AN ANALOGUE COMPUTER FOR THE DETERMINATION OF THE EARTHQUAKE RESPONSE OF BUILDINGS IN BENDING AND SHEAR MODES

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1. INTRODUCTION

Analogue computers, specially designed for the determination of the response of buildings when subjected to earthquakes, are unusual pieces of equipment. Yet the problem of earthquake-resistant design is sufficiently important to warrant considerable expenditure of effort on devising some equipment that will enable a systematic study of building response to be undertaken at reasonable cost. Alford, Housner and Martel [1] appear to have been the first to have used electrical circuitry in earthquake studies. Their single-storey analogue enabled the velocity spectra of various earthquakes to be determined. Unfortunately, the acceleration spectra determined by their method were approximate only, being reasonably accurate in the middle of the frequency range, but considerably in error towards the extremities. Murphy, Bycroft and Harrison [2] developed an analogue for multi-storey buildings that deflect in shear only and much valuable work has been done with this equipment [3] [4]. Further work is reported elsewhere in these proceedings [5].

This paper is concerned with the development of an analogue computer for the investigation of the response of buildings that deflect in bending, or in shear, or in bending and shear. Although primarily designed for buildings of up to 10 storeys, a greater number of storeys can be set up by suitably representing, for example, two storeys in the building by one in the analogue. Such a representation can be an excellent approximation if the optimum setting-up procedure is adopted. Foundations can also be represented in so far that their essential properties can be expressed in terms of mass, stiffness and damping. Literature on the subject of analogues for bending structures is scarce. To date the authors have come across only two references to analogues that are valid at other than a single frequency. McCann and MacNeal [6] describe a passive network employing transformers and Squires and Hughes [7] describe circuitry employing operational amplifiers. Both of these analogues are derived from the finite-difference approximation to the differential equation for a continuous beam. The first analogue is inherently more accurate than the second although not so flexible for systematic study.

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The analogue described here is designed primarily to represent exactly the usual idealized buildings of engineering seismology, viz., massive rigid floors of thickness small compared with the inter-storey distance, connected rigidly to massless elastic columns, and vibrating in one dimension. To this basic analogue can be added circuitry to represent torsional effects and yield in the columns. Viscous damping can be very simply included if required, although the existence of viscous damping in actual buildings is highly doubtful. The accuracy of the analogue computer, within the restrictions of the above assumptions, depends only on the quality of the passive components employed. Since no transformers are included high accuracy can be achieved.

The principal advantage of the electrical analogue as opposed to the mechanical analogue described elsewhere in these proceedings is the flexibility of setting-up a problem, including the effects of shear deflection and non-linear damping. The two procedures for analysing the same problem are in many respects complementary and one can be used to check the other.

2. The Basic Analogue

2.1 Bending Elastic Columns

It is assumed that the columns or walls between successive floors can be replaced by a single elastic massless beam of uniform Young's modulus and second moment of area in so far as stiffnesses and the forces and moments exerted on the neighbouring floors are concerned. Small displacements of such a beam satisfy the equation

$$\frac{\partial^4 y}{\partial x^4} = 0.$$

From this it follows, with reference to the sign convention of fig. 1, that

$$y = c_3 x^3 + c_2 x^2 + c_1 x + c_0 ,$$

so that

$$\psi = \frac{\partial y}{\partial x} = 3c_3 x^2 + 2c_2 x + c_1 \qquad \text{etc.,}$$

where the c's, which are functions of time, can be expressed in terms of y_1, ψ_1, M_1, S_1 . Remembering that $M = EI\frac{\partial^2 y}{\partial x^2}$ and $S = \frac{\partial M}{\partial x}$, we find that the coordinates at end 2 of the beam can be simply expressed in terms of those at end 1 in the form

$$\begin{pmatrix} y_2 \\ \psi_2 \\ M_2 \\ S_2 \end{pmatrix} = \begin{pmatrix} 1 & \ell & \ell^2/(2EI) & \ell^3/(6EI) \\ 0 & 1 & \ell/(EI) & \ell^2/(2EI) \\ 0 & 0 & 1 & \ell \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \psi_1 \\ M_1 \\ S_1 \end{pmatrix}, \qquad (1)$$

where ℓ = the length of the beam.

These relations can be represented electrically by the theoretical circuit of fig. 2. Space does not permit an account of its derivation, but the correctness of this analogue can be readily verified by elementary circuit analysis, and a slight rearrangement of equations (1). For example, it is readily established that

$$y_2 = y_1 + 1/2\,\ell(\psi_1 + \psi_2) - \frac{1}{12}\frac{\ell^3}{EI} S_1 .$$

In the figure, arrows denote the positive direction of current flow and dots denote differentiation with respect to time.

It is important to realize that in this analogue only the shear forces and bending moments at the ends of the beam are represented. This is adequate for earthquake purposes since only maximum stresses are required and the bending moment is a maximum at one of the ends of the beam, while the shear force is constant throughout the beam.

2.2 Rigid Massive Floors

With reference to fig. 1 the displacements of the floor satisfy the equations

$$\left.\begin{aligned} S_2 - S_3 &= m\ddot{y}_2 \\ M_2 - M_3 &= -I_m \ddot{\psi}_2 \end{aligned}\right\}, \qquad (2)$$

provided the thickness of the floor in the x direction is negligible. Corresponding to the form (1), (2) can be written in the form

$$\begin{pmatrix} y_3 \\ \psi_3 \\ M_3 \\ S_3 \end{pmatrix} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & I_m \frac{\partial^2}{\partial t^2} & 1 & \cdot \\ -m\frac{\partial^2}{\partial t^2} & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} y_2 \\ \psi_2 \\ M_2 \\ S_3 \end{pmatrix} .$$

Here, m is the mass of the floor and Im its moment of inertia about an axis normal to the xy plane and passing through its centre of mass.

2.3 Shear Effects

The analysis of shear effects follows that given by Trail-Nash and Collar [8]. Neglecting so-called shear-lag effects, the describing equation is

$$\frac{\partial y}{\partial x} = \frac{\partial z}{\partial x} - \frac{1}{\sigma} S , \tag{3}$$

where σ is the shear stiffness for unit length of beam, y is the total displacement and z the displacement in bending alone. Equations (1) still hold if y is now replaced by z, and ψ is interpreted as $\partial z/\partial x$. Thus the analogue for the beam alone is the same as in fig. 2 (a) provided y is replaced by z. From (3) we have

$$y_2 - y_1 = z_2 - z_1 - \frac{\ell}{\sigma} S_1 . \tag{4}$$

It is then readily verified that (2), and (4) are correctly represented by the network of fig. 3.

We note in passing that, if the beams are infinitely stiff in bending, i.e., if

$$(EI)^{-1} = 0,$$

then the coils labelled $\ell(2EI)^{-1}$ in fig. 3 become short circuits and ψ is everywhere the same. In the case of a building, the base is clamped so that $\psi = 0$. Thus the current generator in the lower part of the diagram is zero and the analogue degenerates to that of ref. 2, as might be expected.

2.4 Boundary Conditions

The boundary conditions to be applied are:

(i) at the base of the building the slope is zero and the displacement is the same as that of the ground during the earthquake,

(ii) at the top of the building the bending moment and shear force are zero.

These conditions are readily imposed as shown in fig. 4. It will be seen that the earthquake ground velocity is fed into the analogue in the form of current as in ref. 2.

≠ This equation differs from that in ref. 8 owing to the different sign conventions employed.

3. The Analogue Computer

3.1 Practical Circuitry

Space does not permit more than a very brief mention of the practical circuitry here. Details will appear elsewhere.

The coils range from 10 to 100 mH and are wound on ferroxcube pot cores. The block diagram for a single storey is shown in fig. 5. It will be noticed that the voltages on the signal lines are fed through cathode followers into adder circuits which in turn drive the current generators. This circuitry is required to satisfy the specifications,

gain:	$< 1\%$	variation in range 0 - 10 kc/s,
phase shift:	$< 1^\circ$	" " " " ,
output impedance:	> 10 Megohm	" " " .

3.2 Signal Generation

The method of earthquake-signal generation is in principle unchanged from that described in ref. 2.

3.3 Scaling and Setting-up Procedure

In deciding at what voltage and current levels and frequency range to operate, it is necessary to introduce scaling factors that convert from the mechanical to the electrical quantities. For example one needs to decide how many volts should correspond to 10^6 lb of shear force. Introduction of scaling leaves the basic circuitry unchanged but changes the labels of the various electrical quantities. Time is also scaled according to the relation

real time $= \rho \times$ analogue time, so that

$$\frac{\partial}{\partial t_{real}} = \frac{1}{\rho}\frac{\partial}{\partial t_{analogue}} \qquad (5)$$

In practice it is much more convenient to vary capacitance and transconductance than to vary inductance. For this reason the available inductors span a range of 10 : 1 (which covers the range of masses of the floors of most buildings). On the other hand, the capacitors span a range of 400 : 1 and the transconductances span a range of 10 : 1. This permits the operator considerable flexibility in arranging the significant natural frequencies of the building to lie in the range of frequencies for the best electrical performance.

3.4 Measurement and Recording

The two quantities of most interest to structural engineers are the maximum values of shear force and bending moment occurring at the various storeys. These are measured as voltage and current respectively. The voltage is fed to a peak-voltage measuring device, thence to a pen recorder. The current is measured with the aid of a current transformer employing a very high-μ core and a tertiary feed-back winding, thence on to a pen recorder. The twenty pieces of information are recorded simultaneously together with the number of times the 'electrical earthquake' has been speeded up from the original. Should they be required, displacement and angular orientation can be measured by recording the integrated current of the lower, and integrated voltage of the upper, line, respectively, of fig. 3.

3.5 Damping

Compared with the assessment of mass and stiffness in a real structure the assessment of damping is poorly developed. It is most unlikely that viscous damping in the structure itself is at all significant. Much more probable is the occurrence of hysteretic damping, both in the structural materials and in the foundation soils. The analogue has been designed to incorporate hysteretic damping of the so-called square-law type shown in fig. 6 and normally associated with elastic displacement combined with plastic yield. The theoretical circuit for realisation of this is shown in fig. 7. Again for reasons of space limitation all practical details have had to be suppressed. Viscous damping, both of the interstorey type and the absolute type [2], can be included by adding appropriate resistors in series with the capacitors labelled $\ell^3/(12EI)$, in parallel with the coils labelled $\ell/(2EI)$, and in series with the coils labelled m in fig. 3. As explained in ref. 2, it is possible to simulate a constant damping-v.-frequency characteristic over a range of frequencies according to the function $\alpha/\omega + \beta\omega$, where ω is angular frequency, for ω in the neighbourhood of $\sqrt{\alpha/\beta}$. This has the practical advantage that it enables one to set up a prescribed amount of damping in a particular mode more quickly than with yield , although possibly with less precision in some cases. Until more information on the dynamic properties of soils and structural materials, including pre-stressed concrete, is available it will be difficult to carry out valid analyses on real structures. What is really required is a set of stress-strain curves for the various materials for shear, bending, and elongation deflections evaluated for various amplitudes and for various cycling frequencies varying from 0 to 10 c/s. It is most probable that such characteristics would be readily simulable electrically, if needs be with more sophisticated developments of the circuit of fig. 7.

3.6 Future Development

If the centres of mass of the massive elements do not lie on the neutral axis (or plane) of the elastic elements, then coupling between the horizontal linear motion and torsional motion about a vertical axis will exist. This effect can be simulated with the aid of only passive components. A theoretical circuit for these effects would be an LC ladder network coupled on to the existing shear-displacement network (the lower part of the circuit diagram of fig. 2). However, detailed circuit design and construction for this effect have not yet been undertaken.

A further effect to consider would be the influence of flexing floors. In principle, circuitry to simulate this and combined with effects already simulated, could be designed and built. However, at this stage, except for especially important buildings that would be worth detailed and accurate analysis, the ensuing complication develops to expensive proportions. For the purposes of developing a handbook the number of combinations of different parameters would be so large that a digital computer would be needed to programme the operation of the analogue computer!

Finally, it should be noted that, though designed primarily for building analysis, the analogue is not limited to this field. The behaviour of bending structures under the various boundary conditions of pinned, clamped or free end can be analysed with this equipment.

4. Practical Results

At the time of writing it is still too early to report in detail on the accuracy attainable with the equipment. Present indications are that 2% is likely to be commonly attainable. The comparisons of measured values of mode shape and natural frequency with calculated values of a uniform bending building are most promising and indicate considerable improvement over existing analogue equipment.

From Table 1 it will be seen that the maximum errors in normal mode shape are 0.25% and 1.77% of the maximum shear forces in modes 1 and 2, respectively. The errors in frequency determination are not more than 1 part in 300. These figures have been obtained after each inductance, capacitance and transconductance had been set to within 1% of its nominal value.

As an example of the use of the computer, we consider an 11-storey reinforced-concrete bearing-wall building. The structural walls consist of a central spine wall intersecting a series of transverse walls. The building is supported on two rows of piles, 33 ft long, which are assumed to be gripped immovably over their lower 16½ ft and to be capable of elastic longitudinal deflection over the remaining 16½ ft. For the purposes of analogue simulation, the piles may be replaced by a bending beam (incapable of shear deflection) of length 16½ ft. Other details of the building are given in Table 2.

Since there are effectively 12 storeys, it is necessary to 'double up' on certain storeys in the analogue. The foundations and the top storey were represented by a single storey in the analogue, while the remaining 10 storeys were represented by 5 storeys in the analogue. Damping was arbitrarily set up for 2% of critical in the first normal mode. The first two natural frequencies were found to be 2.75 c/s and 14.4 c/s, respectively, from which it can be deduced, with the aid of the curves given in ref. 5, that only the first normal mode has a significant response to an earthquake. The velocity record of the earthquake Taft, California, 21/7/52, component N 21 E, was fed into the computer and the shear forces at the base of the foundations and between the 6th and 7th floors determined. Results are given in Table 2.

In the analogue

1 henry	represents	10^6 slugs
0.2 mA/V	"	9 ft inter-storey height
0.0182 μfd	"	7.7×10^{-11} ft/lb

These figures imply a time-scale factor of $\rho = 65$.

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NOMENCLATURE

y total deflection of a bending and shearing structure

z deflection in bending alone

x distance measured along the beam

ψ angular orientation of the mass or end of the beam

M bending moment

S shear force

l interstorey height

E Young's modulus

I 2nd moment of cross-sectional area

m mass

I_m moment of inertia of the mass about an axis through its centre of mass and normal to the xy-plane

r shear stiffness for unit length of beam

ρ time scaling factor

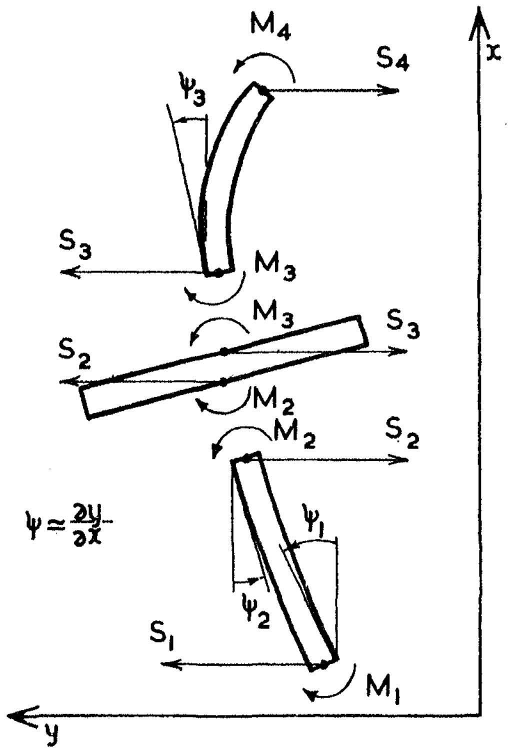

Fig. 1 Sign convention for the cordinates for the beams and masses. Arrows denote the positive directions of moment and force at the various points in the composite structure.

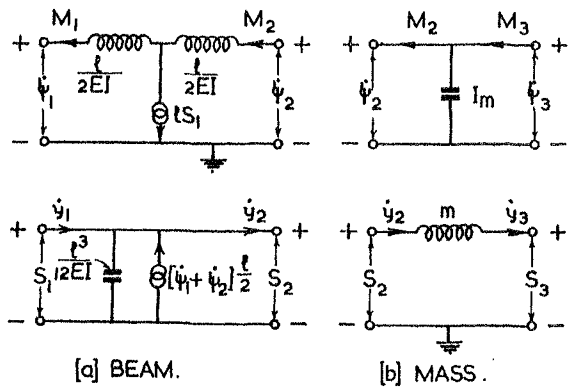

Fig. 2 Theoretical analogue for a single storey deflecting in pure bending.

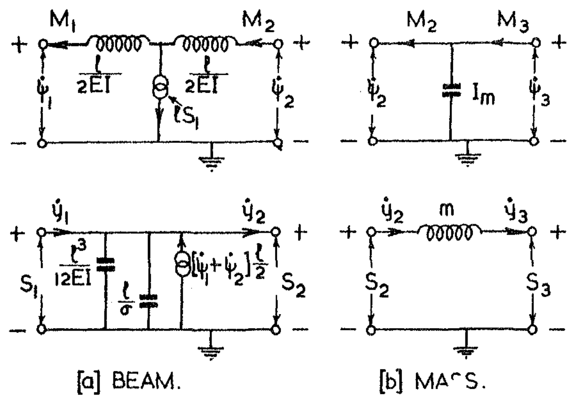

Fig. 3 Theoretical analogue for a single storey deflecting in bending and shear.

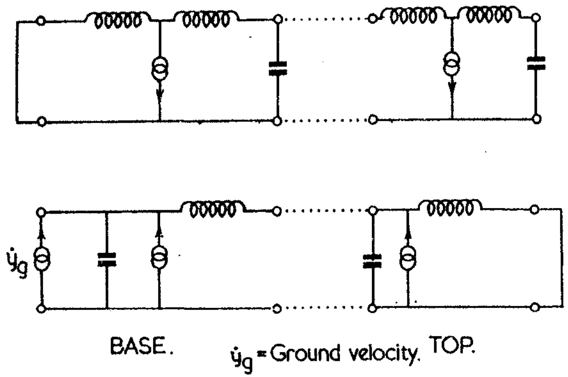

Fig. 4 Imposition of the boundary conditions, clamped-free.

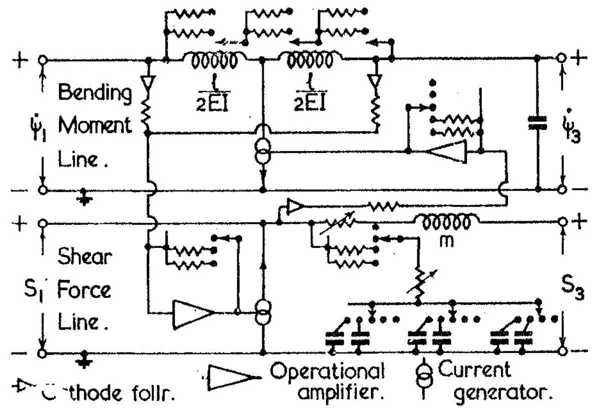

Fig 5 Block diagram of the practical circuitry for a single storey.

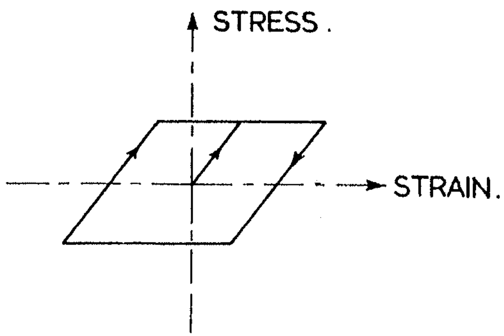

Fig. 6 Yield characteristic.

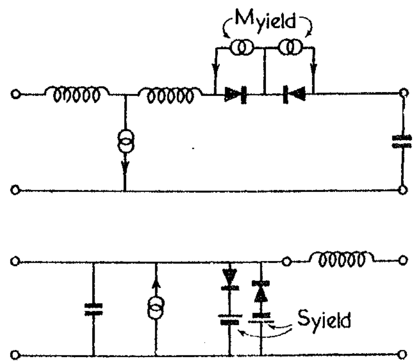

Fig. 7 Theoretical analogue for yield in a single storey.

TABLE 1

Experimental and calculated results for the shear force ratios in the first two modes of a uniform 10-storey building, deflecting in pure bending

Floor No.	Mode 1			Mode 2		
	Experimental	Calculated	Error as percentage of maximum value	Experimental	Calculated	Error as percentage of maximum value
1	1.000	1.000 0	0.00	1.000	1.000 0	0.00
2	0.995	0.996 3	- 0.13	0.960	0.962 4	- 0.24
3	0.980	0.982 1	- 0.21	0.840	0.838 7	+ 0.13
4	0.950	0.951 7	- 0.17	0.618	0.618 9	- 0.09
5	0.900	0.900 3	- 0.03	0.327	0.326 3	+ 0.07
6	0.825	0.824 3	+ 0.07	0.002	0.008 8	- 0.68
7	0.720	0.720 6	- 0.06	- 0.255	-0.272 7	+ 1.77
8	0.585	0.587 4	- 0.24	- 0.444	-0.456 6	+ 1.26
9	0.424	0.423 2	+ 0.08	- 0.480	-0.491 8	+ 1.18
10	0.225	0.227 5	- 0.25	- 0.336	-0.344 6	+ 0.86
Natural Frequency	0.442 c/s	0.442 c/s	< 0.23 %	2.80 c/s	2.80 c/s	< 0.35 %

TABLE 2

Details of building set up on the computer

	Each Storey	Foundations
Mass	4×10^4 slugs	4.05×10^4 slugs
Interstorey height	9 ft	16.5 ft
EI	21.6×10^{12} lb-ft^2	28×10^{12} lb-ft^2
Shear stiffness	1.3×10^{10} lb-ft^{-1}	∞
Moment of inertia	3.5×10^6 slugs ft^2	1.4×10^6 slugs ft^2

Natural frequencies: fundamental 2.75 c/s

1st overtone 14.4 c/s

2 % critical damping

Measured values of shear force:

at base of foundation: 7.4×10^6 lb

between 6th and 7th floors 4.95×10^6 lb