VIBRATIONS IN FOUNDATION, STRUCTURE
AND IN ITS VICINITY ON THE ELASTIC GROUND

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Part I (Toriumi)

Introduction

Vibrations in foundations may be due to either one of two causes, viz.: they may be caused by a train of waves passing through the ground, or they may be set up by fluctuating load on the foundation itself. These problems are similar essentially irrespective of the sign in propagation of waves, and of the sort of foundation for machine or building.

We treat these problems theoretically as a propagation of tremors over the surface of a semi-infinite elastic ground. Thus the vibrations in foundation of machine, vibrations of elastic structure on the elastic ground and the distribution of vibration in vicinity of these foundations and moreover the interference in vibration between two foundations etc. can be solved in series.

1. Fundamental

For the elastic solid in cylindrical coordinates, the displacements are given by

\[ U = \left[ A \frac{\partial^2 H_m(\theta r)}{\partial r^2} e^{\pm i \omega t} + B \frac{\partial H_m(\theta r)}{\partial r} e^{\pm i \omega t} + C \frac{H_m(\theta r)}{r} e^{\pm i \omega t} \right] e^{i \omega t} \cos \theta \theta \quad \text{etc.} \]

where \( A_m \), \( B_m \) and \( C_m \) are unknown coefficients. The stresses are given by

\[ \sigma_r = \mu \left[ A_m \frac{\partial^2 \theta^2 e^\pm i \omega t}{\partial \theta^2} + B_m \frac{\partial \theta e^\pm i \omega t}{\partial \theta} + C_m \theta^2 e^\pm i \omega t \right] e^{i \omega t} \cos \theta \theta \quad \text{etc.} \]

Horizontal force transmitted to the ground \( Q_r = \pi r^2 f_0 \) acts on the circular area with radius \( r_0 \) on the surface of the ground. (Fig. 1)

B.C. 1) \( \theta = 0 : \sigma_r = 0 \)

2) \( \theta = 0, \ r > r_0 : \ T_{rr} = T_{r\theta} = 0 \)

3) \( \theta = 0, \ r = r_0 : \ T_{rr} = -q_0 \cos \theta e^{i \omega t}, \ T_{r\theta} = q_0 \sin \theta e^{i \omega t} \)

\( A_m, B_m \) and \( C_m \) in eq. 1 are decided by the B.C. adapting Fourier-Bessel integral. Thus horizontal displacement at the center of circular area \( (r=0) \) is given by

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\[ U = \frac{e^{\gamma_0}}{2 \mu} e^{i \epsilon t} \int_0^\infty \frac{F(\xi)}{\beta F(\xi)} J_1(\xi \gamma_0) d\xi \]
\[ = \frac{Q_H}{2 \mu \pi \gamma_0} \left( f_1 \cos pt - f_2 \sin pt \right) \quad \text{(Real Part)} \]

The real part \( f_1 \) and the imaginary part \( f_2 \) of the above integral evaluated in the case \( \lambda = \mu \) is shown in Fig. 2 in which abscissa denotes frequency parameter \( Q_0(= \gamma_0 m/2\mu) \).

We get similar results for vertical and rocking vibrations (Fig. 3 and 4).

\[ \omega_{z, y=0} = \frac{Q_0}{2 \mu \gamma_0} \left( f_1 \cos pt - f_2 \sin pt \right) \]

\[ \gamma = \frac{\omega_{y=0}}{\gamma_0} = \frac{4M_m}{\pi \gamma_0} \left( f_1 \cos pt - f_2 \sin pt \right) \]

2. Vibrations in foundation

When additional mass is set on the circular area and unbalanced force \( P_H \) (independent of circular frequency \( \rho \)) acts upon it, equation of horizontal vibration of this foundation is given by

\[ Q_H \cos pt + \frac{\partial U}{\partial t^2} = P_H \cos (pt + \epsilon) \]

where \( \epsilon \) is phase angle of \( P_H \) to \( Q_H \).

Substituting eq. 3 to eq. 6, we obtain

\[ Q_H = P_H \left[ (1 - \frac{\rho_0}{\rho H_0} f_1) \right]^{\frac{1}{2}} \]

Substituting eq. 7 again to eq. 3, we obtain the amplitude \( A_H \) of displacement of the foundation as under

\[ A_H = \frac{P_H}{2 \mu \pi \gamma_0 \left( (1 - \frac{\rho_0}{\rho H_0} f_1) + (\frac{\rho_0}{\rho H_0} f_2) \right)} \]

The phase angle of displacement of the foundation to unbalanced force \( P_H \) is given by

\[ \phi_H = \frac{1}{\tan^{-1} \left( \frac{-f_2}{f_1 - \frac{\rho_0}{\rho H_0} (f_1^2 + f_2^2)} \right)} \]

The behaviors of \( A_H \) and \( \phi_H \) to frequency parameter \( \omega_0 \) are shown in Fig. 5 and 6 at some values of mass parameter \( \rho(=\rho_0/\rho H_0) \). We get similar results for vertical and rocking vibrations. From these curves we get generalized results as under--

The semi-infinite elastic solid itself has no natural frequency, however if an additional mass is set at the finite area of the surface, resonant curves appear. But by the dispersion to the infinite solid, damping acts to resonance.
3. Vibrations in vicinity of foundation

Distributions of vibrations in vicinity of foundations of machine are peculiar. Distribution of statical displacements are featureless (Fig. 7), however it turns to complex type in the case of vibration. This problem is in domain where the surface wave is formed, and this phenomenon seems to have relation essentially to the problem of sound field near the radiator although the generated waves are more complex.

From the equation of vibration in semi-infinite elastic solid, horizontal components U, V of displacement at any point \( z=0; \gamma, \delta \) on the surface and under ground (z-axis) are given by

\[
\begin{align*}
U_{z=0}(Y, \theta) &= \frac{\gamma \rho \sigma}{2 \mu} e^{i\omega t} \int_{0}^{\infty} \left[ \frac{\sigma (0, k)}{\rho F(k)} J_{1}(kr) J_{1}(kr) \right] d \kappa - (2a^2 \omega - \nu) \int_{0}^{\infty} \frac{\sigma (0, k)}{\rho F(k)} J_{1}(kr) J_{1}(kr) d \kappa \\
V_{z=0}(Y, \theta) &= \frac{\gamma \rho \sigma}{2 \mu} e^{i\omega t} \int_{0}^{\infty} \frac{\sigma (0, k)}{\rho F(k)} J_{1}(kr) J_{1}(kr) d \kappa \\
U_{\gamma=0}(Z) &= \frac{\gamma \rho \sigma}{2 \mu} e^{i\omega t} \int_{0}^{\infty} \left[ -z \kappa \kappa (e^{-\kappa^2} - e^{-k^2}) + \frac{\sigma (0, k)}{\rho F(k)} \right] J_{1}(kr) J_{1}(kr) d \kappa
\end{align*}
\]

The distribution diagrams of U and V in four lines (three on the surface and one under ground) are shown in Fig. 8. From the phase angle at some points \( \phi-V/\psi \) curves are drawn and the wave velocity is decided. (Fig. 9)

Results:
1) On line 1-1 S-wave velocity appears and the distribution of amplitude is simple.
2) On line 2-2 from origin to \( r/r_o = 8 \sim 12 \) P-wave velocity appears and then changes to Rayleigh wave velocity at a distance. This changing point of wave velocity is comparatively distinct and it is nearly situated in one length of P-wave from the origin.
3) On line 3-3 standing for lines ( \( 0^\circ \sim 90^\circ \) ), U and V components viz. u and v components reduced to cylindrical coordinates coexist at some amplitude ratio. Besides velocities of these u and v components are the same as on line 2-2 and 1-1 respectively. So the locus at any point becomes to loop.
4) On line 4-4 it shows S-wave velocity and the distribution of amplitude is simple and nearly equal to statical case.

4. Interference in vibration between two foundations

By combining the steady vibration in foundation itself (division 2) with the distribution of vibration in vicinity of foundation (division 4), the interferences in vibration between two foundations are solved.

Though there are many cases in combinations, here two cases A and B as under are treated. (Fig. 10)
1) Two foundations \(a\) and \(b\) are the same.

2) Relations to direction of wave propagation and direction of wave vibration are indicated in Fig. 10.

These are simple cases in many combinations, because the wave of interference is only S-wave in these cases and line 1-1 in Fig. 8 is of simple character in many radial lines from the origin.

Case-A:

It is symmetrical and the behaviours of two foundations \(a\) and \(b\) are the same. The velocity of propagation of tremors \(U_1\) is optional.

Original ground motion;

\[ U_1 = A_1 e^{j \omega t} \]

Term of compensation in foundation \(a\) to \(U_1\); \(U_2 = A_1 E (f_1^a + f_2^a) e^{j \omega (\varphi - \varphi)} \)

Propagation of term of compensation in vicinity of foundation \(a\);

\[ U_2 = -K A_1 E (f_1^a + f_2^a) e^{j \omega (\varphi - \varphi - \delta)} \]

Term of compensation in foundation \(b\) to \(U_1\); \(U_2 = K A_1 E (f_1^b + f_2^b) e^{j \omega (\varphi - \varphi - \delta)} \)

GENERAL TERM OF COMPENSATION:

\[ \alpha U_2 = (-)^\alpha A_1 K^{(\alpha+\gamma)} E (f_1^b + f_2^b) e^{j \omega (\varphi - \varphi - \delta)} \]

GENERAL TERM OF PROPAGATION:

\[ \alpha U_2 = (-)^\alpha A_1 K^{\gamma} E (f_1^a + f_2^a) e^{j \omega (\varphi - \varphi - \delta)} \]

where \(\varphi = \frac{\omega}{c} \frac{f_1^a}{f_2^a}, \delta = p d / v_0\). Thus horizontal displacements in foundation \(a\) & \(b\) are given by:

\[ U = U_1 + (\alpha U_1 + \alpha U_2) + (\alpha U_1 + \alpha U_2) + \cdots + (\alpha U_1 + \alpha U_2) + \cdots \]

\[ = A_1 e^{j \omega t} \left[ 1 - G e^{-j \omega (\varphi - \varphi)} \right] \]

\[ \times \left[ \frac{1 + \delta e^{j \omega (\varphi - \varphi)}}{1 + \delta e^{j \omega (\varphi - \varphi)}} \right] \]

where \(G = E (f_1^a + f_2^a)^{\frac{1}{2}}, \kappa\) : Amplitude at distance \(d\) from origin on line 1-1 (Fig. 8)

The series in eq.11 converges in domain \(|K| < 1\). As horizontal displacement in single foundation is given by \(U = U_1 + U_2\), so the ratio of terms of compensation is shown by

\[ \frac{\text{Term of compensation in case-A}}{\text{Term of compensation in single}} = \frac{1 + \kappa e^{-j \omega}}{1 + K G e^{-j \omega (\varphi - \varphi)}} = \alpha + \beta. \]

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Evaluating \( a_i \) \& \( b_i \), in complex number of eq. 12, the ratio of amplitude of case-A to single is shown by \( (a_i^2 + b_i^2)^{1/2} \) and phase-lag is shown by \( \tan^{-1} b_i/a_i \).

**Case-B:**

In this case, foundations \( a \) \& \( b \) are different in phase to receive the original wave, so the effects of interference in \( a \) \& \( b \) are not the same. On the contrary to case-A, case-B is not independent of the velocity of propagation of tremors. It seems to be suitable here to adopt the velocity of S-wave. Further it is supposed that there are no damping in amplitude of original wave in the distance \( d \) between two foundations as original tremors is very strong.

By the similar process to case-A, ratio of term of compensation for foundation \( b \) to single is given by

\[
\gamma_b = \frac{(1 + K) - K \alpha e^{-2(\theta + \phi)} - K \alpha e^{-2(\theta + \phi + 2\theta)}}{1 - K \alpha^2 e^{-2(\theta + \phi)}}
\]

and for foundation \( a \)

\[
\gamma_a = \frac{1 + K \alpha e^{-2\theta} - (1 + K) K \alpha e^{-2(\theta + \phi + 2\theta)}}{1 - K \alpha^2 e^{-2(\theta + \phi)}}
\]

In Figs. 17 \& 12, the curves of \( \gamma \) and phase-lag of eq. 12, 13 \& 14 evaluated for frequency parameter \( a_0 = 0 \) \& \( d/\gamma_0 = 2 \sim 20 \) are shown.

**Results:**

1) The curves decrease damping oscillationally to the distance of two foundations in both cases A and B as expected.

2) But they differ in pitch about two times, that is the pitch in case-A agrees with about one wave length of S-wave and the one in case-B agrees with a half wave length of S-wave.

3) In case-B, foundation \( a \), which receives farther the ground motion is affected relatively more notably than foundation \( b \), and foundation \( b \) is forward in phase always to compare with single foundation.
Part II (Satō and Yamaguchi)

Shear vibration of a continuous circular column, excited by ground motion, was discussed considering the coupling effect of the vibrating body and the elastic foundation, the calculation being based on the theory of Toriumi.

Maximum values of the response curves of rocking motion are given taking various values of the ratio of the density of structures and that of the foundation.

Empirical formulas are also presented which hold between the parameters giving maximum response.

5. Shear vibration of continuous circular columns

In the present study structures are assumed to be continuous, and purely horizontal disturbance with amplitude \(A_H\) and frequency \(p\) travel vertically upward and is reflected at the surface.

At the free surface the displacement should be \(U_1 = 2A_H \exp (izpt)\), but the extra displacement \(U_p = U \exp (izpt)\) must be added because of the existence of a structure upon the ground. Displacement of the structure at height \(\chi\) relative to the base is expressed by \(\gamma = \gamma \exp (izpt)\).

The equation of motion connecting above quantities is

\[
\frac{d^2}{d\chi^2} \left( U_1 + U_0 + \gamma \right) = \frac{\mu_0}{f_0} \frac{d^2\gamma}{d\chi^2}
\]

Boundary conditions are:

At the base of the structure \(\chi = 0\) : \(\gamma = 0\) and \(E_0 \mu_0 \frac{d\gamma}{d\chi} = A_H\)

At the top of the structure \(\chi = 2L_0\) : \(E_0 \mu_0 \frac{d\gamma}{d\chi} = 0\)

Combining these expressions we have the following equation, which gives the response at the base of structure

\[
U_{\text{base}} \exp (-iz\chi) = \left( 2A_H + U \right) / 2A_H
\]

\[
= \frac{1}{1 - \frac{1}{2} \frac{f_1 V_0}{2 V_0 a_0} (f_1 + izf_2) \tan \left( 2 \frac{V_0 a_0}{V_0 V_0 a_0} \right)}
\]

and at the top

\[
U_{\text{top}} \exp (-iz\chi) = \left( 2A_H + U + \gamma \right) x = 2L_0 / 2A_H
\]

\[
= U_{\text{base}} \exp (-iz\chi) / \cos \left( 2 \frac{V_0 a_0}{V_0 V_0 a_0} \right)
\]
Results of numerical computation of these quantities are given in Figs. 13-18.

6. Empirical formulas

There is a certain relation between the maximum amplitude of the base movement and the corresponding frequency. (See Fig. 19) This relation is easily obtained as follows with a good accuracy. From (16)

\[ V_{HB} \exp (-i \phi_h) = \frac{1}{\sqrt{1 - \frac{1}{2} \left( \frac{f_1}{f_2} \right)^2 \tan^2 \left( 2 \frac{V_s}{V_{50}} a_o \right)}} \]

\[ = \frac{1}{\sqrt{1 - c_o \left( \frac{f_1}{f_2} \right)^2}} \]

\[ = \exp (-i \phi_h) / \sqrt{1 - (c_o f_1)^2} \]

If \( f_2 \) does not change too rapidly, \( V_{HB}^{\text{MAX}} \) is obtained putting simply \( 1 - c_o f_1 = 0 \) in the above expression. Then we have

\[ V_{HB}^{\text{MAX}} = \frac{1}{c_0 f_2} = \frac{f_1}{f_2} \approx 2.45 / a_o \]

For the top of the structure we have the next relation from Fig. 20,

\[ V_{HT}^{\text{MAX}} \approx 11.5 \left( \frac{V_s}{V_{50}} \right) / a_o \]

\[ V_{HT}^{\text{MAX}} \approx 8.5 \left[ \left( \frac{a_o}{V_s} \right) a_o \right] \]

from which the next formula is obtained.

\[ a_o \text{MAX} \approx 0.74 \left( \frac{V_s}{V_{50}} \right) \]

Therefore the period which gives the maximum \( V_{HT}^{\text{MAX}} \) is

\[ T_{\text{MAX}} \approx 8.5 \left( \frac{a_o}{V_s} \right) \]

Similar formula was obtained before by K. Suyehiro. He did not consider the coupling effect of the structure and foundation, but only assumed a definite displacement at the base of the structure. His formula has the same form with (23), but the coefficient was not 8.5 but 8.0.

Simple formulas were not obtained for the case other than \( P_o / \rho = 1/4 \), but the effect of \( P_o / \rho \) can be seen in Figs. 21 and 22.

7. Maximum response for the rocking motion

The calculation of rocking motion was carried out assuming parameters covering wider range than before. However, simple relations like our previous study were not deduced. Only the formula for \( T_{\text{MAX}} \) is obtained as follows.

\[ T_{\text{MAX}} \approx 12 \sqrt{R_o / \rho} \left( \frac{a_o}{V_s} \right) \]

Results are given in the Figs. 23-25.
Bibliography


2. I. Toriumi, "Vibrations in vicinity of foundations of machines", ibid., 6, 296 (1958)

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Nomenclature

\[ a_s = \rho_s \sqrt{f} / \rho = \rho_s / v_s = 2\pi v_s / (\text{wavelength of S-waves}) \]

\[ E_s \] Cross section of the structure

\[ t_1, t_2 \] Cf. eq. 3 etc.

\[ l \] Height of the center of gravity of the structure

\[ Q_H \] Horizontal force propagated to the foundation

\[ R \] Radius of the structure

\[ V_s, V_0 \] Velocity of S-waves propagated in the foundation and the structure

\[ \mu, \mu_o \] Rigidity of the foundation and the structure respectively

\[ \rho, \rho_o \] Density of the foundation and the structure respectively

\[ \varphi_H \] Phase difference between the incident waves and the oscillation of structure

\[ \rho_{NB} \] Amplitude of the base of structure

\[ \rho_{NT} \] Amplitude of the top of structure
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Fig. 1

\[ f_0 = \frac{Q_0}{\kappa \gamma_0^2} \]

Fig. 2 \( f_1, f_2 \) curves for horizontal vibration

\[ a_{\alpha} = (p \gamma_{\alpha} \beta \alpha) \]

Fig. 3 \( f_1, f_3 \) curves for vertical vibration

Fig. 4 \( f_1, f_4 \) curves for rocking vibration
Fig. 5 Amplitude curves for horizontal vibration

Fig. 6 Phase angle curves for horizontal vibration

Fig. 7 Contour line for horizontal displacement (statical)

Fig. 8 Amplitude distribution in vicinity of foundation
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**Fig. 9** Wave velocity in vicinity of foundation for horizontal vibration

**Fig. 10**

**Fig. 11** $\gamma$ & phase-lag curves for case-A in Fig. 10

**Fig. 12** $\gamma$ & phase-lag curves for case-B in Fig. 10
Fig. 13  Amplitude of the base of structure for the case $V_s/V_{so}=1.0$

Fig. 14  Amplitude of the top of structure for the case $V_s/V_{so}=1.0$
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Fig. 15 Amplitude of the base of structure for the case \( \delta_0/\nu_0 = 1.0 \)

Fig. 16 Amplitude of the top of structure for the case \( \delta_0/\nu_0 = 1.0 \)
Fig. 17 Phase difference between the incident waves and the oscillation of structure in the case of $V_s/V_{s0} = 1.0$

Fig. 18 Phase difference between the incident waves and the oscillation of structure in the case of $I_s/I_{s0} = 1.0$

Fig. 19 Frequency which give the maximum amplitude of the base of structure
Maximum amplitudes of the top of structure for various density ratio
Fig. 23 Maximum amplitude of the center of gravity in horizontal motion.

Fig. 24 Maximum displacement of the center of gravity caused by angular motion.

Fig. 25 Frequency which gives the maximum amplitude in rocking motion.
M. Hakuno, University of Tokyo:

In applying your computed data in practice, what is the most important consideration to be taken?

I. Toriumi:

I tried to apply these theoretical results to several real foundations of horizontal type compressor machines. So the check is for horizontal vibration.

From by experience, the values of displacement in foundation evaluated from my theoretical results are about two times larger than the measured ones. So the ground seems as if it is more hard.

I think this is caused from two factors mainly that is,

1. the distribution of displacement under the area of foundation was not found to be uniform in my calculation, even though the stress distribution is uniform.

2. the lower half of the real foundation is imbedded in the ground.