NONLINEAR VIBRATIONS OF BUILDING STRUCTURES

BY NORIHIRA ANDO[1)]

## PREFACE

The dynamical force-displacement relation of a building is complicated in its dependency on the number of repetition of the displacement as well as on the magnitude of displacement and velocity. Hence, there are many difficulties in the prediction of the effective rigidity and the damping of a building structure in a state of violent motions due to a strong motion earthquake. In spite of the development of the response spectrum techniques, one of difficulties seems to be in this respect in introducing a substantial design basis from the studies based upon them.

A building structure producing a large deflection beyond its elastic limit is generally recognized as a hysteretic system. Though the dynamical properties of restoring force of a building structure is complicated and have many varieties, some typical hysteretic forms of the force-displacement relation may be assumed. The response computing techniques for the nonlinear systems must be developed, for which the fundamental theoretical basis will be desirable.

In this paper, studies on vibrating systems with some typical forms of the hysteretic force-displacement relations are presented. In Chap. I, theory of stationary vibrations of hysteretic single degree of freedom systems is treated. In Chap. II, solutions of transient vibrations of the hysteretic bilinear single degree of freedom systems are shown. In Chap. III, a method to discriminate the stability of the stationary vibrations of hysteretic single degree of freedom systems is given. The discriminant of the stability are explicitly obtained about the systems treated in the previous chapters. In Chap. IV, the transient vibrations of hysteretic bilinear two degrees of freedom system are solved.

## CHAPTER I
## FORCED STATIONARY VIBRATIONS OF HYSTERETIC SINGLE DEGREE OF FREEDOM SYSTEMS

### 1.1 Equation of motion

Let the force-displacement relation for the forced stationary vibrations of a single degree of freedom structure be as shown in Fig. 1. The equation of motion for a vibration excited by a stationary harmonic force is, for non-negative velocity,

$$(1.1) \qquad m\frac{d^2y}{dt^2} + f(y,a) = -P\cos(pt+\phi)$$

where $\phi$ is the phase angle between the force and the displacement. The spring stiffness for a small displacement is

$$(1.2) \qquad c = \left[\frac{\partial f(y,a)}{\partial y}\right]_{y=o,a=o}, \quad \text{from which } \omega = \sqrt{\frac{c}{m}}$$

1)Assist. Prof. of Structural Engineering, Yokohama National University, Yokohama, Japan

is the natural circular frequency of the small amplitude vibration. Applying new variables

(1.3) $\xi = \frac{y}{a},\ \tau = pt,\ \lambda = \frac{p}{\omega},\ n = \frac{ca}{P},$

Eq.(1.1) can be transformed into the following dimensionless form:

(1.4) $$\ddot{\xi} + \frac{1}{\lambda^2}\psi(\xi) = -\frac{1}{n\lambda^2}\cos(\tau+\phi)$$

where

(1.5) $$\psi(\xi) = \frac{f(y,a)}{ca} = \frac{f(a\xi,a)}{ca}.$$

In Eq.(1.4) and throughout this paper, dots over a quantity refer to differentiations with respect to the variable $\tau$. The notation n is a measure of amplitude, the unit of which is the statical displacement of an elastic system having the spring stiffness c, produced by an external force P.

Provided that a harmonic type stationary vibration occurs; for which the initial and the terminal conditions of one half cycle during which the velocity takes non-negative values can be written as follows:

(1.6) $\tau = 0$: $\xi = -1,\ \dot{\xi} = 0$;

(1.7) $\tau = \pi$: $\xi = 1,\ \dot{\xi} = 0$.

The equation of motion (1.4) satisfying conditions (1.6) can be replaced by the following equivalent integral equations:

$$\begin{cases} \xi = -\cos\tau + \dfrac{1}{2n\lambda^2}(\sin\tau - \tau\cos\tau)\sin\phi - \tau\sin\tau\cos\phi \\ \qquad + \dfrac{1}{\lambda^2}\displaystyle\int_0^\tau \left[\lambda^2\xi(\tau') - \psi\{\xi(\tau')\}\right]\sin(\tau-\tau')\,d\tau', \\ \dot{\xi} = \sin\tau + \dfrac{1}{2n\lambda^2}\{\tau\sin\tau\sin\phi - (\sin\tau+\tau\cos\tau)\cos\phi\} \\ \qquad + \dfrac{1}{\lambda^2}\displaystyle\int_0^\tau \left[\lambda^2\xi(\tau') - \psi\{\xi(\tau')\}\right]\cos(\tau-\tau')\,d\tau'. \end{cases}$$

In order that the above equations satisfy the terminal conditions (1.7), we must have

(1.8) $$\begin{cases} \sin\phi = -\dfrac{2n}{\pi}\displaystyle\int_0^\pi \left[\lambda^2\xi(\tau) - \psi\{\xi(\tau)\}\right]\sin\tau\,d\tau, \\ \cos\phi = \dfrac{2n}{\pi}\displaystyle\int_0^\pi \left[\lambda^2\xi(\tau) - \psi\{\xi(\tau)\}\right]\cos\tau\,d\tau. \end{cases}$$

### 1.2 Hysteretic system with quadratic force-displacement relation

For the following force-displacement relation:

(1.9) $$f(y) = c(y - \tfrac{1}{2}\beta y^2)$$

maintained throughout the interval corresponding to the non-negative velocity, the dimensionless restoring force (1.5) becomes

(1.10) $$\psi(\xi) = \tfrac{1}{2}\beta a(1-\xi^2) + \sqrt{1-\beta^2a^2}\,\xi.$$

For a small amplitude vibration, the displacement may be assumed as sinusoidal form $\xi = -\cos\tau$. Inserting this in Eq.(1.10) and carrying out the integration in Eqs.(1.8), we have

(1.11) $$\frac{\sin\phi}{n} = \frac{4}{3\pi}\beta a,\quad \frac{\cos\phi}{n} = \sqrt{1-\beta^2a^2} - \lambda^2.$$

From the above relations, the equations of the response and the phase angle curves will easily be derived.

The equation of the response curve becomes a quadratic equation of $(\lambda^2)$, and its discriminant vanishes at the resonant state. Hence, we have the resonance amplitude and the resonance phase angle, respectively, as follows:

$$(\beta a)_{Res.} = \sqrt{\frac{3\pi}{8}\frac{2\beta P}{c}}, \quad \phi_{Res.} = \frac{\pi}{2}.$$

The response and the phase angle curves are calculated and shown in Fig. 5. In the figure, $c/2\beta P$ is the ratio of the maximum value of $f(y)$ to the exciting force amplitude P. The response curves are somewhat similar to those of so-called softening spring type nonlinear systems.

When the amplitude increases largely, the wave form diverges from the sinusoidal one, and the exact solution should be obtained by numerical integration of the equation of motion. The wave forms of the stationary vibrations are calculated, from which the response and the phase angle curves are obtained as shown in Fig. 6. The full lines in the figure are the results, and the dotted lines are given by Eqs.(1.11). The latter curves show good approximation, but the exact wave forms differ considerably from the sinusoidal one, especially for low frequency excitation.

1.3 Hysteretic bilinear system

Hysteretic bilinear restoring force shown in Fig. 2 can be written as follows:

$$(1.12)\quad \begin{cases} -a \leqq y \leqq -(a-2e): & f(y,a) = c(y+a-e), \\ -(a-2e) \leqq y \leqq a: & f(y,a) = ce = f_o, \end{cases}$$

where e is the elastic limit displacement, and $f_o$ is the ultimate strength of the system. From Eqs.(1.12), the dimensionless restoring force defined by Eq.(1.5) becomes

$$(1.13)\quad \begin{cases} -1 \leqq \xi \leqq -(1-\frac{2f_o}{Pn}): & \psi(\xi) = \xi+1-\frac{f_o}{Pn}, \\ -(1-\frac{2f_o}{Pn}) \leqq \xi \leqq 1: & \psi(\xi) = \frac{f_o}{Pn}. \end{cases}$$

For the amplitude larger than e, the system behaves as a nonlinear system. For an interval of a harmonic type stationary vibration, for which the initial and the terminal conditions (1.6) and (1.7) have been given, let the transition time of the stationary motion from the elastic part of the restoring force to the nonlinear part be

$$(1.14)\quad \tau = \alpha\pi.$$

It is evident that $\alpha$ is some positive value not larger than the unity. Inserting Eqs.(1.13) into Eq.(1.4) and using the conditions of continuity of displacement and velocity at the time $\tau = \alpha\pi$, we obtain the following relations:

$$(1.15)\quad \begin{cases} \frac{f_o}{P} = \frac{1}{|1-\lambda^2|}\frac{C(\alpha,\lambda)}{\{A^2(\alpha,\lambda)+B^2(\alpha,\lambda)\}^{\frac{1}{2}}}, \quad \phi = \tan^{-1}\frac{A(\alpha,\lambda)}{B(\alpha,\lambda)}, \end{cases}$$

$$n = \frac{f_o}{P}\left\{1+\frac{(1-\alpha)^2\pi^2}{4\lambda^2}\right\} - \left\{(1-\alpha)\pi - \sin\alpha\pi\right\}\frac{\sin\phi}{2\lambda^2} - (1+\cos\alpha\pi)\frac{\cos\phi}{2\lambda^2},$$

where

$$A(\alpha,\lambda) = (1-\alpha)\pi(\cos\alpha\pi - \cos\frac{\alpha\pi}{\lambda}) + \sin\alpha\pi(1+\cos\frac{\alpha\pi}{\lambda}) - \lambda(1+\cos\alpha\pi)\sin\frac{\alpha\pi}{\lambda},$$

$$B(\alpha,\lambda) = (1-\alpha)\pi\sin\alpha\pi - (1+\cos\alpha\pi)(1+\cos\frac{\alpha\pi}{\lambda}) - \lambda\left\{(1-\alpha)\pi+\sin\alpha\pi\right\}\sin\frac{\alpha\pi}{\lambda} + 2\lambda^2(1+\cos\frac{\alpha\pi}{\lambda}),$$

$$C(\alpha,\lambda) = (1+\cos\alpha\pi)(1-\cos\frac{\alpha\pi}{\lambda}) - 2\lambda\sin\alpha\pi\sin\frac{\alpha\pi}{\lambda} + \lambda^2(1-\cos\alpha\pi)(1+\cos\frac{\alpha\pi}{\lambda}).$$

In case of the elastic resonance $\lambda = 1$, taking the limit $\lambda \to 1$ in the above formulas, we obtain the corresponding formulas which take the place of Eqs.(1.15). Since Eqs.(1.15) are formulated as to contain the variables $\phi$ and n separated and explicitly given by the second and the third equations, numerical calculations are facilitated.

The response and the phase angle curves for constant amplitude external forces are calculated by Eqs.(1.15) and shown in Fig. 7. The response curve for the exciting force $P < f_o$ has a resonance point. The resonance phase angle is nearly equal to $\pi/2$. After some lengthy computations, we know a fact that the slope of any response curve varies continuously when the frequency is varied passing through the transition points from elastic to nonlinear range.

Since the effect of sinusoidal ground motion

(1.16) $y_o = a_o \sin pt$

can be considered as equivalent to the effect of exciting force having an amplitude $P = mp^2 a_o$, using the ratio

(1.17) $\nu = \frac{a_o}{e}$, from Eqs.(1.3) we have $n = \frac{a}{e}\frac{1}{\nu\lambda^2}$, $\frac{f_o}{P} = \frac{1}{\nu\lambda^2}$.

Substituting these relations into Eqs.(1.15), we obtain the response to the ground motion. The response and the phase angle curves for the ground motion are shown in Fig. 8. The resonance phase angle is somewhat larger than $\pi/2$. The slope of the response curve varies continuously through the transition points from the elastic to the nonlinear range.

# CHAPTER II
# TRANSIENT VIBRATIONS OF HYSTERETIC BILINEAR SINGLE DEGREE OF FREEDOM SYSTEMS

## 2.1 Equation of motion

For the hysteretic bilinear restoring force shown in Fig. 4, the course A-C-B represents an interval of non-negative velocity. At the end of this interval B, the velocity turns its sign, and the next interval represented by the dotted lines in the figure is an interval of no positive velocity. The motion during the latter interval can be inverted into an interval which is similar to the former one by inverting the sign of the excitation as well as the signs of the abscissa and the ordinate of the restoring force diagram. So that it is sufficient to consider an interval represented by the course A-C-B of the figure, and we will call this interval as one half wave of the transient vibration.

For the above mentioned half wave, the restoring force $f(y)$ in the equation of the motion excited by a ground motion $y_o$

$$(2.1)\qquad m\frac{d^2y}{dt^2} + f(y) = -m\frac{d^2y_o}{dt^2}$$

can be written as follows:

$$(2.2)\qquad \begin{cases} e\eta_o \leqq y \leqq e\eta_p: & f(y) = cy, \\ e\eta_p \leqq y: & f(y) = \gamma^2 c\left\{y + \left(\frac{1}{\gamma^2} - 1\right)e\eta_p\right\}. \end{cases}$$

In Eqs.(2.2), $\gamma^2$ is a constant giving the ratio of the slope of the restoring force diagram in the nonlinear range to that in the elastic one. Defining the dimensionless displacement and restoring force

$$(2.3)\qquad \eta = \frac{y}{e},\ \chi(\eta) = \frac{f(y)}{ce} = \frac{f(e\eta)}{ce},$$

and using the notations $\tau$, $\Lambda$ in Eqs.(1.3), the equation of motion (2.1) can be transformed into the following form:

$$(2.4)\qquad \ddot{\eta} + \frac{1}{\Lambda^2}\chi(\eta) = -\frac{1}{\Lambda^2 f_o}\left(m\frac{d^2y_o}{dt^2}\right)$$

where

$$(2.5)\qquad \begin{cases} \eta_o \leqq \eta \leqq \eta_p: & \chi(\eta) = \eta, \\ \eta_p \leqq \eta: & \chi(\eta) = \gamma^2\left\{\eta + \left(\frac{1}{\gamma^2} - 1\right)\eta_p\right\}. \end{cases}$$

In Eq.(2.4), the right hand side is the ratio of the inertia force acting on a rigid mass $m$ to the ultimate strength of the system $f_o$ divided by the square of period ratio $\Lambda^2$.

<u>2.2 Transient motions</u>

Transient vibrations excited by the sinusoidal ground motion (1.16) are shown in Fig. 9 and Fig. 10, in which $\nu$ is the ratio defined by Eq.(1.17). The transient waves of the system with the constant $\gamma^2 = 0$ are shown in Fig. 9, and those of the system with the constant $\gamma^2 = 0.2$ are shown in Fig. 10.

The transient waves shown in Fig. 9 has a tendency to converge to a harmonic type stationary vibration 'monotonically' and very rapidly. The solutions shown in the figure attain almost stationary states at their end of the transient waves. It is seen in the figure that there may be a tendency in a transient motion to yield a large amount of displacement after a number of repetition of vibrational waves, due to a creeping of the displacement towards one direction. *This tendency becomes more remarkable for greater* frequency and larger amplitude of the ground motion.

The transient waves shown in Fig. 10 are somewhat different from those of the system above mentioned. There are some fluctuations in the waves, due to the fact that there exists always some restoring action on the vibrating mass to coincide with the center of the restoring force O shown in Fig. 4. The phenomena of the creeping of the displacement are not so remarkable as in the system above mentioned. The transient motion does not converge to the stationary state 'monotonically'. But the transient vibrations have still a tendency to converge to the corresponding harmonic type stationary state.

# CHAPTER III
# STABILITY OF FORCED STATIONARY VIBRATIONS OF HYSTERETIC SINGLE DEGREE OF FREEDOM SYSTEMS

## 3.1 Discriminant of stability

A method to discriminate the stability of harmonic type stationary vibrations are shown in this chapter, in regard to the systems with the restoring force $f(y,a)$ which is determined by the magnitude of the restoring force $-f(a,a) = f(a,a)$ at both the extremities of the half cycle, and does not depend directly on the value of the displacement.

The dimensionless restoring force for the stationary state have been defined by Eq.(1.5). The restoring force for an interval of an unstationary motion with non-negative velocity which will be produced by imposing an infinitesimal variation on a stationary state, can be expressed by $f(y,a+\Delta a_o)$, provided that the origin of the displacement y is appropriately chosen. The relation between the restoring force for the stationary state $f(y,a)$ and that for the disturbed motion $f(y,a+\Delta a_o)$ is shown in Fig. 1. Let the displacement of the disturbed motion be $y+\Delta y$, and applying the same notation as in Eqs.(1.3), then the restoring force $f(y+\Delta y,a+\Delta a_o)$ for this motion can be transformed into

$$(3.1) \qquad \psi(\xi+\Delta\xi) = \psi(\xi)+\psi'(\xi)\Delta\xi-\frac{\partial f(y,a)}{c\partial a}\Delta\xi_o, \quad \text{where}$$

$$\psi'(\xi) = \frac{d\psi(\xi)}{d\xi} = \frac{\partial}{\partial(\frac{y}{a})}\frac{f(y,a)}{ca}, \quad \Delta\xi = \frac{\Delta y}{a}, \quad \Delta\xi_o = -\frac{\Delta a_o}{a}.$$

Let the variation of the restoring force $\Delta f_o$ be given at the initial state of the one half cycle. The corresponding variation $\Delta a_o$ is not a real variation, but that corresponds to the real variation in the restoring force $\Delta f_o$ as shown in Fig. 1. The signs of $\Delta a_o$ and $\Delta\xi_o$ are opposite, because an increment in the amplitude means a decrease in the initial displacement, and vice versa.

For an interval during which the velocity takes non-negative values, and initial and terminal velocities are zero, the initial conditions for any unstationary motion as a disturbed state of the stationary vibration can be represented by

$$(3.2) \qquad \tau = 0: \ \xi' = -1+\Delta\xi_o, \ \dot{\xi}' = 0, \ \phi' = \phi+\Delta\phi_o,$$

where $\Delta\phi_o$ is an infinitesimal variation of the phase angle.

By Eq.(3.1) and the equation of stationary vibration (1.4), the equation of motion of the disturbed state occuring in an interval of non-negative velocity can be transformed into

(3.3) $\ddot{u} + \frac{1}{\lambda^2}\psi'(\xi)u = \frac{1}{\lambda^2}g(\tau)\Delta\xi_o, \quad u = \Delta\xi - \dot{\xi}\Delta\phi_o, \quad g(\tau) = \frac{\partial f(y,a)}{c\partial a}.$

Now, let a particular solution of the differential equation (3.3) be $u_o\Delta\xi_o$, and the fundamental solutions of Hill's equation

$$\ddot{u} + \frac{1}{\lambda^2}\psi'(\xi)u = 0$$

satisfying the following initial conditions:

$$\tau = 0: \; u_1 = 0, \; \dot{u}_1 = 1; \; u_2 = 1, \; \dot{u}_2 = 0$$

be $u_1$ and $u_2$. Then the general solution of Eq.(3.3) is given by linear combination of this particular solution $u_o$ and the fundamental solutions $u_1$, $u_2$. Determining the arbitrary constants in the general solution by the initial conditions (3.2), we find

(3.4) $u = \Delta\xi - \dot{\xi}\Delta\phi_o = \left[u_o + \{1-u_o(0)\}u_2 - \dot{u}_o(0)u_1\right]\Delta\xi_o - \alpha u_1\Delta\phi_o$

where

$$\alpha = \ddot{\xi}_{\tau=o} = -\ddot{\xi}_{\tau=\pi} = \frac{n\,\psi(1)-\cos\phi}{n\lambda^2}$$

is the acceleration of the stationary vibration at the initial state of one half cycle corresponding to the interval of non-negative velocity.

At the time $\tau = \pi$, the acceleration of the disturbed motion becomes

$$-\alpha + \frac{d^2}{d\tau^2}(\Delta\xi)_{\tau=\pi}.$$

Hence, setting the time $\tau = \pi + \Delta\tau_1$, when the velocity becomes zero, we may write

(3.5) $\frac{d}{d\tau}(\Delta\xi)_{\tau=\pi} = \alpha\Delta\tau_1.$

We considered up to now an interval of motion in which the velocity takes non-negative values and the initial and the terminal velocities are zero. The next interval corresponding to no positive values of the velocity can be inverted into an interval which is similar to the first one by inverting the sign of the exciting force as well as the signs of the abscissa and the ordinate of the restoring force diagram. In this sense, during the next interval, the sign of velocity can be treated as non-negative. The variation of the phase angle at the initial state of the next interval in the above mentioned sense is

(3.6) $\Delta\phi_1 = \Delta\tau_1 + \Delta\phi_o.$

Representing, the variation of the displacement at the time $\tau = \pi$ by $\Delta\xi_1'$, from Eq.(3.5) and Eq.(3.6) we have

$$u(\pi) = \Delta\xi_1', \quad \dot{u}(\pi) = \frac{d}{d\tau}(\Delta\xi)_{\tau=\pi} - \ddot{\xi}_{\tau=\pi}\Delta\phi_o = \alpha\Delta\phi_1.$$

Combining these formulas with Eq.(3.4), we obtain

$$(3.7) \quad \begin{cases} \Delta\phi_1 = -\dot{u}_1(\pi)\Delta\phi_o + \frac{1}{\alpha}\left[\dot{u}_o(\pi) + \{1-u_o(0)\}\dot{u}_2(\pi) - \dot{u}_o(0)\dot{u}_1(\pi)\right]\Delta\xi_o, \\ \Delta\xi_1' = -\alpha u_1(\pi)\Delta\phi_o + \left[u_o(\pi) + \{1-u_o(0)\}u_2(\pi) - \dot{u}_o(0)u_1(\pi)\right]\Delta\xi_o. \end{cases}$$

The variation of displacement at the end of one half cycle is given by

(3.8) $\Delta a_1' = a\Delta\xi_1'$.

The magnitude of the restoring force at the end of one half cycle of the disturbed motion will become $f(a+\Delta a_1', a+\Delta a_o)$. Considering a stationary vibration producing an amplitude $a+\Delta a_1$, in which the same magnitude of the maximum restoring force as that of the terminal state of the disturbed motion will be attained, we may put

$$f(a+\Delta a_1', a+\Delta a_o) = f(a+\Delta a_1, a+\Delta a_1).$$

Differentiating this formula, we have

$$\frac{\partial f(a,a)}{\partial y}\Delta a_1' + \frac{\partial f(a,a)}{\partial a}\Delta a_o = \left\{\frac{\partial f(a,a)}{\partial y} + \frac{\partial f(a,a)}{\partial a}\right\}\Delta a_1$$

where the meaning of the derivatives are

$$\frac{\partial f(a,a)}{\partial y} = \left[\frac{\partial f(y,a)}{\partial y}\right]_{y=a}, \quad \frac{\partial f(a,a)}{\partial a} = \left[\frac{\partial f(y,a)}{\partial a}\right]_{y=a}.$$

If we apply the following notations:

(3.9) $\Delta\xi_1 = -\dfrac{\Delta a_1}{a}$

to the above relation, we obtain

(3.10) $-\psi'(1)\Delta\xi_1' = -\dfrac{\partial f(a,a)}{c\partial a}\Delta\xi_o + \left\{\psi'(1) + \dfrac{\partial f(a,a)}{c\partial a}\right\}\Delta\xi_1.$

Since the value of $\Delta\xi_1$ in Eq.(3.9) can be considered to be the variation of the displacement at the initial state of the next half cycle following that under consideration, $\Delta a_1$ and $\Delta\xi_1$ should have the opposite signs as shown in the equation.

Since the displacement in the expression of the restoring force is always measured from the origin of the coordinate axis, the relation between the variations of the amplitude $\Delta a_o$, $\Delta a_1$ and the variations of the restoring force amplitude $\Delta f_o$, $\Delta f_1$ are

$$\Delta f = f(a+\Delta a, a+\Delta a) - f(a,a) = \left\{\frac{\partial f(a,a)}{\partial y} + \frac{\partial f(a,a)}{\partial a}\right\}\Delta a,$$

or

(3.11) $\Delta\psi = -\dfrac{\Delta f}{ca} = \left\{\psi'(1) + \dfrac{\partial f(a,a)}{c\partial a}\right\}\Delta\xi,$

in which the suffixes 0, 1 are omitted for brevity. In this formula, the variation of the restoring force $\Delta\psi$ and the variation of the restoring force amplitude $\Delta f$ should have the opposite signs.

Inserting Eq.(3.10) and Eq.(3.11) into Eqs.(3.7), we have

$$(3.12)\quad \left\{\begin{aligned} \Delta\phi_1 &= -\dot{u}_1(\pi)\Delta\phi_o + \frac{\dot{u}_o(\pi) + \{1-u_o(0)\}\dot{u}_2(\pi) - \dot{u}_o(0)\dot{u}_1(\pi)}{\alpha\left\{\psi'(1) + \dfrac{\partial f(a,a)}{c\partial a}\right\}}\Delta\psi_o \\ &= a_{11}\Delta\phi_o + a_{12}\Delta\psi_o, \\ \Delta\psi_1 &= \alpha\psi'(1)u_1(\pi)\Delta\phi_o \\ &\quad - \frac{\psi'(1)\left[u_o(\pi) + \{1-u_o(0)\}u_2(\pi) - \dot{u}_o(0)u_1(\pi)\right] - \dfrac{\partial f(a,a)}{c\partial a}}{\psi'(1) + \dfrac{\partial f(a,a)}{c\partial a}}\Delta\psi_o \\ &= a_{21}\Delta\phi_o + a_{22}\Delta\psi_o. \end{aligned}\right.$$

Eqs.(3.12) show that the following linear relation hold between the variations of the phase angle and the restoring force at the initial state of j-th half cycle and the variations of the corresponding quantities at the initial state of the following (j+1)-th half cycle:

$$(3.13)\quad \begin{Bmatrix} \Delta\phi_{j+1} \\ \Delta\psi_{j+1} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} \Delta\phi_j \\ \Delta\psi_j \end{Bmatrix} = [A] \begin{Bmatrix} \Delta\phi_j \\ \Delta\psi_j \end{Bmatrix}, \quad j = 0,\ 1,\ 2,\ \ldots.$$

If the series (3.13) does not diverge when j tends to infinity, the unstationary motion under consideration continues in the neighborhood of the corresponding stationary vibration, i.e. the stationary vibration is stable, and vice versa.

In case of the matrix [A] has two different eigenvalues $\mu_1$ and $\mu_2$, [A] can be transformed into a diagonal matrix

$$\begin{Bmatrix} \Delta\phi'_{j+1} \\ \Delta\psi'_{j+1} \end{Bmatrix} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{Bmatrix} \Delta\phi'_j \\ \Delta\psi'_j \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} \Delta\phi'_j \\ \Delta\psi'_j \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \end{bmatrix}^{-1} \begin{Bmatrix} \Delta\phi_j \\ \Delta\phi_{j+1} \end{Bmatrix}.$$

If this series does not diverge, the series (3.13) does not diverge and vice versa. Hence, the conditions of stability of the stationary state are

$$(3.14)\quad |\mu_1| \leqq 1,\ |\mu_2| \leqq 1.$$

If the characteristic polynomial of [A] takes a double root $\mu$, [A] can be transformed into a triangular matrix

$$\begin{Bmatrix} \Delta\phi'_{j+1} \\ \Delta\psi'_{j+1} \end{Bmatrix} = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix} \begin{Bmatrix} \Delta\phi'_j \\ \Delta\psi'_j \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} \Delta\phi'_j \\ \Delta\psi'_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \Delta\phi_j \\ \Delta\phi_{j+1} \end{Bmatrix}.$$

Hence, the condition of stability should be

$$(3.15)\quad |\mu| < 1.$$

### 3.2 Application to hysteretic system with quadratic force-displacement relation

If we assume $\beta a$ as a small quantity in a small amplitude vibration, and neglecting the powers of $\beta a$ above the first, we have the following result in regard to the system treated in Sec. 1.2, Chap. I.

$$\mu_{1,2} = -(1-\beta a)(\cos\frac{\pi}{\lambda} \pm i \sin\frac{\pi}{\lambda}) = -(1-\beta a)e^{\pm i\frac{\pi}{\lambda}}$$

Hence, we have the relation

$$|\mu_{1,2}| = 1-\beta a \leqq 1,$$

from which we can conclude that the stationary vibration is always stable at least for the small amplitude, and moreover, any unstationary motion in the neighborhood of the stationary vibration always converge to the stationary state.

### 3.3 Application to hysteretic bilinear systems

For the restoring force of the hysteretic bilinear system in a stationary motion, the relation between the restoring force and the displacement shown in Fig. 3 can be written as follows:

$$\begin{cases} -a \leqq y \leqq -(a-2e): & f(y,a) = c\{y + (1-\gamma^2)(a-e)\}, \\ -(a-2e) \leqq y \leqq a: & f(y,a) = c\{\gamma^2 y + (1-\gamma^2)e\}. \end{cases}$$

For the system with the above mentioned restoring force, the following result will be derived:

$$(3.16)\quad \mu_{1,2} = -\left(\frac{1+\gamma^2}{2}+\frac{1-\gamma^2}{2}\cos\frac{\alpha\pi}{\lambda}\right)\cos\theta \pm\sqrt{\left(\frac{1-\gamma^2}{2}\right)^2\left(1-\cos\frac{\alpha\pi}{\lambda}\right)^2-\left(\frac{1+\gamma^2}{2}+\frac{1-\gamma^2}{2}\cos\frac{\alpha\pi}{\lambda}\right)^2\sin^2\theta}\,,$$

where
$$\frac{1-\gamma^2}{2}+\frac{1+\gamma^2}{2}\cos\frac{\alpha\pi}{\lambda} = \left(\frac{1+\gamma^2}{2}+\frac{1-\gamma^2}{2}\cos\frac{\alpha\pi}{\lambda}\right)\cos\left\{\theta-\frac{\gamma(1-\alpha)\pi}{\lambda}\right\},$$
$$\sin\frac{\alpha\pi}{\lambda} = \left(\frac{1+\gamma^2}{2}+\frac{1-\gamma^2}{2}\cos\frac{\alpha\pi}{\lambda}\right)\sin\left\{\theta-\frac{\gamma(1-\alpha)\pi}{\lambda}\right\}.$$

In Eq.(3.16), $\alpha$ is the quantity defined by Eq.(1.14). It will easily be verified that regardless of $\mu_{1,2}$ being real or imaginary, there always holds the following relation:

$$|\mu_{1,2}| \leqq 1.$$

From the above result, we can conclude that the stationary vibrations are always stable. Furthermore, at least in the neighborhood of a stationary state, any unstationary motion has a tendency to converge to the corresponding stationary vibration, excepting the special case $|\mu_{1,2}| = 1$. The relation between the values of $\alpha$ and $|\mu_{1,2}|$ are calculated by using Eq.(3.16), and shown in Fig. 11. In the figure, $\mu_{1,2}$ take real values in the range of $\alpha$ in which two values of $|\mu_{1,2}|$ exist, and the values of $\mu_{1,2}$ become imaginary in the remaining range of $\alpha$. It is also seen in case of $\lambda \leqq 1$ that one of $|\mu_{1,2}|$ takes the maximum value unity for some value of $\alpha$ in the real range.

Inserting $\gamma^2 = 0$ into Eq.(3.16), we have the relations

$$\mu_1 = -\cos\frac{\alpha\pi}{\lambda},\quad \mu_2 = -1$$

which should be the discriminant of stability for the system with the restoring force shown in Fig. 2. It is to be noted that the second relation represents the fact that the state of motion is indeterminate in regard to the absolute amount of the displacement, since there is no center of restoring force in the system. Using the results of calculation of stationary vibration, the value of $\mu_1$ is computed and shown in Fig. 12.

The discriminant of stability (3.14) or (3.15) is a measure of convergency of a disturbed motion to original stationary state. The transient vibration may be regarded as a disturbed state of the stationary motion. From this point of view, the properties of the transient waves of the bilinear systems can be explained numerically, based upon the theory of stability above mentioned.

## CHAPTER IV
## TRANSIENT VIBRATIONS OF HYSTERETIC BILINEAR TWO DEGREES OF FREEDOM SYSTEM

### 4.1 Equation of motion

For an interval of the motion of a two storied structure during which the restoring force for the first story traces the course A-C-B shown in Fig. 13, which corresponds to non-negative values of the relative velocity of the first story, the equations of the motion excited by a ground motion $y_o$ can be written as follows:

$$(4.1)\quad \begin{cases} m_1 \dfrac{d^2y_1}{dt^2} + f_1(y_1) - f_2(y_2) = -m_1 \dfrac{d^2y_o}{dt^2}, \\ m_2 \dfrac{d^2y_1}{dt^2} + m_2 \dfrac{d^2y_2}{dt^2} + f_2(y_2) = -m_2 \dfrac{d^2y_o}{dt^2}. \end{cases}$$

It is sufficient to consider only the above mentioned interval of motion, and we will call this interval of the motion as one half wave of the transient vibration. The interval corresponding to the no positive relative velocity of the first story can be inverted into an interval which is similar to the above mentioned half wave.

For the hysteretic bilinear restoring forces shown in Fig. 13, introducing the following dimensionless quantities:

$$(4.2)\quad \eta_s = \frac{y_s}{e_s},\ \tau = pt,\ \lambda_s = \frac{p}{\omega_s} = p\sqrt{\frac{m_s}{c_s}};\ s = 1, 2,$$

and the dimensionless restoring forces:

$$(4.3)\quad \chi(\eta_s) = \frac{f_s(y_s)}{c_s e_s} = \frac{f_s(e_s\eta_s)}{c_s e_s},$$

the equations of motion (4.1) can be transformed into

$$(4.4)\quad \begin{cases} \ddot{\eta}_1 + \dfrac{1}{\lambda_1^2}\chi(\eta_1) - \dfrac{m_2e_2}{m_1e_1}\dfrac{1}{\lambda_2^2}\chi(\eta_2) = -\dfrac{1}{\lambda_1^2 f_{o1}}(m_1 \dfrac{d^2y_o}{dt^2}), \\ \dfrac{e_1}{e_2}\ddot{\eta}_1 + \ddot{\eta}_2 + \dfrac{1}{\lambda_2^2}\chi(\eta_2) = -\dfrac{1}{\lambda_2^2 f_{o2}}(m_2 \dfrac{d^2y_o}{dt^2}), \end{cases}$$

where $f_{o1}$, $f_{o2}$ are the ultimate strength of the respective stories, and the restoring force terms are given by

$$(4.5)\quad \begin{cases} |\eta_s| \leqq 1:\ \chi(\eta_s) = \eta_s, \\ \eta_s \geqq 1:\quad \chi(\eta_s) = 1;\ s = 1, 2, \\ \eta_2 \leqq -1:\ \chi(\eta_2) = -1. \end{cases}$$

### 4.2 Transient motions

From the solutions of the equations of motion (4.4), the transient motions can be computed. The transient waves of a system with the constants $m_1 = m_2$, $c_1 = c_2$, $f_{o1} = f_{o2}$ excited by the sinusoidal ground motion (1.16) are shown in Fig. 14, in which the notations $\lambda = \lambda_s$, $\nu = a_o/e_s$ are used.

The wave forms of the relative displacement of the first story are quantitatively similar to that of the single degree of freedom system studied in Chap. II. As previously mentioned, there may be a tendency of creeping of the relative displacement of the first story towards one direction, which becomes more remarkable for greater frequency and larger amplitude of the ground motion. There rarely occurs the tendency of creeping of relative displacement in the second story.

The effective damping in the first story is considered to reveal its remarkable effect in the response of the second story. However, the hysteresis damping is essentially differs from a viscous type damping in its effect on the wave forms, which are very complicated in the second story even in case in which there occurs an almost elastic vibration in this story.

Due to the effective damping above mentioned, there is a tendency to converge to a stationary state in a transient wave. Most transient waves shown in the figure almost attain the stationary state at their end of wave sequences. However, it is to be noted that the stationary state to which the transient wave converges is not always a harmonic type vibration. In fact, the transient waves for $\nu = 1, 2; \Lambda = 1.2$ converge to the non-harmonic type stationary states, while the remaining waves converge to the harmonic type stationary vibrations. In the non-harmonic type stationary state, there appear alternatively two kinds of half cycles with different amount of the total displacement and the time of duration. The tendency of the creeping of the relative displacement tends to vanish in a sequence of the transient waves with the lapse of sufficient time, if the stationary state is the harmonic type. However, for a transient motion for which the non-harmonic type stationary vibration corresponds, the converged stationary state itself should have a tendency of creeping, due to the differences in the total displacements of successive half cycles.

## BIBLIOGRAPHY


(1) Committee of Seismic Tests of Structures, Japan, "Report of the Committee," No. 1, 1951; No. 2, 1953.

(2) Den Hartog, J. P., "Mechanical Vibrations," McGraw-Hill Book Co., Inc., New York, 1940.

(3) Jacobsen, L. S. and Ayre, R. S., "Engineering Vibrations," McGraw-Hill Book Co., Inc., New York, 1958.

(4) Stoker, J. J., "Nonlinear Vibrations," Interscience Publishers, Inc., New York, 1950.

(5) Timoshenko, S., "Vibration Problems in Engineering," D. Van Nostrand Co., Inc., New York, 1955.


## NOMENCLATURE

$a$ = stationary amplitude

$c$ = linear spring stiffness

$e$ = elastic limit deformation

$f(y,a)$ = restoring force, function of $y$ and $a$

$f(y)$ = restoring force, function of y

$f_o$ = ultimate strength

$m$ = mass

$n$ = $\frac{ca}{P}$, dimensionless amplitude

$p$ = circular frequency of excitation

$P$ = amplitude of external force

$t$ = time

$y$ = lateral displacement of story

$y_o$ = ground motion

$\eta$ = $\frac{y}{e}$, dimensionless lateral displacement of story

$\lambda$ = $\frac{p}{\omega}$, period ratio

$\nu$ = $\frac{a_o}{e}$

$\xi$ = $\frac{y}{a}$, dimensionless lateral displacement of story

$\tau$ = pt

$\chi(\eta)$ = $\frac{f(y)}{ce}$, dimensionless restoring force

$\psi(\xi)$ = $\frac{f(y,a)}{ca}$, dimensionless restoring force

$\omega$ = $\sqrt{\frac{c}{m}}$, natural circular frequency of linear system

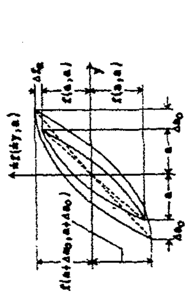

FIG. 1

Fig. 1
Restoring force diagram
of hysteretic systems

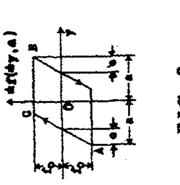

FIG. 2

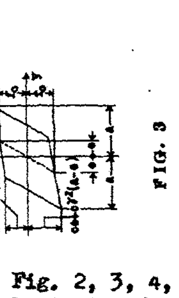

FIG. 3

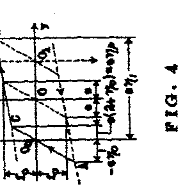

FIG. 4

Fig. 2, 3, 4,
Restoring force diagrams
of hysteretic bilinear
systems

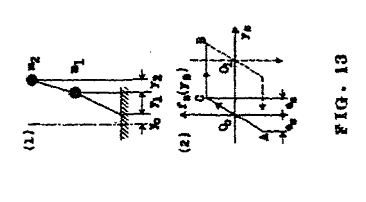

FIG. 13

Fig. 13
Restoring force diagram
of a hysteretic bilinear
two degrees of freedom
system

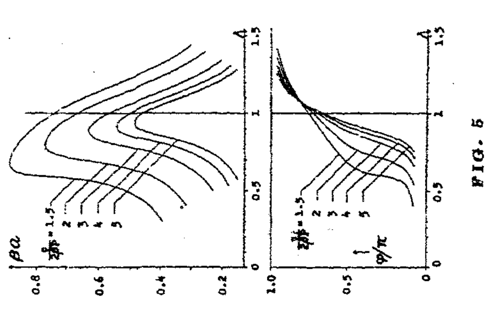

FIG. 5

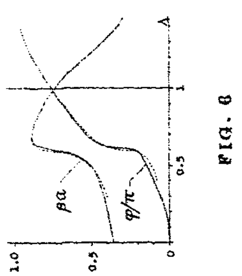

FIG. 6

Fig. 5, 6
Response curves of a
hysteretic system

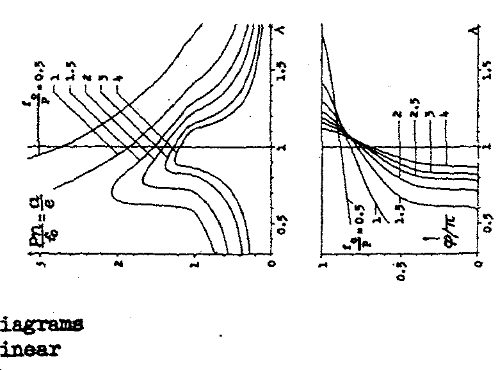

FIG. 7

Fig. 7
Response curves of a hysteretic
bilinear system

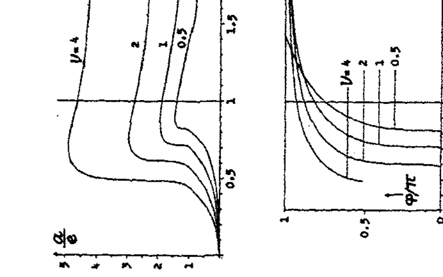

FIG. 8

Fig. 8
Response curves of a hysteretic
bilinear system

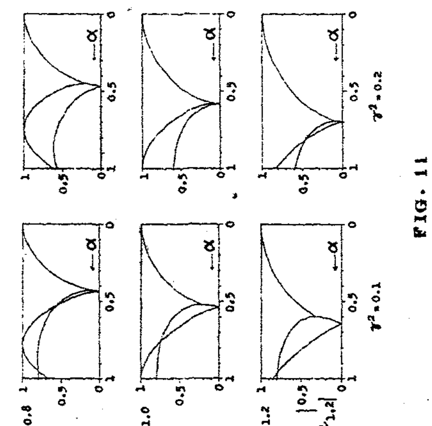

FIG. 11

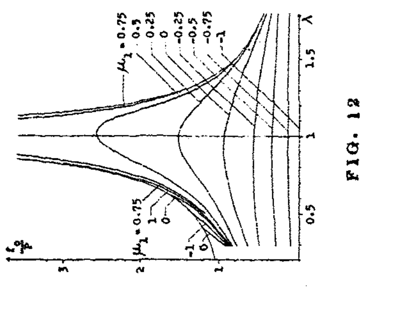

FIG. 12

Fig. 11, 12
Stability curves of hysteretic bilinear systems

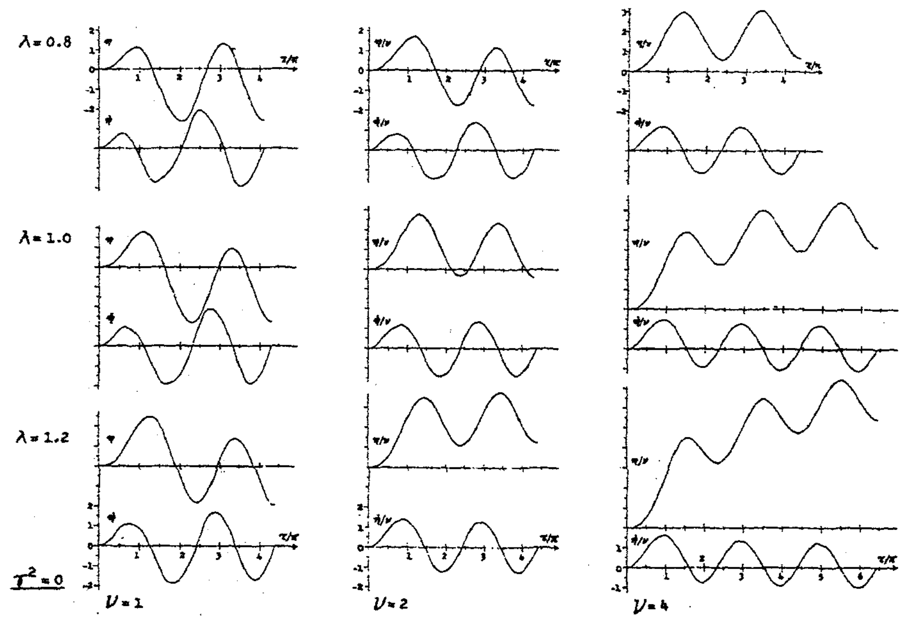


FIG. 9

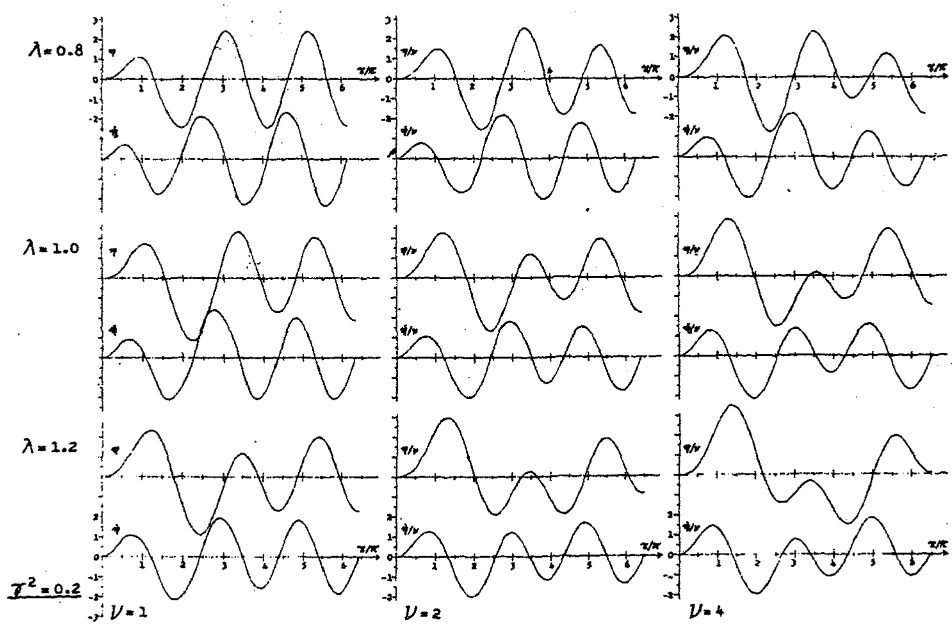


FIG. 10

Transient waves of hysteretic bilinear systems

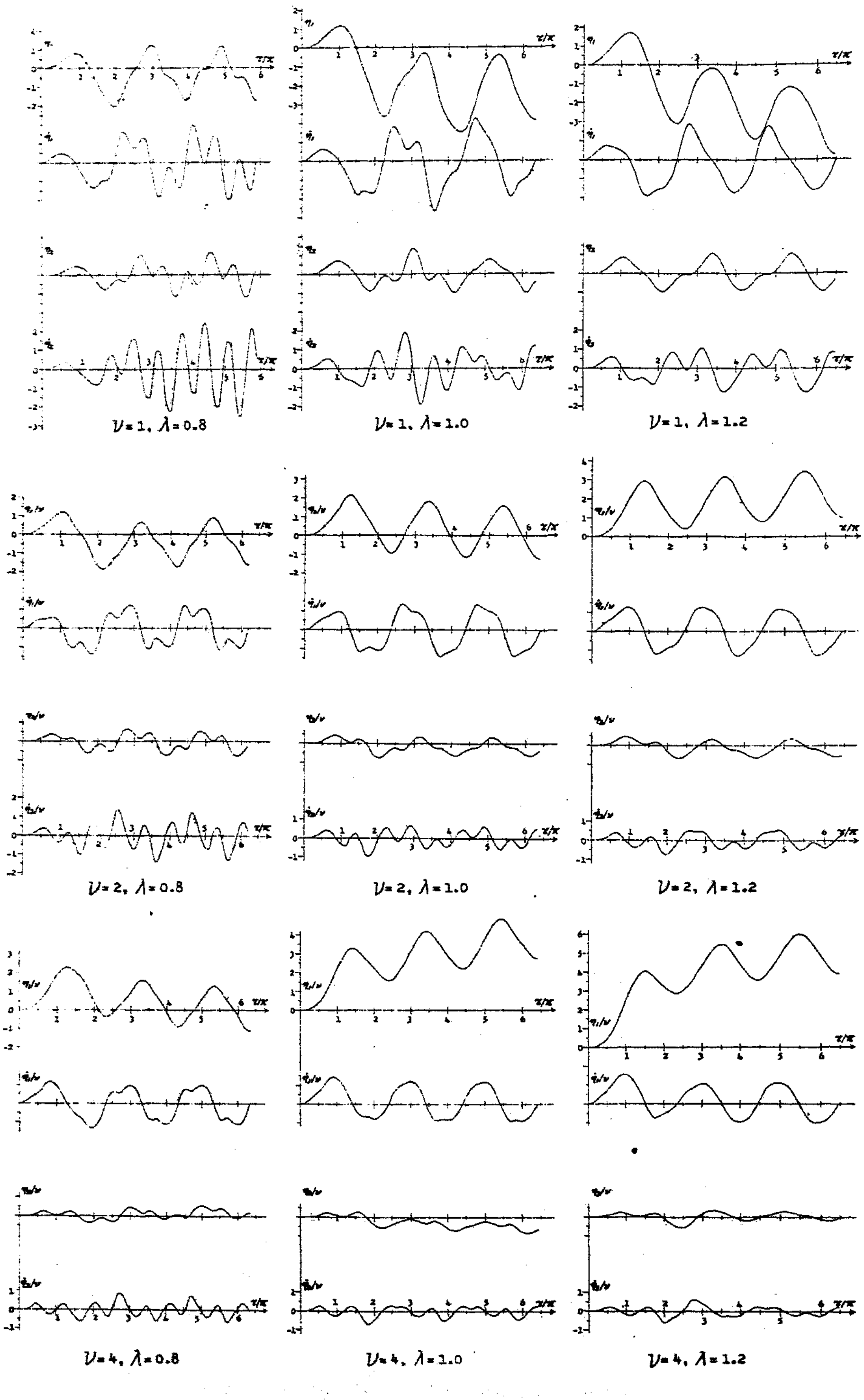


FIG. 14 **Transient waves** of a hysteretic **bilinear two degrees of freedom system**