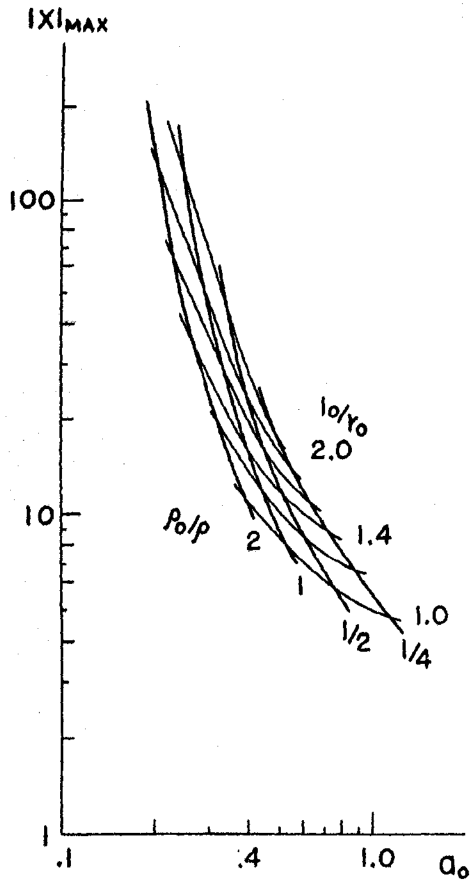

Fig.23 Maximum amplitude of the center of gravity in horizontal motion.

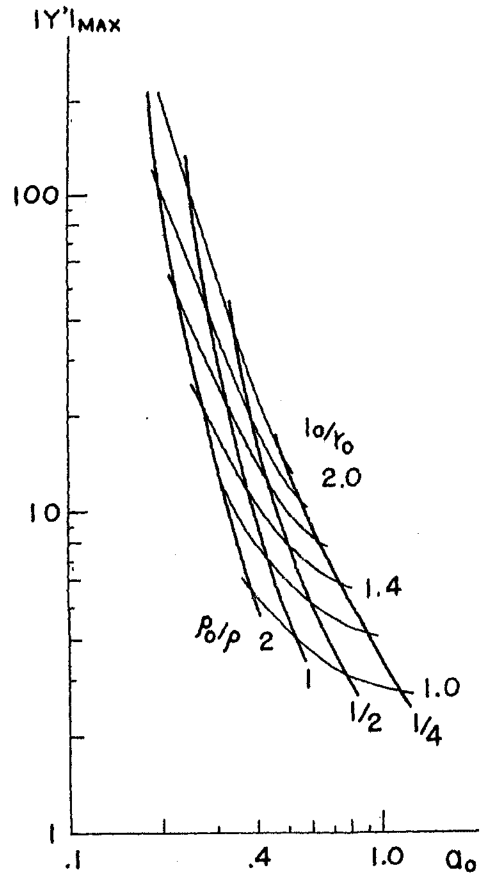

Fig. 24 Maximum displacement of the center of gravity caused by angular motion.

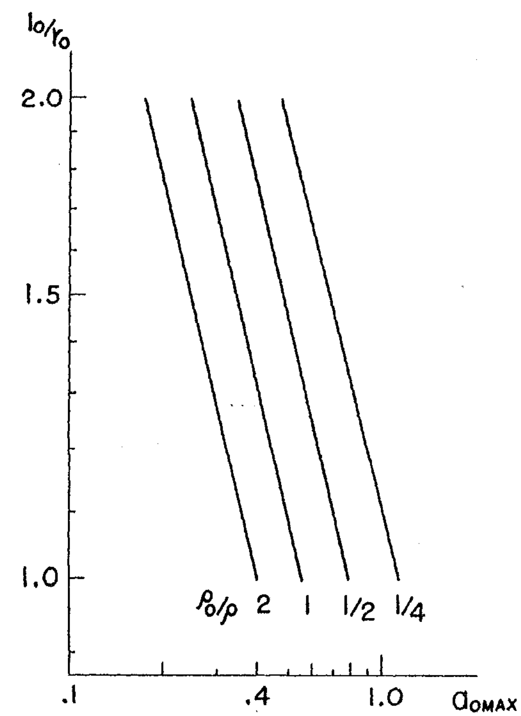

Fig. 25 Frequency which gives the maximum amplitude in rocking motion.

DISCUSSION

M. Hakuno, University of Tokyo:

In applying your computed data in practice, what is the most important consideration to be taken?

I. Toriumi:

I tried to apply these theoretical results to several real foundations of horizontal type compressor machines. So the check is for horizontal vibration.

From by experience, the values of displacement in foundation evaluated from my theoretical results are about two times larger than the measured ones. So the ground seems as if it is more hard.

I think this is caused from two factors mainly that is,

1. the distribution of displacement under the area of foundation was not found to be uniform in my calculation, even though the stress distribution is uniform.

2. the lower half of the real foundation is imbedded in the ground.