THEORETICAL RESEARCH ON THE TRANSIENT VIBRATION OF STRUCTURES SUBJECTED TO EARTHQUAKE

by
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INTRODUCTION

In dealing with the vibration of structures subjected to earthquake, induced free vibration cannot be neglected because of the short duration and irregularity of earthquake waves. In 1926, the late Dr. Suehiro pointed out the fact in his article,[1] dealing with the bending vibration and the shearing vibration assuming that their rigidity is uniformly distributed and that earthquake wave is sinusoidal.

In 1941, Drs. Sezawa and Kanai published an operational solution of the vibration of damped one-mass system, subjected to earthquake.[2] Dr. Hayashi solved the problem of the vibrations of damped multi-mass system subjected to external force, by means of operational calculus and matrix in 1942,[3] and Mr. Sugai obtained the mathematical solution of the vibration of damped bar excited by transient forces.

While, in the above-mentioned articles, external forces are introduced in terms of forced action of differential equation, we have obtained general solution by taking motion of ground as term of forced action.

We confirm that research on transient vibration is of great value to that of resistance proof of structures against earthquake. In this paper structures are classified into three kinds, that is, multi-mass system, bending bar and shear beam to simplify analysis. We take into account damping property, based on internal friction, and rigidity and density varying accounding to the height, on the assumption that motion of ground be of sinusoidal wave.

This paper devotes itself to the research on the transient vibration of structures and, at the same time, on the stationary vibration which would be rearched through it.

CHAPTER I. OPERATIONAL CULCULUS OF THE VABRATION OF STRUCTURE.

Art. 1. Vibration of multi-mass system

Assumption and notation are as follows.
(a) m_i ...concentrated mass of the i-th story, in case that the mass of any story of structure would be concentrated to the floor, which is assumed rigid.

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(b) k_i ...spring constant between m_{i-1} and m_i

C_i ...internal friction between m_{i-1} and m_i , which is assumed to be proportional to the relative velocity, expressed by proportional constant.

Let ξ , y and ξ_0 be absolute, relative coordinates and ground motion respectively, as Fig. 1.1(a) shows, the equations of the vibrations of multi-mass system are given as follows,

$$\left.\begin{aligned}
&m_1\ddot{\xi}_1 + C_1(\dot{\xi}_1-\dot{\xi}_0) - C_2(\dot{\xi}_2-\dot{\xi}_1) + k_1(\xi_1-\xi_0) - k_2(\xi_2-\xi_1) = 0\\
&m_2\ddot{\xi}_2 + C_2(\dot{\xi}_2-\dot{\xi}_1) - C_3(\dot{\xi}_3-\dot{\xi}_2) + k_2(\xi_2-\xi_1) - k_3(\xi_3-\xi_2) = 0\\
&\qquad\text{- - - - - - - - - -}\\
&m_{n-1}\ddot{\xi}_{n-1} + C_{n-1}(\dot{\xi}_{n-1}-\dot{\xi}_{n-2}) - C_n(\dot{\xi}_n-\dot{\xi}_{n-1})\\
&\qquad\qquad + k_{n-1}(\xi_{n-1}-\xi_{n-2}) - k_n(\xi_n-\xi_{n-1}) = 0\\
&m_n\ddot{\xi}_n + C_n(\dot{\xi}_n-\dot{\xi}_{n-1}) + k_n(\xi_n-\xi_{n-1}) = 0
\end{aligned}\right\} \qquad (1.1)$$

These equation are to be found in the article by Drs. Tanabashi, Mizuhara and Taniguchi.

Let the natural period, the circular frequency and the damping constant of i-th story be T_i , ω_i and h_i respectively, the following relations hold,

$$\omega_i = 2\pi/T_i \ , \quad k_i/m_i = \omega_i^2 \qquad (1.2)$$

$$C_i/m_i = 2h_i\omega_i \qquad (1.3)$$

Putting $m_{i+1}/m_i = \mu_i$ and substituting (1.2), (1.3) into eqs. (1.1), eqs. (1.1) become

$$\left.\begin{aligned}
&\ddot{y}_1 + (2h_1\omega_1 + 2h_2\omega_2\mu_1)\dot{y}_1 - 2h_2\omega_2\mu_1\dot{y}_2 + (\omega_1^2+\omega_2^2\mu_1)y_1 - \omega_2^2\mu_1 y_2 = -\ddot{\xi}_0\\
&\ddot{y}_2 - 2h_2\omega_2\dot{y}_1 + (2h_2\omega_2 + 2h_3\omega_3\mu_2)\dot{y}_2 - 2h_3\omega_3\mu_2\dot{y}_3 - \omega_2^2 y_1\\
&\qquad\qquad + (\omega_2^2+\omega_3^2\mu_2)y_2 - \omega_3^2\mu_2 y_3 = -\ddot{\xi}_0\\
&\qquad\text{- - - - - - - - - -}\\
&\ddot{y}_{n-1} - 2h_{n-1}\omega_{n-1}\dot{y}_{n-2} + (2h_{n-1}\omega_{n-1} + 2h_n\omega_n\mu_{n-1})\dot{y}_{n-1} - 2h_n\omega_n\mu_{n-1}\dot{y}_n\\
&\qquad - \omega_{n-1}^2 y_{n-2} + (\omega_{n-1}^2 + \omega_n^2\mu_{n-1})y_{n-1} - \omega_n^2\mu_{n-1}y_n = -\ddot{\xi}_0\\
&\ddot{y}_n - 2h_n\omega_n\dot{y}_{n-1} + 2h_n\omega_n\dot{y}_n - \omega_n^2 y_{n-1} + \omega_n^2 y_n = -\ddot{\xi}_0
\end{aligned}\right\} \qquad (1.4)$$

(1.4) can be expressed in the form of matrix as follows,

$$[A]\{\ddot{y}\} + [B]\{\dot{y}\} + [C]\{y\} = -\ddot{\xi}_0\{1\} \qquad (1.5)$$

where[A] indicates unit matrix of n-th order, [B] and[C]square matrixes of n-th order and { } column matrix of n-th order.

On condition that structure is place at origin of absolute coordinates and is set at rest, the solution of (1.5) can be expressed in the form of Laplace transformation:

$$y(p) = \int_0^\infty e^{-pt} y(t)\,dt \qquad (1.6)$$

as follows,

$$\{y(p)\} = -\frac{[S(p)]}{\Delta(p)}p^2\xi_0(p)\{1\} \tag{1.7}$$

where $\Delta(p)$ is determinant of $[\Delta(p)]$, $[S(p)]$ is adjoint matrix of $[\Delta(p)]$, $(\Delta(p)=[A]p^2+[B]p+[C])$, $\xi_0(p)$ is assumed as follows,

$$\xi_0(p) = \frac{a_0}{p-iq} \sim a_0 e^{iqt} = a_0(\cos qt + i\sin qt) \tag{1.8}$$

On the assumtion that any root of $\Delta(p)=0$, which is indicated by α_ν, is different from each other, let an element of adjoint matrix $[S(\alpha_\nu)]$ be $S_{j\mu}(\alpha_\nu)$, then the application of the inversion theorem

$$y_j(t) = a_0\sum_{\mu=1}^{n}\frac{S_{j\mu}(iq)q^2}{\Delta(iq)}e^{iqt} + a_0\sum_{\mu=1}^{n}\sum_{\nu=1}^{2n}\frac{S_{j\mu}(\alpha_\nu)\alpha_\nu^2}{\Delta^{(1)}(\alpha_\nu)(iq-\alpha_\nu)}e^{\alpha_\nu t} \tag{1.9}$$

Let determinant obtained by substitutiong unit into any element of the j-th column of $\Delta(p)$ be Q_j (j=1, 2, 3, ..., n), so $\sum_{\mu=1}^{n} S_{j\mu}(\alpha_\nu)$ can be proved to be reduced to Q_j.

Then eq. (1.9) may be expressed as follows,

$$y_j(t) = a_0\frac{Q_j(iq)}{\Delta(iq)}q^2e^{iqt} + a_0\sum_{\nu=1}^{2n}\frac{Q_j(\alpha_\nu)}{\Delta^{(1)}(\alpha_\nu)(iq-\alpha_\nu)}\alpha_\nu^2 e^{\alpha_\nu t} \tag{1.10}$$

Art. 2. Bending Vibration

If we make coordinates as Fig. 1.1(b) shows, the differential equation of vibration may be given as

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} + b^2\frac{\partial^5 y}{\partial t\,\partial x^4} = -\frac{\partial^2\xi_0}{\partial t^2} \tag{1.11}$$

where $$a^2 = EJ/\rho, \quad b^2 = \gamma J/\rho \tag{1.12}$$

E......Young's modulus of the bar
J......Moment of inertia of the corss section
γ......Coefficient of internal friction
ρ......Density

Now we assume that any of them is constant along the bar. We apply the Laplace transformation to (1.11).

As boundary conditions the following relations may hold,

$$\left.\begin{aligned} x=0 &: \quad y(p)=0, \quad \frac{dy(p)}{dx}=0 \\ x=l &: \quad \frac{d^2y(p)}{dx^2}=0, \quad \frac{d^3y(p)}{dx^3}=0 \end{aligned}\right\} \tag{1.13}$$

So, we get,

$$y(p) = \frac{V(qx)}{2U(ql)}\xi_0(p) \tag{1.14}$$

where q can be given from $p^2+(a^2+k^2p)q^4=0$, and

$$
\begin{aligned}
V(qx) = & (1+\cos ql\cosh ql)(\cos qx+\cosh qx-2) \\
& -\sin ql\sinh ql\,(\cos qx-\cosh qx) \\
& +(\cos ql\sinh ql+\sin ql\cosh ql)(\sin qx-\sinh qx)
\end{aligned}
\tag{1.15}
$$

$$U(ql) = 1+\cos ql\cosh ql \tag{1.16}$$

Applying the inversion theorem to (1.14), by making use of Bromwich integral,

$$y(t)=\frac{1}{2\pi i}\int_{c+i\infty}^{c-i\infty}\frac{e^{zt}V(qx)\xi_0(z)}{2U(ql)}dz \tag{1.17}$$

where z is an expression of p in complex number and c **indicates a finite** positive number.

If we give the motion of ground as follows

$$\left.\begin{aligned}
\xi_0(t) &= a_0\sin\omega t \;\sim\; \frac{a_0\omega}{z^2+\omega^2} && (t \geqq 0) \\
\xi_0(t) &= 0 && (t<0)
\end{aligned}\right\} \tag{1.18}$$

we can obtain the value of (1.17) by "Residue" calculation. The part of forced vibration may be expressed as

$$y_e(t)=\frac{a_0}{2g}\sqrt{\xi_6^2+\eta_6^2}\,\sin(\omega t+\delta) \tag{1.19}$$

where g, ξ_6 and η_6 are functions of ω, a and b, δ is given from $\tan\delta = \eta_6/\xi_6$.

The part of free vibration can be expressed as

$$y_f(t)=-a_0\omega\sum_{n=1}^{\infty}\sqrt{a_{23}^2+b_{23}^2}\;e^{-\mu_n t}\sin(\nu_n t-\varepsilon_n) \tag{1.20}$$

where a_{23}, b_{23} are constants consisting of a, b, μ_n, ν_n and l.

ν_n ..... circular frequency of n-th order

μ_n ..... damping factor of n-th order

ε_n is given from $\tan\varepsilon_n = a_{23}/b_{23}$.

Art. 3. Shearing Vibration

Let us consider the case that masses of multi-mass system are so densely distributed that we can take the system as a continuous body. In such a case, the theory adopted in multi-mass system could be applied to shearing vibration of a continuous body. The equation of shearing vibration can be expressed as

$$\frac{\partial^2\xi}{\partial t^2}-k^2\frac{\partial^3\xi}{\partial t\,\partial x^2}-a^2\frac{\partial^2\xi}{\partial x^2}=0 \tag{1.21}$$

where b is a constant depending on the damping factor and mass, a a constant depending on the rigidity and mass, and both are assumed to be unvariable through the whole height.

On condition that the structure is placed at origin of absolute coordinates and set at rest at the beginning, we apply the Laplace transformation to (1.6). Putting

$$\left.\begin{aligned} &\xi(p)=\xi_0(p) && \text{for } x=0 \\ &\frac{\partial \xi(p)}{\partial x}=\int_0^{\infty} e^{-pt}\frac{\partial \xi(t)}{\partial x}dt=0 && \text{for } x=l \end{aligned}\right\} \qquad (1.22)$$

we get

$$\xi(p)=\frac{\cosh \alpha(l-x)}{\cosh \alpha l}\xi_0(p) \qquad (1.23)$$

where α is given from $\alpha^2=p^2/(b^2p+a^2)$

Substituting (1.8) into (1.23) and applying the inversion theorem, we get

$$\xi(t)=\frac{M(iq)}{\left(\frac{dN(p)}{dp}\right)_{p=iq}}e^{iqt}+a_0\sum_{n=1}^{\infty}\frac{M(p_n)}{\left(\frac{dN(p)}{dp}\right)_{p=p_n}}e^{p_nt} \qquad (1.24)$$

where

$$\left.\begin{aligned} M(p)&=\cosh \alpha(l-x)=\cosh\frac{p}{\sqrt{b^2p+a^2}}(l-x) \\ N(p)&=(p-iq)\cosh \alpha l=(p-iq)\cosh\frac{p}{\sqrt{b^2p+a^2}}l \end{aligned}\right\} \qquad (1.25)$$

The first term of (1.24) represents forced vibration, the second free vibration. Substitutiong (1.25) into (1.24), we get ξ_e, which indicates a term of forced vibration.

$$\xi_e=a_0\sqrt{u^2+v^2}\cos(qt+\delta)+i\,a_0\sqrt{u^2+v^2}\sin(qt+\delta) \qquad (1.26)$$

where

$$u=\frac{\cos\beta\cosh\gamma\cos\beta\left(1-\frac{x}{l}\right)\cosh\gamma\left(1-\frac{x}{l}\right)+\sin\beta\sinh\gamma\sin\beta\left(1-\frac{x}{l}\right)\sinh\gamma\left(1-\frac{x}{l}\right)}{(\cos\beta\cosh\gamma)^2+(\sin\beta\sinh\gamma)^2}$$

$$v=\frac{\cos\beta\cosh\gamma\sin\beta\left(1-\frac{x}{l}\right)\sinh\gamma\left(1-\frac{x}{l}\right)-\sin\beta\sinh\gamma\cos\beta\left(1-\frac{x}{l}\right)\cosh\gamma\left(1-\frac{x}{l}\right)}{(\cos\beta\cosh\gamma)^2+(\sin\beta\sinh\gamma)^2}$$

β and γ are functions of a, b, l and q. As a term of free vibration we get

$$\begin{aligned} \xi_f=&\,a_0\sum_{n=1}^{\infty}\sqrt{(\theta_n+\theta_n')^2+(\varphi_n+\varphi_n')^2}\,e^{-k_nt}\cos(\omega_nt+\psi_n) \\ &+i\,a_0\sum_{n=1}^{\infty}\sqrt{(\theta_n-\theta_n')^2+(\varphi_n-\varphi_n')^2}\,e^{-k_nt}\sin(\omega_nt+\varepsilon_n) \end{aligned} \qquad (1.27)$$

where

$$\tan\psi_n = \frac{\varphi_n + \varphi_n'}{\theta_n + \theta_n'} \quad , \quad \tan\varepsilon_n = \frac{\varphi_n - \varphi_n'}{\theta_n - \theta_n'}$$

k_n is the damping factor of n-th order and ω_n the circular frequency of n-th order. θ_n , θ_n' , φ_n and φ_n' are functions of a, b, q, l, and x. The whole vibration is expressed by $\xi(t) = \xi_e + \xi_f$, and the relative displacement to the ground may be expressed by $y(t) = \xi(t) - \xi_o(t)$.

CHAPTER II. COMPARISON OF THE DEFLECTION CURVE OF STRUCTURE IN TRANSIENT STATE TO THAT IN STATIONARY STATE

Art. 1. Comparison of the deflection curve of the bending vibratory body to that of shearing vibratory body in case that their rigidity is uniformly distributed.

We have made calculation of (1.19) and (1.20), letting the natural period of 1st oder T_I be 1 sec, the damping factor of 1st order $k_I = 0.03$, ground motion of sinusoidal wave, the ratio of T_I to the period of ground motion T is 4/3, 1,3/4, and T_{II}/T unity (T_{II} indicates the natural period of 2nd order). Fig.2.1 shows how the deflection curve changes in course of time and suggests the following conclusions. The deflection curves in transient and stationary state at the time when the maximum displacement appears have resemblance in shape in case that T_I/T falls in a range from 0.7 to 1.4 and $T_{II}/T = 1$. Therefore, the deflection curve of free vibration could be considered to resemble those in shape. The curvature of the deflection curve is the most acute at the lowest part for $T_I/T = 1$ and at the middle part or at the lowest part for $T_{II}/T = 1$.

If groung motion keeps the amplitude unchanged, the curvature is more acute for $T_{II}/T = 1$ than for $T_I/T = 1$ in a range from the beginning to the 6th wave, but after that, on the contrary, it becomes more acute for $T_I/T = 1$.

For shearing vibratory body, too, the same conclusions as for bending vibratory body mentioned above have been drawn.

Art. 2. Effect of the rigidity and density of shearing vibratory body on its period and deflection curve.

We take now the structure which is wide in the drection of vibration compared with its height as shearing vibratory body.

Generally, a structure decreases in rigidity and density with the height above ground. Now we take up two cases:

Case 1. Decreasing linearly with the height.
Case 2. Decreasing in steps with the height.

§ 1 Mathematical Solution

The differential equation of vibration for the case that rigidity G and density ρ are functions of the height x is as follows,

$$\rho(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left\{G(x)\frac{\partial y}{\partial x}\right\} \tag{2.1}$$

Assuming a solution of this equation of the form $y = X(x)\cdot T(t)$ where $X(x)$, $T(t)$ are functions of x only and of t only respectively, we see

$$\frac{d}{dx}\left\{G(x)\frac{dX}{dx}\right\} + \lambda\rho(x) = 0 \tag{2.2}$$

where λ is a eigenvalue.

Let l be the height of structure and the boundary conditions be expressed as follows,

$$\left.\begin{aligned} x = 0 &: \quad X = 0 \\ x = l &: \quad \frac{dX}{dx} = 0 \end{aligned}\right\} \tag{2.3}$$

For case 1, we assume first,

$$G(x) = c\left\{1 + n\left(1 - \frac{x}{l}\right)\right\} \tag{2.4}$$

$$\rho(x) = a\left\{1 + m\left(1 - \frac{x}{l}\right)\right\} \tag{2.5}$$

where c is the rigidity on the top, a the density on the top and m, n are positive numbers.

In this paper, we have dealt with 18 cases which represent the combinations for n=0,1,2,4,9,15, m=0, o.5,1, to get eigenvalues of 1st, 2nd and 3rd order for each case. (See Fig. 2.2(a))

For case 2, changes of G (x) are illustrated in Fig.(2.2(b)(c)) for example. We have adopted 11 cases among combinations of G (x) and $\rho(x)$.

The numerical solutions have been given by the Analogue-Computer equipped in Tsurumi Laboratory, Toshiba Co. Ltd. The results are illustrated in Figs. 2.3, to 2.10.

§ 2 Natural Period

We have obtained the following conclusions, based on these figures and some considerations.

(1) In case that the ratio of the rigidity at the bottom to that on the top, we call it rigidity ratio later, is nearly unity, the slight change of the rigidity has a great effect on the period, but the change of the density ratio has only a little effect on it.

(2) The increase of the rigidity at the bottom brings about the reduction of the period and its decrease results in the increase of the period.

(3) If the curve of T_I is replaced on logarithmic graph paper, their curve can be shown to be almost linear. So, as an approximate expression for T_I, we obtain

$$(n+1)\,T_I^{\,2.4} = 7.0m + 28$$

(4) The period of higher order can be obtained by using T_I

$$T_N = (0.7 \sim 0.8)\, T_I / N \tag{2.6}$$

where N denotes the order number.

In (2.6), 0.7 should be adopted only when the rigidity and density remain almot unchanged through the whole height and when the structure has no walls between columns. Otherwise, 0.8 should be used.

(5) The rigidity of the structure, whose density is uniformly distributed, equivalent to the structure changing linearly in both rigidity and density is illustrated in Fig. 2.5.

§ 3 Deflection curve X and dX/dx

Considering the same structure as in the preceeding section we deal with the deflection curve, and dX/dx which are represented in vibrations of 1st to 3rd order and the shearing force of 1st order.

(a) Comparison among the distribution curves of deflection, and shearing force in the structure changing linearly in rigidity and density.

For six cases combined with G $1\sim1$, $1\sim5$, $1\sim16$ for rigidity and ρ $1\sim1$, $1\sim2$ for density, the curve of deflection and dX/dx are illustrated in Figs. 2.6 and 2.7. Fig. 2.6 shows that the deflection curve for G $1\sim1$ is concave through the whole height and G $1\sim16$ convex at the lower part. Fig. 2.7 shows that for G $1\sim1$, dX/dx decreases with higher level and its maximum is at the bottom and that for G $1\sim5$, G $1\sim16$ maximum dX/dx takes place at higher level from the bottom.

(b) Comparison between structures changing linearly and in steps in rigidity and density by curves of deflection and dX/dx. For ρ $1\sim2$, distribution curves of deflection and dX/dx of 1st order are shown in Figs. 2.8 and 2.9, respectively. The Figures state that there is little difference between them for deflection and shearing force and that difference becomes larger with the higher value of rigidity ratio for dX/dx.

(c) Vibration characteristics of the structures with great change in rigidity at the lowest part.

Discontinuity is apparently found in curves of deflection and dX/dx at a point where rigidity changes greatly and such a discontinuity can not be found in that of shearing force.

(d) Position of maximum dX/dx

The positions at which maximum dX/dx takes place are illustrated in Fig. 2.10 by combinations of G and ρ . We can apparently see that the positions of maximum dX/dx become higher as the rigidity increases at the lowest part.

Art. 3. Effect of the rigidity and density of bending vibratory body on the period and the deflection curve.

We take the slender structure, which is high compared with its width as bending vibratory body. Here we deal with bending vibratory body with its rigidity and density increasing with the lower level of the height, looking into the natural period, deflection curve and bending moment distribution.

§ 1 Mathematical analysis

Putting $\rho A = m(x)$, $EJ = B(x)$, the quation of free vibration is expressed

$$m\frac{\partial^2 y}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left\{B\frac{\partial^2 y}{\partial x^2}\right\} \tag{2.7}$$

If we assume a solution of (2.7) of the from $y = X(x)\cdot T(t)$, where $X(x)$, $T(t)$ are functions of x only and of t only respectively, we see that

$$-\frac{\frac{d^2T}{dt^2}}{T} = \frac{\frac{d^2}{dx^2}\left\{B(x)\frac{d^2X}{dx^2}\right\}}{m(x)\cdot X} = \lambda \tag{2.8}$$

So, from (2.8) we get an ordinary differential equation

$$\frac{d^2}{dx^2}\left\{B(x)\frac{d^2X}{dx^2}\right\} - \lambda m(x)\, X = 0 \tag{2.9}$$

where λ is a eigenvalue. Boundary conditions are as follows,

$$X = 0, \quad \frac{dX}{dx} = 0 \qquad \text{for} \quad x = 0 \tag{2.10}$$

$$\frac{d^2X}{dx^2} = 0, \quad \frac{d^3X}{dx^3} = 0 \qquad \text{for} \quad x = l \tag{2.11}$$

Now, we assume

$$B(x) = c + d_0(l - x) \tag{2.12}$$

$$m(x) = a + b(l - x) \tag{2.13}$$

where c and a correspond to EJ and ρA on the top respectively. We have obtained an approximate solution by adopting Galerkin's method for natural period and by Stodola's method for deflection.

§ 2 Natural period

The functions which satisfy (2.10), (2.11) are defined as

$$\left.\begin{aligned} Y_1 &= 6\left(\frac{x}{l}\right)^2 - 4\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4 \\ Y_2 &= 20\left(\frac{x}{l}\right)^2 - 10\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^5 \end{aligned}\right\} \tag{2.14}$$

We have made calculation for 18 cases obtained from combinations of rigidity and density as selected in the previous article. We have drawn the following conclusions.

(1) Natural period of 1st and 2nd order decrease as rigidity ratio increases and are independent of density ratio.

(2) As an approximate expression for T_I, we obtain

$$Y = -\log_{10} 2.9\cdot X + \log_{10} N_1 \qquad \text{for that of 1st order} \tag{2.15}$$

$$Y = -\log_{10} 2.7\cdot X + \log_{10} N_2 \qquad \text{for that of 2nd order} \tag{2.16}$$

Where $X = \log_{10} n$, $Y = \log_{10} T$ and n is rigidity ratio, N_1 and N_2 are taken as 1.8, 1.9, 2.0 and 0.25, 0.26, 0.30 respectively for density ratio 1, 1.5, 2.

(3) The period of 2nd order can be approximately obtained by using T_I

$$T_{II} = 0.3\,\frac{T_I}{2} \tag{2.17}$$

(4) The rigidity of the structure, whose density is uniformly distributed, equivalent to the structure changing linearly in both rigidity and density is obtained as Fig. 2.5.

§ 3 Deflection curve

We have obtained the deflection curve of 1st order by Stodola's method from (2.9). Fig. 2.11 shows the deflection curve of 1st order, for example, for density ratio 1.5. Fig. 2.12 shows the distribution of bending moment along the height.

CHAPTER III. POSITION OF THE MAXIMUM dX/dx IN THE MULTI-STORY BUILDING AND RELATION BETWEEN dX/dx AND DISPLACEMENT RELATIVE TO THE GROUND AND THAT OF ONE MASS SYSTEM

Art. 1. Position of the Maximum dX/dx in the Multi-story Building

(1) Two masses system

We put mass ratio m_2/m_1, spring constant ratio k_2/k_1, are assumed as in the Table 3.1. As to the value of damping constant $h_1 = h_2 = 0.01$, 0.03, 0.05 are considered and as to the natural period $T_1 = T_2 = 0.1$ sec. is adopted.

Transient vibration of the system is calculated for various period ratios in the case of sinusoidal ground motion.

When the period of the ground motion is larger, extremely smaller than, or nearly equal to the foundamental period of the building, dX/dx is more acute for the lower story than for the upper in the course of the whole vibration. On the contrary, when the period of ground motion is nearly equal to that of the second order of the building, the upper story is more disadvantageous than the lower.

(2) Three, four masses systems and shearing vibratory body.

In the similar way numerical calculations are continued. As the result of the examination of dX/dx of any story of these systems in the transient and stationary condition in the case of sinusoidal ground motion, we see that the maximum dX/dx in multi-story building takes place at an intermediate position in the case when the period of ground motion is near by the foundamental period of the building.

Art. 2. Relation between dX/dx of any story of the multi-story building and the displacement of one mass system relative to the ground.

Representing the vibration of the multi-story building by bending vibration, shearing vibration and vibration of multi-mass system, in the case of bending and shear we consider the relative displacement between two points, and that of two masses in the case of multi-mass system. These are assumed to be dX/dx of any story of the building and the quotient of the Maximum dX/dx in the transient by that in the stationary state is denoted by α. Similarly, the maximum displacement of these systems above mensioned relative to the ground in the transient is divided by that in the stationary, and this ratio is denoted by β. These α and β are compared with those in the case of one mass system. Consequently, the value of α and β are, so far as the value of the period ratio T_I/τ is about 0.7 to 1.5, equal to those of one mass system having same period ratio and damping constant except the lowest story.

At the lowest story only, the value of α and β of the building increases by 20% over those of one mass system on the condition that the value of the period ratio T_I/τ is larger than unity, and, on the contrary, decreases by 20% on the condition that the value of T_I/τ is smaller than unity.

(3) Relation between the maximum displacement of multi-story buildings and that of one mass system relative to the ground.

Let us compare the maximum relative displacement at middle point for the shearing vibration, that of middle body in the system of odd discrete bodies, mean value of the middle two bodies in the system of even discrete bodies with that of one mass system.

By way of illustration, we have a few examples adopted here among the systems having three or four degrees of freedom in the case when the period ratio $T_I/\tau=4/3$. These are shown in Tab. 3.2 and 3.3. In the case of bending vibration let us compare the maximum displacement at three quarters point from the base of the body relative to the ground with the maximum displacement of one mass system in the transient. This is shown in Tab. 3.4.

From discussion above mensioned, we see that the maximum relative displacement at the middle part in height of the building corresponds to that of one mass system in the transient and stationary respectively, where the period ratio T_I/τ (τ is the period of ground motion) falls within the range about 0.7 to 1.5. Similary, in the case of slender structures which are relarged as a bending bar the displacement at a three quarters point from the base corresponds to that of one mass system.

CONCLUSION

In case building is so perfect or little cracked, that its damping constant is within the range from 0 to 0.16, we must expect that the maximum displacement in the transient relative to the ground is more than about 3 times larger than half amplitude of ground motion at the center of gravity and its neiborhood, and more than about 6 times larger at the top of building, when subjected to an earthquake motion in the condition that the period ratio is from 0.75 to 1.5.

In case the building is cracked that its damping constant is considered as large as 0.3, we must expect that the maximum displacement is from about 3.5 times to twice larger than half amplitude of ground motion at the top in the condition that the period ratio is from 0.75 to 1.5, and when the damping constant is to be taken up as large as 0.4, we must expect that the maximum displacement at the top of the building is from about 3 to 1.5 times larger.

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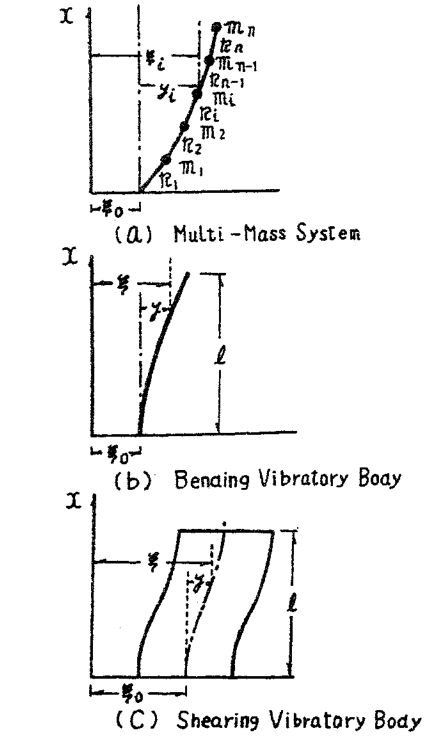

Fig. 1.1 Vibratory System and Bodies

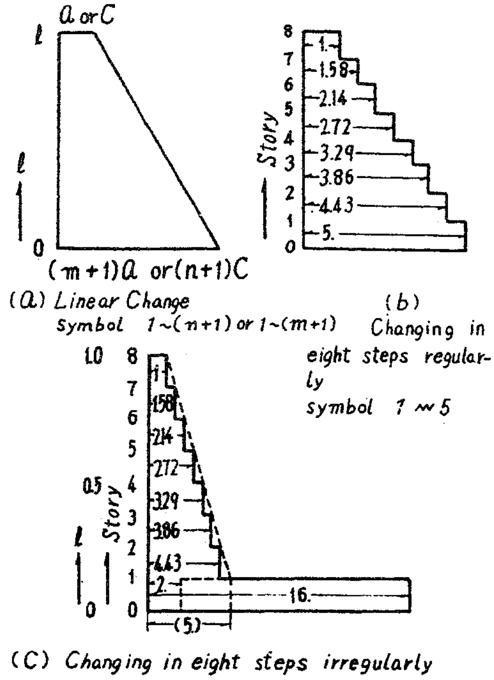

Fig. 2.2 Change in G and ρ

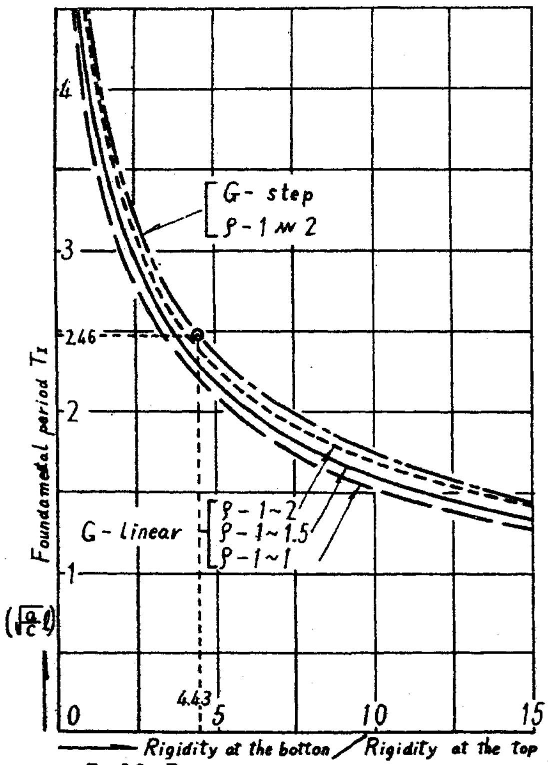

Fig. 2.3 Foundamental period

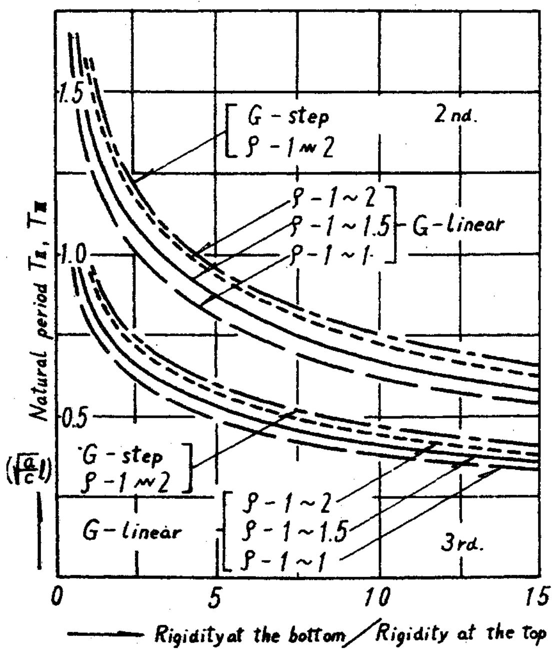

Fig. 2.4 Natural periods of second and third order

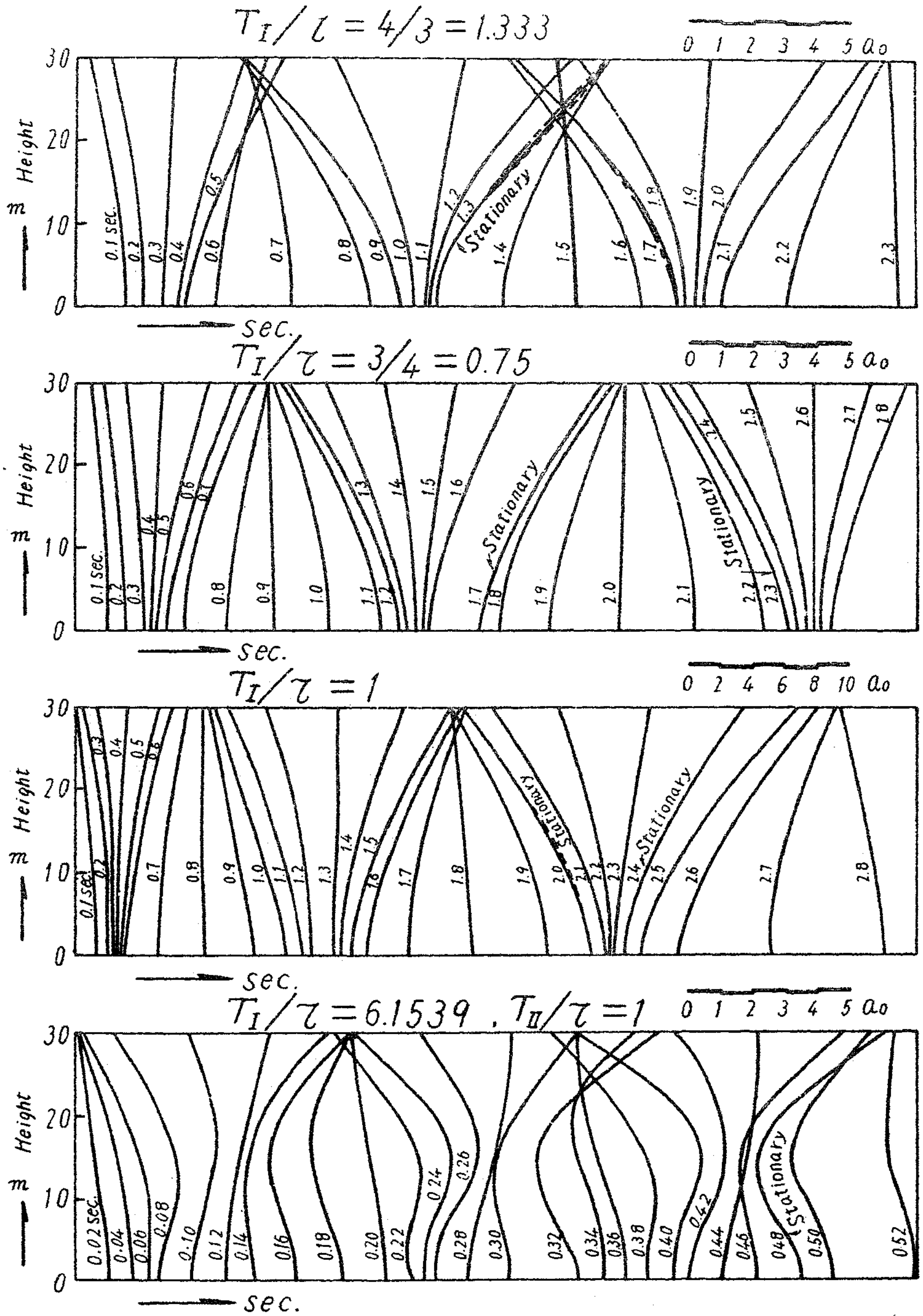

Fig. 2.1 Deflection curves of bending vibratory body in the transient state.

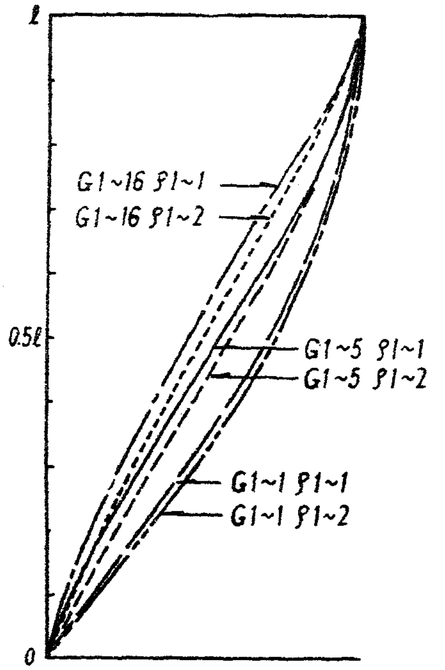

Fig. 2.6 Deflection curves of 1st order
(Shearing vibration)

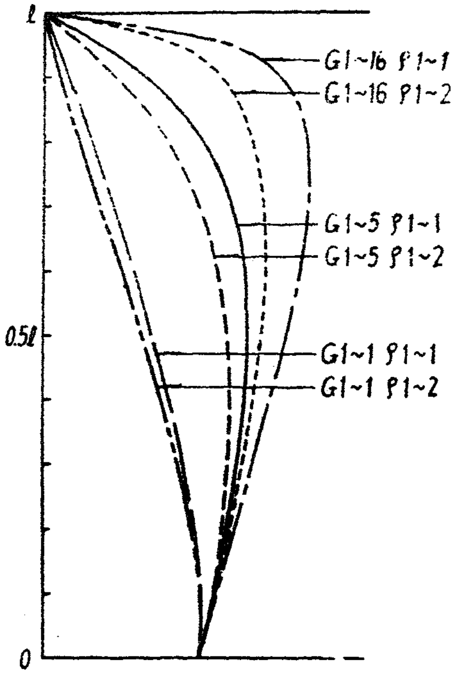

Fig. 2.7 Curves of dX/dx of 1st order
(Shearing vibration)

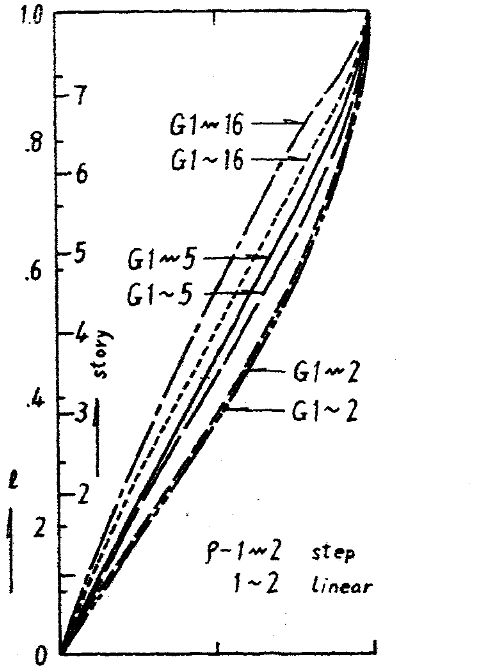

Fig. 2.8 Comparison of linear change to step change by the deflection curve of 1st order.

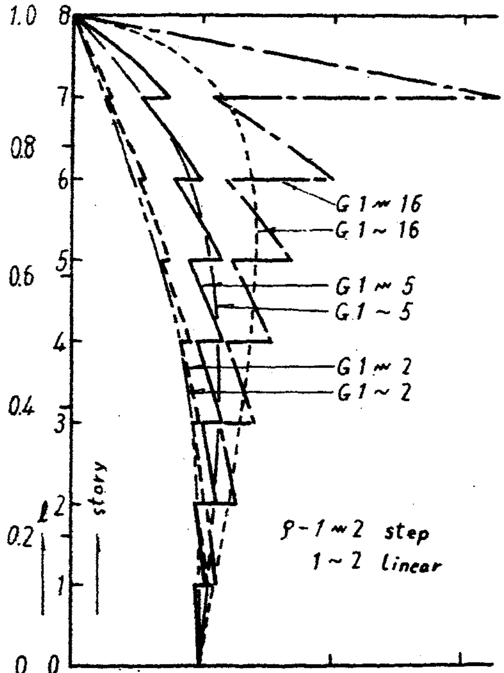

Fig. 2.9 Comparison of linear change to step change by dX/dx curve of 1st order.

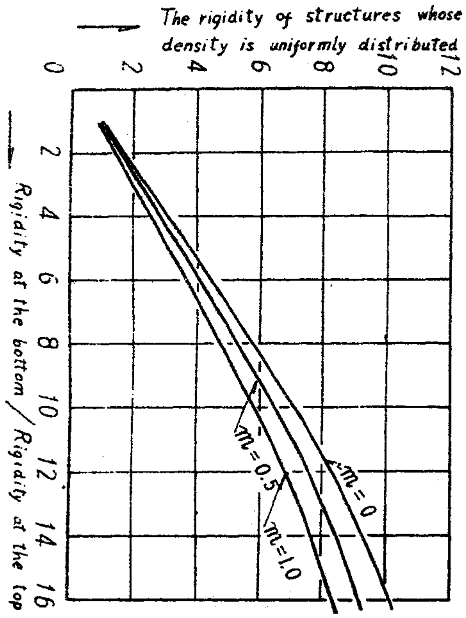

Fig. 2.5

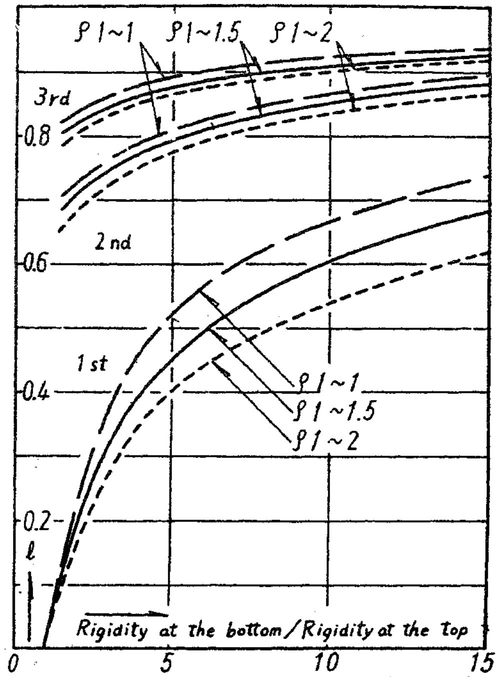

Fig. 2.10 Position of the maximum dX/dx by 1st, 2nd and 3rd order.

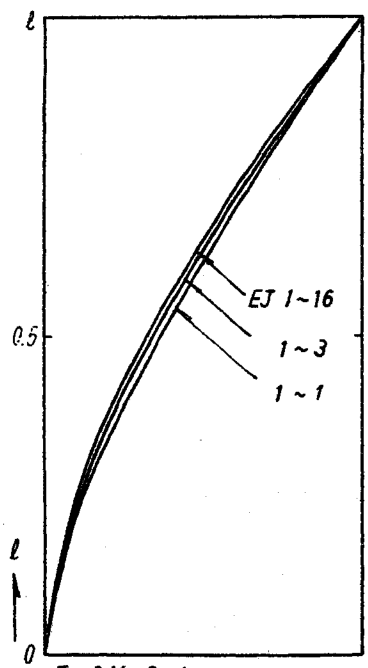

Fig. 2.11 Deflection curves of first order for bending vibration. (ρ bottom / ρ top = 1.5)

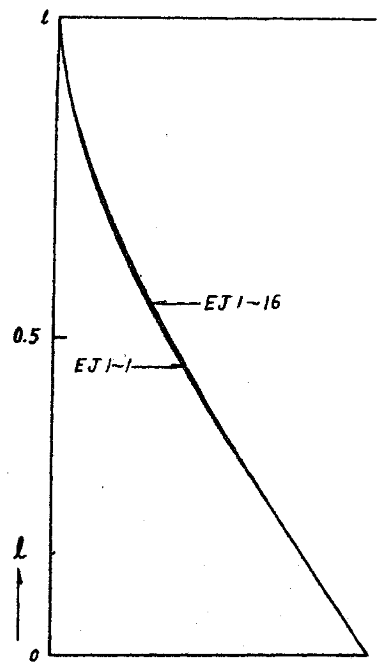

Fig. 2.12 Bending moment distribution (ρ bottom / ρ top = 1.5)

TABLES

Table 3.1 Mass ratio and spring constant ratio.

	1st kind	2nd kind	3rd kind
m_2/m_1	1/1.5	1/1	1.5/1
k_2/k_1	1/1.5	1/1	1.5/1

Table 3.2 Relation between the relative displacement of the three bodies system and one mass system, where a_0 denotes half amplitude of the ground motion.

	Transient	Stationary
displacement of the middle body (a_0)	3.75	2.25
displacement of one mass system (a_0)	3.8	2.3

Table 3.3 Relation between the relative displacement of the four bodies system and one mass system.

	Transient	Stationary
mean value of the middle two bodies (a_0)	3.6	2.15
displacement of mass system (a_0	3.8	2.3

Table 3.4 Relation between the relative displacement of the bending bar and one mass system.

Period Ratio T_1/τ	4/3		3/4		1
	Trans.	Stat.	Tran.	Stat.	Tran. Stat.
displacement at three quarters point from the base (a_0)	3.6	2.3	2.5	1.3	17
displacement of one mass system (a_0)	3.6	2.3	2.5	1.3	16