The influence of concentrated damages on the dynamic behaviour of framed structures

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SUMMARY
Structural health monitoring techniques are often based on static or dynamic response-based damage detection methods since the occurrence of damage can alter both the static and the dynamic behaviour of structures. The first needed step for successfully solving an inverse damage identification problem, appears to be the definition of a reliable model of the damaged structures and the evaluation of an accurate solution of the direct problem.
In this paper a new powerful approach, aimed to evaluate the frequencies and the vibration modes of frame structures in presence of concentrated damage, is presented. This approach is based on a closed form solution, previously derived by the authors, relative to the vibration modes of a beam with an arbitrary number of concentrated damage represented by means of internal elastic hinges. The exact explicit solution possesses the same analytical structure of the undamaged beam, being a function of four integration constants only, regardless of the number of the damaged cross sections. On the basis of the explicit expressions provided for the eigenmodes, the exact dynamic stiffness matrix of the multi-cracked Euler-Bernoulli beam is derived. The knowledge of the exact dynamic stiffness matrix of the multi-cracked beam, as a function of the end degrees of freedom only, represents a fundamental result since it allows the direct evaluation of the global dynamic stiffness matrix of frame structures in presence of an arbitrary number of concentrated damages along its members. The great advantage of the proposed approach, with respect to the classical approach based on the knowledge of the dynamic stiffness matrix of the undamaged beam, is that the degrees of freedom of the overall frame structure are exactly the same of the equivalent undamaged structure irrespective of the number of the concentrated damages. This fact represents an advantage both from a computational cost and an implementation effort. Moreover, it facilitates the numerical investigation relative to the evolution of damage in complex frame structures when subjected to dynamic loadings.

Keywords: damaged frame structures, concentrated damages, dynamic stiffness method.

1. INTRODUCTION

In the last decade, several studies have been conducted aimed at the detection and identification of damages in civil and mechanical structures and to assess their general health or their residual load carrying capacity both in static and in dynamic context. These structural health monitoring techniques are often based on static or dynamic response-based damage detection methods since the occurrence of damage can alter both the static and the dynamic response of structures. Damage identification on the basis of dynamic measurements is usually conducted by means of approximate numerical procedures in view of the difficulty in obtaining exact solutions of both the direct and inverse analysis problems. An overview on the use of inverse methods in damage detection and location, using measured vibration data, is provided in (Friswell, 2007). Inverse problems combine an initial model of the structure and measured data to improve the model or test a hypothesis. As a consequence, the first needed steps for successfully solving an inverse problem, appears to be the definition of a reliable model, consistent with the real structure, and the evaluation of an accurate solution of the direct problem. In view of the difficulties connected with the solution of damage identification problems for complex structures, many of the studies available in the specific literature, aimed to obtain a solution of the direct or inverse problems, have been devoted to very simple structural systems. In particular, greater attention has been devoted to the solution of the direct analysis problems of vibrating beams in the presence of single or multiple concentrated cracks, (Morassi, A., 1993; Dimarogonas, A.D. 1996;

Li, Q.S., 2002; Binici, B., 2005; Caddemi et al., 2009, 2011; Xiaoping, Z. et al. 2010).
The identification problem becomes extremely more complicated in the case of complex damaged frame structures. The approaches presented in the literature, aimed to investigate the direct or the inverse problem of frame structures in presence of concentrated damage, are generally based on finite element approximate models (Nikolakopoulos, P. G., 1991; Dado, M.H.F. et al., 2003).
The only available approach for obtaining an exact evaluation of the eigen-properties of multi-cracked frame structure, consistent with a distributed parameter systems modeling, is based on the use of the dynamic stiffness matrix of the undamaged beam and the subdivisions of the cracked beams in more elements whose number is associated to the number of damaged cross section, an example of this approach is reported in reference (Greco, A. et al. 2012). This procedure, although providing an exact solution, determines a significant increase of the overall degrees of freedom of the structures that is proportional the number n of damages. As a consequence the study of the evolution of concentrated damages in structures requires an a priori definitions of the sections that can be subjected to damage and the associated degrees of freedom.
In this paper a new powerful approach, aimed to evaluate the frequencies and the vibration modes of cracked frame structures, is presented. This approach is based on a closed form solution, derived by the authors (Caddemi et al., 2009), relative to the vibration modes of the multi-cracked beam, which possess the same analytical structure of the undamaged beam, being a function of four integration constants only, regardless of the number of the cracked cross sections. On the basis of the explicit expressions provided for the eigenmodes, the exact dynamic stiffness matrix of the multi-cracked Euler-Bernoulli beam is derived. As it is well known, the knowledge of the exact dynamic stiffness matrix of the multi-cracked beam, represents a fundamental result since it allows the direct evaluation of the global dynamic stiffness matrix of frame structures in presence of an arbitrary number of concentrated damages along its members. The great advantage of the proposed approach, with respect to the classical approach based on the knowledge of the dynamic stiffness matrix of the undamaged beam, is that the degrees of freedom of the overall frame structure are exactly the same of the equivalent undamaged structure irrespective of the number of the concentrated damages. This fact represents an advantage both from a computational cost and an implementation effort. Moreover, it facilitates the numerical investigation relative to the evaluation of the influence of damage in complex frame structures towards the identification inverse problem solution. Once the global dynamic stiffness matrix of a damage frame structure is evaluated, the solution of the nonlinear eigenvalue problem requires a safe and reliable procedure to be solved. In the paper, the Wittrick & Williams algorithm (Williams, F.W. et al. 1970), which gives all required eigenvalues with the desired accuracy, is adopted. An efficient procedure for the evaluation of the exact solution of the direct problem represents a further and important step towards the more complicated solution of the inverse identification problem of the damage detection in frame structures.

2. THE CLOSED-FORM SOLUTION OF THE CRACKED EULER-BERNOULLI BEAM

The model adopted in this study is based on the concept that a concentrated crack or a concentrated damage locally affect the flexural stiffness of the beam and that its influence can be modelled through generalised functions. According to a model already considered by the same authors (Caddemi et al., 2009) the governing differential equation of a multi-cracked uniform beam may be written in the following form.

$$\left[E_o I_o\left(1-\sum_{i=1}^{n}\hat{\gamma}_i\delta(x-x_{oi})\right)u''(x,t)\right]''+m\ddot{u}(x,t)=0 \tag{2.1}$$

where prime denotes differentiation with respect to the spatial coordinate x along the beam axis and the superimposed dot differentiation with respect to time t. The n singularities, given by Dirac's deltas centred at abscissa $x_{oi}, i=1,\ldots,n$, represent n concentrated damages and the parameters $\hat{\gamma}_i, i=1,\ldots,n$ multiplying Dirac's deltas are directly related to the rotational stiffness of an equivalent

internal spring, as specified in reference paper (Caddemi et al., 2009).
This model is equivalent to consider a straight beam with n elastic rotational springs.
By considering the non-dimensional coordinate $\xi = x/L$, and observing that the time dependence is harmonic, the differential Eq. (2.1) reduces the following form:

$$\left[\left(1-\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{oi})\right)\phi''(\xi)\right]'' - \omega^2\frac{mL^4}{E_oI_o}\phi(\xi)=0 \tag{2.2}$$

By performing the double differentiation with respect to ξ of the first term containing the Dirac's delta distribution, and after simple algebra, eq. (2.2) can be written in the following inhomogeneous form:

$$\phi^{IV}(\xi)-\alpha^4\phi(\xi)=B(\xi) \tag{2.3}$$

where the dimensionless frequency parameter $\alpha^4=\omega^2\frac{mL^4}{E_oI_o}$ has been introduced and the function $B(\xi)$ collects all the terms with the Dirac's deltas and their derivatives as follows:

$$B(\xi)=\left[\sum_{i=1}^{n}\gamma_i\phi^{IV}(\xi)\delta(\xi-\xi_{oi})+2\sum_{i=1}^{n}\gamma_i\phi'''(\xi)\delta'(\xi-\xi_{oi})+\sum_{i=1}^{n}\gamma_i\phi''(\xi)\delta''(\xi-\xi_{oi})\right] \tag{2.4}$$

The general explicit solution of Eq. (2.3) has been derived by the authors in (Caddemi et al., 2009) by making use of the generalised function theory and may be given the following form:

$$\begin{aligned}\phi(\alpha,\xi)=C_1&\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\sin\alpha\xi\right\}+\\ +C_2&\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\upsilon_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\cos\alpha\xi\right\}+\\ +C_3&\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\zeta_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\sinh\alpha\xi\right\}+\\ +C_4&\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\eta_i\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\cosh\alpha\xi\right\}\end{aligned} \tag{2.5}$$

where $U(\xi-\xi_{oi})$ is the well known unit step (Heaviside) function and the terms $\mu_i(\alpha),\nu_i(\alpha),\varsigma_i(\alpha),\eta_i(\alpha)$ are given by:

$$\begin{aligned}\mu_j(\alpha)&=\frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\mu_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]-\alpha^2\sin\alpha\xi_{oj}\\ \upsilon_j(\alpha)&=\frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\upsilon_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]-\alpha^2\cos\alpha\xi_{oj}\\ \zeta_j(\alpha)&=\frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\zeta_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]+\alpha^2\sinh\alpha\xi_{oj}\\ \eta_j(\alpha)&=\frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\eta_i\left[-\sin\alpha(\xi_{oj}-\xi_{oi})+\sinh\alpha(\xi_{oj}-\xi_{oi})\right]+\alpha^2\cosh\alpha\xi_{oj}\end{aligned} \tag{2.6}$$

The integration constants C_1, C_2, C_3, C_4 can be easily evaluated by imposing the relevant boundary conditions. The first and second derivatives of the eigen-mode can be obtained by means of single and double differentiation of Eq.(2.5) by making use of the distributional derivatives of the unit step (Heaviside) function. Eq. (2.5) can also be re-written in the more compact form:

$$\phi(\alpha,\xi) = C_1 f_1(\alpha,\xi) + C_2 f_2(\alpha,\xi) + C_3 f_3(\alpha,\xi) + C_4 f_4(\alpha,\xi) = \sum_{k=1}^{4} C_k f_k(\alpha,\xi) \tag{2.7}$$

with

$$\begin{aligned}
f_1(\alpha,\xi) &= \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i(\alpha)\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\sin\alpha\xi \\
f_2(\alpha,\xi) &= \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\upsilon_i(\alpha)\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\cos\alpha\xi \\
f_3(\alpha,\xi) &= \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\zeta_i(\alpha)\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\sinh\alpha\xi \\
f_4(\alpha,\xi) &= \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\eta_i(\alpha)\left[\sin\alpha(\xi-\xi_{oi})+\sinh\alpha(\xi-\xi_{oi})\right]U(\xi-\xi_{oi})+\cosh\alpha\xi
\end{aligned} \tag{2.8}$$

It is worth noting that the solution expressed by Eq (2.8) is valid for the overall beam and for any number and positions of cracks, furthermore, it preserves the same analytical structure of the undamaged beam being a function of four integration constants only irrespective of the number of the cracked cross sections. This form of the solution makes possible the derivation of the exact dynamic stiffness matrix of a multi-cracked beam, as reported in the following paragraph.

3. THE DERIVATION OF THE DYNAMIC STIFFNESS MATRIX

In order to obtain the dynamic stiffness matrix of the multi-cracked beam, two sets of boundary conditions have to be specified: the kinematic boundary conditions, to be expressed in terms of transverse displacements and flexural rotations, and the natural boundary conditions, to be expressed in terms of shear forces and bending moments.

By assuming end displacements and forces positive downwards, end rotations and moments positive clockwise, the kinematic boundary conditions provide the following relationships between the end degrees of freedom and the integration constants:

$$\begin{bmatrix} u_o \\ L\varphi_o \\ u_1 \\ L\varphi_1 \end{bmatrix} = \begin{bmatrix} u(0) \\ Lu'(0) \\ u(L) \\ Lu'(L) \end{bmatrix} = \begin{bmatrix} \phi(0) \\ \phi'(0) \\ \phi(1) \\ \phi'(1) \end{bmatrix} = \begin{bmatrix} f_1(0) & f_2(0) & f_3(0) & f_4(0) \\ f_1'(0) & f_2'(0) & f_3'(0) & f_4'(0) \\ f_1(1) & f_2(1) & f_3(1) & f_4(1) \\ f_1'(1) & f_2'(1) & f_3'(1) & f_4'(1) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{3.1}$$

Expression (3.1) can be written in matrix form as:

$$\mathbf{v} = \mathbf{W} \cdot \mathbf{c} \tag{3.2}$$

Similarly, by applying the natural boundary conditions, the following relationships between the end forces and the integration constants must be enforced:

$$\begin{bmatrix} T_o L \\ M_o \\ T_1 L \\ M_1 \end{bmatrix} = \begin{bmatrix} -T(0)L \\ M(0) \\ T(L)L \\ -M(L) \end{bmatrix} = \frac{E_o I_o}{L^2} \begin{bmatrix} \phi'''(0) \\ -\phi''(0) \\ -\phi'''(1) \\ \phi''(1) \end{bmatrix} = \frac{E_o I_o}{L^2} \begin{bmatrix} f_1'''(0) & f_2'''(0) & f_3'''(0) & f_4'''(0) \\ -f_1''(0) & -f_2''(0) & -f_3''(0) & -f_4''(0) \\ -f_1'''(1) & -f_2'''(1) & -f_3'''(1) & -f_4'''(1) \\ f_1''(1) & f_2''(1) & f_3''(1) & f_4''(1) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{3.3}$$

Expression (3.3) can be written in matrix form as:

$$\mathbf{F} = \mathbf{Q} \cdot \mathbf{c} \tag{3.4}$$

By solving Eq. (3.2) with respect to **c** and substituting the result in Eq. (3.4), the dynamic stiffness matrix **K** of the Euler Bernoulli beam in presence of an arbitrary number of concentrated cracks is obtained as follows:

$$\mathbf{F} = \mathbf{Q} \cdot \mathbf{W}^{-1} \cdot \mathbf{v} = \mathbf{K} \cdot \mathbf{v} \tag{3.5}$$

where

$$\mathbf{K} = \mathbf{Q} \cdot \mathbf{W}^{-1} \tag{3.6}$$

It is worth noticing that the coefficient of the dynamic stiffness matrix, after many algebraic manipulations, can be written in explicit form. Here for the sake of brevity only the computational procedure has been described.

The global dynamic stiffness matrix of a frame structure composed of beams with concentrated cracks can be obtained by a standard assemblage procedure.

The use of the dynamic stiffness, or flexibility, functions in connection with the Wittrick and Williams algorithm (Williams, F.W. et al. 1970) has the advantage of providing exact solutions, as opposed to the approximate ones obtainable by the finite element.

4. THE APPLICATION OF THE WITTRICK & WILLIAMS ALGORITHM

The Wittrick & Williams (WW) algorithm was developed over 40 years ago and has been applied with increasing sophistication to problems in structural mechanics ever since. The use of stiffness or flexibility functions in connection with the Wittrick & Williams algorithm has the advantage of providing exact solution, as opposed to the approximate ones obtained by well adapted techniques such as the finite element methods. In what follows the main steps of the Wittrick & Williams algorithm are summarized.

Once the global dynamic stiffness matrix of the entire structure is derived, the central step in the Wittrick & Williams algorithm is the evaluation of the number J of natural frequencies that are lower than a specified frequency value. It is apparent that the knowledge of J allows the evaluation of any required frequency, with the desired accuracy, by means of simple iterative procedures. According to the Wittrick & Williams algorithm, this number may be expressed as the sum of two terms

$$J = J_k + J_0 \tag{4.1}$$

where J_k is the number of negative eigenvalues of the global dynamic stiffness matrix evaluated at the specified frequency value ω^* and J_0 is the number of frequencies of vibration of the beams with both ends clamped which are lower than ω^*. The evaluation of J_0 must be performed through a summation extended to the number N_{beams} of the beams that compose the system, as follows:

$$J_0 = \sum_{r=1}^{N_{beams}} J_{b_r} \tag{4.2}$$

where J_{b_r} is the number of frequencies of vibration lower than ω^* for the generic r beam of the structure. If the beam is undamaged, the corresponding value of J_{b_r} can be evaluated by means of the closed form expression

$$J_b = i - \frac{1}{2}\left[1-(-1)^i \operatorname{sign}(D_u)\right]; \quad i = \operatorname{int}\left(\frac{\alpha(\omega^*)}{\pi}\right) \tag{4.3}$$

where D_u is given by

$$D_u = 1 - cC \tag{4.4}$$

The value J_{b_r} corresponding to the cracked beams cannot be evaluated in closed form; this apparent drawback can be overcome by considering the multi-cracked clamped-clamped beam as a substructure to be handled in a different way. Namely, J_{b_r} represents the number of frequencies of the clamped-clamped damaged beam that are lower than the specified frequency value ω^*, and has to be evaluated for each damaged beam of the frame structure.
This can be obtained, at a very low computational cost, by simply applying the Wittrick and William algorithm to each single damaged clamped-clamped beam by assembling the dynamic stiffness matrix introducing equivalent internal elastic hinges, at each damaged section. It is worth noticing that the latter substructuring procedure does not alter the number of the degrees of freedom of the overall frame structure that is coincident with the number of the corresponding undamaged structures, hence the global dynamic stiffness matrix of the damaged frame structure has the same dimension of the corresponding dynamic stiffness matrix of the undamaged structures that can be obtained as particular case by setting all the damages equal to zero.

5. NUMERICAL APPLICATIONS

The applications reported in the following are intended to validate the efficiency of the numerical procedure whose computational cost is strongly reduced, with respect to a standard application of Wittrick & Williams algorithm, since it does not require any additional degree of freedom accounting for the presence of the concentrated damages.

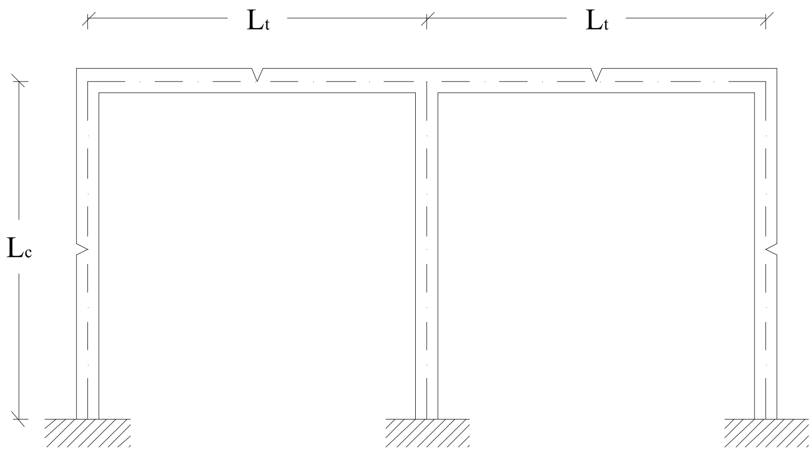

Figure 1. The two span damaged frame.

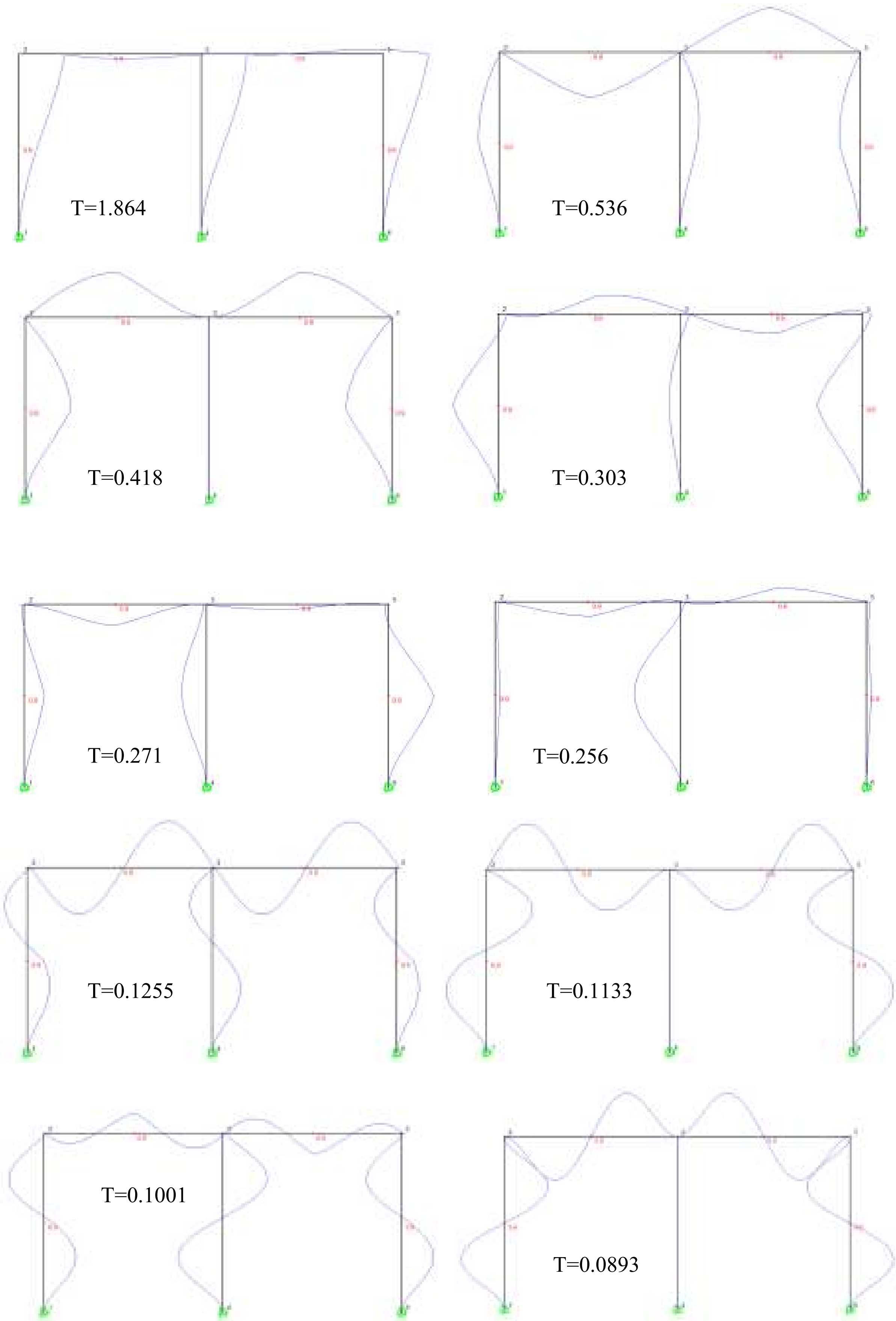

Figure 2. First ten vibration modes and the corresponding frequencies of the two span frame.

The structure under investigation is a two span portal frame in presence of four concentrated cracks located at the middle of the horizontal beams and in the middle of external columns, figure 1. The geometrical and mechanical parameters of the system are reported in table 1.

Table 1. Portal Frame geometrical and mechanical properties.

Elastic Modulus	$E = 206000$ Mpa
Span Length	$L = 12$ m
Column height	$L = 12$ m
Rectangular Section base	$b = 19.8$ mm
Rectangular Section height	$h = 12.2$ mm
Mass density	$\rho = 7.675$ t/m^3

The damaged frame elements, with uniform rectangular cross sections are considered as subjected to concentrated cracks whose depth to the cross section height ratio is equal to 0.9. Each concentrated damage has been modeled by means of a rotational spring, according to a model already applied in the literature (Caddemi, S. et al., 2008, 2009). In figure 2, the first ten vibration modes and the corresponding periods of vibration of the damaged frame are reported.
It is worth noticing that the global dynamic stiffness matrix of the system has been evaluated by assembling the stiffness matrices of each damaged element, and by considering for the overall system only four degrees of freedom (three rotational and one translational), since the axial deformability has been neglected.
The modal shapes are clearly influenced by the presence of the concentrated damages in the middle of each element. It is interesting to observe that the tenth mode is not influenced by the damage since the nodal points of the mode shape are coincident with the positions of the cracks.
Here, for the sake of brevity, only a simple application has been reported.
It is worth noticing that the considered approach make possible a low cost numerical evaluation of the evolution of damage in the eigenproperties of frame structures, since the degrees of freedom of the damaged and undamaged structures are coincident.

6. CONCLUSIONS

In the paper a new powerful approach, aimed to solve exactly the direct problem of the evaluation of the frequencies and the vibration modes of multi-cracked frame structures, is presented. This approach is based on a closed form solution, derived by the present authors, relative to the vibration modes of the multi-cracked beam, which possesses the same analytical structure of the undamaged beam, being a function of four integration constants only, regardless of the number of the cracked cross sections. The exact dynamic stiffness matrix of the multi-cracked beam has been presented and it represents a fundamental result since it allows the direct evaluation of the global dynamic stiffness matrix of frame structures in presence of an arbitrary number of concentrated damages along its members. The great advantage of the proposed approach, with respect to the classical approach based on the knowledge of the dynamic stiffness matrix of the undamaged beam, is that the degrees of freedom of the overall frame structure are exactly the same of the equivalent undamaged structure irrespective of the number of the concentrated damages. This fact represents an advantage both from a computational cost and an implementation effort. Moreover, it facilitates the numerical investigation relative to the evaluation of the influence of damage in complex frame structures towards the identification inverse problem solution.

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