STATISTICAL THEORY OF THE ASEISMIC DESIGN OF STRUCTURES

By

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There are three different approachs to the design of structures on seismic loading. The first one consists in representation of values describing seismic action by means of some deterministic function of time. Different kinds of such function were considered in references [1 - 3]. It is obvious however that any analytical laws have nothing in common with actual chaotic earthquake accelerograms.

The second, semiempirical approach consists in the investigation of the response of the structure to the strongest earthquakes which had taken place in the past and which are typical for seismic region considered. In this direction some investigations and calculations were made [4 - 7]. The value of risk and the life time of the structure expected could not be defined by this method. The structure designed by this semiempirical method may be destroyed by the earthquake having less maximum acceleration but more infavourable spectrum and phase characteristics.

The third approach is based on probability methods. It is well known that seismic forces have random character. In references [8 - 9] the ground acceleration was represented in the form of random set of noncorrelated impulses. This scheme is far from being practicable. The correlation of accelerations in different time moments is of great importance. It is expressed in presence of predominant periods in accelerograms and in heightened sensibility of structures with definite free-oscillation frequency to seismic forces.

A further development of the theory or the aseismic design of structures must take in the direction of a wider use of methods of the probability theory and the mathematical statistics. The difficulties connected with application of statistical methods to this problem are great, but they are typical for many other applications. They principally conclude in the lack of information accumulated by the impossibility of sharp increase of information in the nearest future. The most important problem is to create a theory which could secure an adequate description of response of a structure on seismic loading with minimum quantity of empirical data required.

An approach to the creation of such theory is presented in this paper. It is supposed that seismic action may be described by means of some nonstationary random functions of time depending on a finite number of random parameters. For these parameters which characterize the earthquake as a whole (its magnitude, duration, spectrum etc) density of

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probability distribution may be found. With the values of parameters fixed each nonstationary random function is approximately expressed by deterministic functions of time and stationary random functions the spectral densities of which are also determinated from the analyse of accelerogram. When a complete system of correlation functions for seismic action is known it is possible to find a complete system of correlation functions for generalizated coordinates, bending moments, shearing forces, stresses etc and then the momentary probability density of values mentioned may be found. A further investigation consists in the determination of mean quantity of exeeding of fixed level and in determir-tion of exeeding probability during the earthquake. This probability is conventional; the complete probability may be found by use of probability density for parameters characterizing the earthquake as a whole. The final results for the single-degree-of-freedom system may be presented in the form of curves analogous to the well known acceleration spectra. We obtain a set of these curves depending on the probability of the prescribed term of exploitation.

1. General considerations. Let us introduce the generalized coordinates corresponding to the complete division of unknown functions in the equations if small oscillation of a linear system. Let $u(x,y,z,t)$ be the displacement of each point of system, $u_k(x,y,z)$ are the forms of small free oscillations. Prescribing the displacement in the form

$$u(x,y,z,t)=\sum_{k=1}^{n} f_k(t)\,u_k(x,y,z) \tag{1.1}$$

we obtain a set of independent equations for linear systems with viscous damping:

$$\ddot{f}_k + 2\varepsilon_k \dot{f}_k + \omega_k^2 f_k = -a_k(t) \qquad (k=1,2,\ldots.. n) \tag{1.2}$$

Here ω_k and ε_k are natural frequences and damping coefficients, $a_k(t)$ - generalized accelerations which may be easily expressed by components of the ground accelerations and by the functions $u_k(x,y,z)$. The equations (1.2) are written for the case of viscous damping, but further results will be true for any linear system.

Let us suppose that the generalized accelerations $a_k(t)$ are some nonstationary random functions of time and functions of a finite number of random parameters q_1, q_2, ... q_m, characterizing the earthquake as a whole:

$$a_k = a_k(q_1, q_2, \cdots q_m; t) \tag{1.3}$$

In the capacity of such parameters we may assume the characteristic of accelerogram envelope (maximum acceleration, duration of the earthquake, constants characterizing the increasing and decreasing of acceleration) and the characteristics of spectrum composition (predominant frequency, width of spectrum or some other quantities describing the correlation of acceleration on different time moments etc). For the evaluation of structure survival only strong earthquakes are interesting. Below we shall assume that for the seismic region considered a whole $\mathcal{N}$ -number of strong earthquakes at a definite period of T years and the joint probability density for parameters $q_1, q_2, \ldots q_m$ are known:

$$p_q = p_q(q_1, q_2, \cdots q_m) \tag{1.4}$$

When parameters q_j are fixed the random functions have the same external indications (the same envelope and spectrum) and only phase correlations are different. The change of phase correlations may have an essential influence on earthquake-resistance of structure causing infavourable condensation of maxima etc.

Let us introduce the complete system of correlation functions for : $a_K(t)$

$$K_{a_{\beta_1}a_{\beta_2}\cdots a_{\beta_s}}(t_1, t_2, \cdots t_s) = \overline{a_{\beta_1}(t_1)a_{\beta_2}(t_2)\cdots a_{\beta_s}(t_s)} \tag{1.5}$$

The ensemble average which is taken by fixed values of $q_1, q_2, \ldots q_m$ is destignated with a line. For the calculation of this average a large quantity of accelerograms has to be known. In our case however only a principal possibility of such averaging is important. Further we shall show that under some assumptions the function system (1.5) may be constructed in a simpler way.

If the complete system of correlation functions for generalized accelerations $a_K(t)$ is known the complete system of correlation functions for generalized coordinates $f_K(t)$ is determinated by following formula:

$$K_{f_{\beta_1}f_{\beta_2}\cdots f_{\beta_s}}(t_1, t_2, \ldots t_s) =$$

$$= (-1)^s \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_s} h_{\beta_1}(t_1-\tau_1)\cdots h_{\beta_s}(t_s-\tau_s) K_{a_{\beta_1}a_{\beta_2}\cdots a_{\beta_s}}(\tau_1, \tau_2, \tau_s)\, d\tau_1 d\tau_2 \ldots d\tau_s \tag{1.6}$$

In the formula (1.6) $h_K(t)$ is the impulsive transitional function for the operator by means of which the transformation $a_K(t)$ to $f_K(t)$ is realized. Correlation functions calculated by means of formula (1.6) also depend from parameters $q_1, q_2, \dots q_m$.

2. Simplifying assumptions for generalized accelerations. The composition of complete system of correlation functions (1.5) is a very difficult problem. The analysis of actual accelerograms shows that with the exception of a short initial time section and final time section with small amplitudes the spectral composition is sufficiently little changed. So the generalized accelerations may be represented in a form

$$a_k(t) = A_k(q_1, q_2, \dots q_r; t)\, y_k(q_{r+1}, \dots q_m; t) \tag{2.1}$$

where $A_k(t)$ - the deterministic function of time (the envelope of accelerogram depending on $q_1, q_2, \dots q_r$), $y_k(t)$ - stationary random function of time. Putting formula (2.1) into the formula (1.6) we obtain a formula for the second order correlation functions:

$$K_{f_j f_k}(t_1, t_2) =$$

$$= \int_0^{t_1}\int_0^{t_2} h_j(t_1-\tau_1) h_k(t_2-\tau_2) A_j(\tau_1) A_k(\tau_2) K_{y_j y_k}(\tau_1-\tau_2)\, d\tau_1 d\tau_2 \tag{2.2}$$

It is seen that for the solution of the problem it is sufficient to know the correlation functions for stationary random functions $y_k(t)$ depending on difference $t_1 - t_2$ only. In order to construct these functions immeasurably lesser quantity of records is demanded. Two methods are possible here. By the first one the envelope for given accelerogram must be found and all the ordinates must be divided by $A_k(t)$. So a sufficiently long realisation of stationary (more punctual, quasistationary) random function and the ensemble average may be replaced by time average. For sufficiently long earthquakes with slowly changing envelope parameters another method may be applied. The accelerogram is divided into sections which are sufficiently small so within their limits the function may be considered approximately as a stationary one, but are sufficiently large so each of them could be sufficiently representative. Then the time average for each section will be calculated.

The analysis of accelerogram shows that for the envelope in many cases the exponential law may be accepted:

$$a_k(t) = 0 \quad (t \leq 0), \quad a_k(t) = A_k e^{-ct} y(t) \quad (t > 0) \tag{2.3}$$

The simplest expression for the second correlation function apparently has a form

$$K_{\gamma\gamma}(\tau) = K_0 e^{-\alpha|\tau|} \cos\theta\tau \tag{2.4}$$

where K_0 is some constant, θ is the predominant frequency of the earthquake, α is the parameter characterizing the rate of correlation. The less is the magnitude of α, the stronger is the correlation i.e. the narrower is the diapason frequences in which the most part of seismic energy is concluded. The last is seen especially from the formula for spectral density

$$\Phi_{\gamma\gamma}(\omega) = \frac{1}{2}\,\frac{\alpha K_0}{(\omega-\theta)^2+\alpha^2} + \frac{1}{2}\,\frac{\alpha K_0}{(\omega+\theta)^2+\alpha^2} \tag{2.5}$$

According to the data of I. L. Kortchinsky [3] who worked up the accelerograms of nine weak earthquakes the magnitudes of parameters are in average $c = 0.15\ \mathrm{sec}^{-1}$, $\theta = 10 \div 25\ \mathrm{sec}^{-1}$. The magnitudes of α are not determinated till now.

Putting the formulae (2.3) and (2.4) into the formula (2.2) we shall obtain for the case of equation (1.2)

$$K_{f_j f_k}(t_1,t_2) = \frac{A_j A_k K_0}{\sqrt{(\omega_j^2-\varepsilon_j^2)(\omega_k^2-\varepsilon_k^2)}} \int_0^{t_1}\int_0^{t_2} \exp\left[-\varepsilon_k(t_1+t_2-\tau_1-\tau_2) - c(\tau_1+\tau_2) - \alpha|\tau_1-\tau_2|\right] \sin\left[\sqrt{\omega_j^2-\varepsilon_j^2}\,(t_1-\tau_1)\right] \sin\left[\sqrt{\omega_k^2-\varepsilon_k^2}\,(t_2-\tau_2)\right] \cos\theta(\tau_1-\tau_2)\, d\tau_1\, d\tau_2 \tag{2.6}$$

After integration we obtain the formula describing the alteration of functions $K_{f_j f_k}$. Some interesting numerical results are obtained.

For calculating the method of canonical expansions [10] may be also applied. On the ground of supposition (2.1) the functions $a_k(t)$ and $f_k(t)$ may be presented in the form

$$a_k(t) = \int_{-\infty}^{\infty} Z_k(\omega)\, y_k(\omega,t)\, d\omega, \quad f_k(t) = \int_{-\infty}^{\infty} Z_k(\omega)\, X_k(\omega,t)\, d\omega$$

where $\Xi_k(\omega)$ - random quantities, and canonical functions $\varphi_k(\omega,t)$ are

$$\varphi_k(\omega, t) = A_k(t)e^{i\omega t}$$

If the functions $f_k(t)$ satisfy the equation (1.2) for the finding of $\chi_k(\omega,t)$ we obtain the equation

$$\ddot{\chi}_k + 2\varepsilon_k\dot{\chi}_k + \omega_k^2\chi_k = -\varphi_k(\omega, t)$$

with initial conditions $\chi_k(0) = \dot{\chi}_k(0) = 0$. Then the correlation functions are determinated by means of the formulae

$$K_{f_j f_k}(t_1, t_2) = \int_{-\infty}^{\infty}\Phi_{\varphi_j\varphi_k}(\omega)\chi_j(\omega, t_1)\chi_k^*(\omega, t_2)\,d\omega \qquad (2.7)$$

($\Phi_{\varphi_j\varphi_k}$ is the joint spectral density for φ_j and φ_k, by the asterisk the complex conjugated number is destignated). More preferable are those formulae which are connected with lesser calculations in each specific case.

In conclusion let us note that when we cannot neglect the variation of spectrum composition during the earthquake two generalizations of the formula (2.1) may be done. Firstly the parameters $q_{r+1}, \ldots q_m$ may be considered as slowly varying time functions. Secondly the generalized acceleration may be represented as a sum of terms taking into account the varying of spectral properties by means of selecting the functions $A_k(t)$ and $\varphi_k(t)$.

3. The determination of expected term of structure lifetime. The principle problem of the aseismic design theory is the evaluation of the expected value of the lifetime and the development of methods of designing structures the expecting life term of which will be no less than the prescribed value. The determination of the system of correlation functions is the first step in the direction of the problem solving. The knowledge of them makes possible to find the law of time varying of mathematical expectation and of mean square of generalized coordinates. For engineering purposes however it is insufficient.

The second step consists in the determination of joint probability density for generalized coordinates considered in fixed time moment and by fixed values of random parameters $q_1, q_2, \ldots q_m$. It is the "momentary" probability density $p_t(f_1, f_2, \ldots f_n; t)$. For its determination we must know the complete system of moments for $f_1, f_2, \ldots f_n$ which are equal to the values of the correlation functions $K_{f_{p1}f_{p2}\cdots f_{ps}}(t_1, t_2, \ldots t_s)$ at $t_1 = t_2 = \cdots = t_s = t$. So we must solve some generalized problem of moment theory. This problem is very difficult and thus some simplifications must be made.

It is known that the components of ground accelerations, the horizontal components especially have distribution near to the simmetrical one: both directions of motion are approximately equiprobable. So the mathematical expectation as all the moments of odd orders may be assumed equal to zero. In connections with it the moments of second order and corresponding correlation functions become most significant. For a special case when "momentary" probability density is subjected to normal law it is sufficient to find the second order moments only:

$$p_f(f_1, f_2, \cdots f_n; t) = \frac{1}{\sqrt{2^n \pi^n |K(t)|}} \exp\left[-\frac{1}{2}\sum_{\substack{j=1\\k=1}}^{n} L_{jk}(t) f_j f_k\right] \quad (3.1)$$

Here $|K|$ is the determinant of the matrix K_{jk} , L_{jk} is the reciprocal matrix.

An additional investigation must be fulfiled which would show the approaching of the actual distribution to the normal one. It is well known that the normal distribution takes place when the random value is formed under the influence of a great number of statistical independent sufficiently small random causes. It may be possible that we have such situation by strong earthquake in points situated far enough from epicenter. In the cases where the distribution cannot be considered as a normal one it may be taken as normal for approximately evaluation of not very small probabilities.

"Momentary" probability density permits to find the probability of random event that generalized coordinates in any time moment are contained in the limits between f_k and $f_k + df_k$. But we must obtain the probability of exceeding of some dangerous value F during the whole earthquake. Let us calculate this probability supposing for simplicity that the generalized coordinates are statistical independent (this may be done, for example, when system damping is sufficiently small). Let us assume that $p = (f_k, \dot{f}_k; t)$ is the joint statistical probability density for coordinate and its derivative. Then the mean value of exceeding by the coordinate f_k of the level F in time unit is

$$\int_0^\infty p(F, \dot{f}_k; t) \quad \dot{f}_k \, d\dot{f}_k$$

The magnitudes f_k and $\dot{f}_k$ may be assumed statistically independent. The mean number of the exceedings of the level F during the whole earthquake is determinated as

$$N_0(f_k > F) = \int_0^\infty \int_0^\infty p(F; t)\, p(\dot{f}_k)\, \dot{f}_k \, d\dot{f}_k \, dt \quad (3.2)$$

This quantity may be taken as approximative evaluation of exeeding probability when $N_o < 1$. This probability is obvious a conditional probability. For the normal law of distribution we obtain:

$$P(f_k > F | q_1, q_2, \cdots q_m) = \frac{1}{2\pi}\int_0^\infty \frac{\delta_{\dot{f}_k}(t)}{\delta_{f_k}(t)} \exp\left[-\frac{F^2}{2\delta_{f_k}^2(t)}\right] dt \qquad (3.3)$$

Here δ_{f_k} and $\delta_{\dot{f}_k}$ are mean squares for f_k and $\dot{f}_k$ respectively, determinated by formulae

$$\delta_{f_k}^2 = K_{f_k f_k}(t,t); \; \delta_{\dot{f}_k}^2 = K_{\dot{f}_k \dot{f}_k}(t,t); \; K_{\dot{f}_k \dot{f}_k}(t_1,t_2) = \frac{\partial^2 K_{f_k f_k}(t_1,t_2)}{\partial t_1 \partial t_2}$$

The complete probability of the exceeding of level F is equal to

$$P(f_k > F) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(f_k > F | q_1, \cdots q_m) p_q(q_1, \cdots q_m) dq_1 \cdots dq_m \qquad (3.4)$$

Hence the expected term during which the level F shall be exceeded one time is determinated by the formula

$$T(f_k > F) = \frac{T}{N P(f_k > F)}$$

(T is a period during which strong earthquakes are registered, N is the number of strong earthquakes during this period; probability density p_q is normalizated with re pect to earthquakes mentioned).

All previous conclusions were made for generalized coordinates f_k. Stresses, bending moments, shearing forces are linear functions of generalized coordinates (it is essential that the system as a whole remains linear). Let us assume some value v representing the linear combination of generalized coordinates: $v = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$. Its correlation function is given by means of formula

$$K_{vv}(t_1, t_2) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j c_k \; K_{f_j f_k}(t_1, t_2).$$

Then the formulae (3.3), (3.4) and (3.5) may be applied. For engineering purposes the results must be presented in the form of curves analogous to well known "acceleration spectra". These curves depend on natural period of structure, structural damping, prescribed term of structure exploitation and prescribed failure probability to the end of this term. Two last characteristics must have the same significance in aseismic design as safety coefficient in usual statical design of structures.

4. Conclusion. The consequent of methods of the probability theory to the aseismic design of structures demands the further development of some divisions of applied seismology. The investigation of laws of statistical distribution of earthquake magnitude, duration and spectrum for

different regions of seismic activity must be continued. Maximum acceleration and the dominant earthquake period depends on properties of the ground in the given district. Consequently the investigation of the correlation between the objective characteristics of earthquake (measuring by value of releasing energy), the depth of focus, the distance from epicenter, the maximum accelerations, the predominant periods and characteristics of surrounding soils must be fulfiled. The knowledge of spectral properties is scant. The investigations of correlation between parameters of accelerogram envelope and characteristics of earthquake spectrum are necessary. The problem of the distribution laws for "momentary" probability of accelerations at earthquakes must be investigated too.

Our knowledge of dynamical characteristics of constructions must also be more exact. Because of difficult controlled influence construction foundament and joints the natural periods and especially the damping characteristics are random values too. We note that statistical nature of dynamic characteristics must be easily taken into account by supposing that impulsive transitional functions in formulae of type (1.6) also depend on random parameters. Then these parameters must be included into the number of integration variables in the integral of complete probability (3.4).

The theory developed in this paper is applicable for linear systems. At the same time the behavior of construction in elasto-plastic stage is of great interest. It will allow to explain the often marked effect of "accommodation" of construction to strong seismic action due to sharp increasing of damping and growing of natural period when nonelastic deformations appear. If the nonlinearities are not great the method of statistical linearisation may be used. Thus the proposed method after some generalization may be applied to the nonlinear systems too.

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