Application of Green's Function in Evaluating Soil Spatial Variability During Earthquakes

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SUMMARY:
Green's function has the amazing power of solving inhomogeneous differential equations subject to specific boundary conditions. This paper presents a novel method of applying Green's function in evaluating inhomogeneous soil spatial variability due to earthquake shakings. The idea is based on classic electrodynamics. By solving Maxwell's equations applying Green's function for the boundary conditions, soil's spatial variability can be obtained through measuring electric potential distribution changes during earthquake shakings. Compared with electrical resistivity tomography, this method can be of high resolutions.

Keywords: Spatial variability, earthquake, Green's function, boundary condition, inhomogeneous

1. INTRODUCTION

While electrical measurements have been made possible in soils based on the fundamental electromagnetic concepts, solving for soil properties with high resolutions spatially and temporally could be rather complicated, especially due to complex boundary conditions. Named after the British mathematician George Green who first developed the concept in the 1830s, Green's functions has the amazing power to solve inhomogeneous differential equations subject to specific initial conditions or boundary conditions. Using Green's functions, it is possible to solve Laplace's equation or Poisson's equation subject to initial boundary conditions.

1.1. Electrical measurements in soil

In 1942, Archie relates the in-situ electrical conductivity (the reciprocal of electrical resistivity) of sedimentary rock to its porosity and brine saturation to effectively monitor spatial and temporal variations in soil for evaluating fundamental physical and mechanical properties of soil

$$C_t = C_w \phi^m S_w^n \tag{1.1}$$

where ϕ denotes the porosity, C_t is the electrical conductivity of the fluid saturated rock, C_w represents the electrical conductivity of the brine, S_w is the brine saturation, *m* is the cementation exponent of the rock (usually in the range 1.8–2.0), and *n* is the saturation exponent (usually close to 2). Soil resistivity has been shown empirically to depend on the resistivity of pore fluid, which saturates the particulate soil medium consisting of non-conductive particles, soil porosity, particle shape and size distribution, and the direction of measurements as

$$\rho_s = \rho_w \cdot n^{-\bar{f}} \tag{1.2}$$

where ρ_s is the resistivity of soil, ρ_w is the resistivity of pore fluid, *n* denotes soil porosity, and *f* is the form factor which is a function of the particle shape and grain size distribution, and has been shown empirically and theoretically to be independent on the porosity (Archie 1942; Arulanandan and Sybico 1992). $\bar{f}$ (usually around 1.0 ~ 2.4) is the average form factor, referring to the arithmetic average of form factors measured in three orthogonal directions. The ratio of the soil resistivity to the pore fluid resistivity is defined as the formation factor $F = {}^{\rho_s}\!/_{\rho_w}$, which has been used for determining volume changes during a pressuremeter test, for evaluating the in-situ porosity of non-cohesive sediments, and for evaluation of in-situ density and fabric of soil.

Other soil electrical measurements have been made possible based on the fundamental electromagnetic concepts, such as electrical resistivity tomography for monitoring contaminant evolution as a non-intrusive method (Günzel et al. 2003), porosity measurements using dielectric constant and dielectric dispersion (Arulanandan et al. 1973 and Kutter et al. 1979), and resistivity profiling using electrical needle probes (Cho et al. 2004 and Li et al. 2005).

In this paper, Green's function is applied in evaluating inhomogeneous soil spatial variability due to earthquake shakings based on the concept of classic electrodynamics.

2. THEORY

In a resistive medium such as soil, electrical potential measurements to monitor spatial variations can be made through introducing electromagnetic fields.

The time-harmonic forms of Maxwell's equations are

$$\nabla \times \mathbf{E} = -j\omega\mu\,\mathbf{H} \tag{2.1}$$
$$\nabla \times \mathbf{H} = \sigma\,\mathbf{E} + j\omega\varepsilon\,\mathbf{E} \tag{2.2}$$

where **E** and **H** represents the electric field and the magnetic field, respectively; and ω is the angular frequency.

In an electrostatic field, the differential form of Gauss's law

$$\nabla \bullet \mathbf{E} = \rho / \varepsilon_0 \tag{2.3}$$

and the equation specifying curl **E** as a function of position

$$\nabla \times \mathbf{E} = \mathbf{0} \tag{2.4}$$

which is equivalent to the statement that **E** is the gradient of a scalar function, the scalar potential Φ

$$\mathbf{E} = -\nabla\,\Phi \tag{2.5}$$

can be combined into Poisson equation

$$\nabla^2\Phi = -\rho / \varepsilon_0 \tag{2.6}$$

In regions of space that lack a charge density, the scalar potential satisfies the Laplace equation

$$\nabla^2\Phi = 0 \tag{2.7}$$

The solution for the scalar potential is

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x')}{|x-x'|} d^3x' \tag{2.8}$$

where point x is the potential location and point x' is the point charge source location.

The singular nature of the Laplacian of $1/r$ where $r = |x-x'|$ can be exhibited formally in terms of a Dirac delta function (Jackson 1999).

$$\nabla^2\left(\frac{1}{|x-x'|}\right) = -4\pi\delta(x-x') \tag{2.9}$$

The function $\frac{1}{|x-x'|}$ is only one of the Green functions. In general,

$$\nabla^2 G(x,x') = -4\pi\delta(x-x') \tag{2.10}$$

where

$$G(x,x') = \frac{1}{|x-x'|} + F(x,x') \tag{2.11}$$

with the function F satisfying the Laplace equation inside the volume V

$$\nabla^2 F(x,x') = 0 \tag{2.12}$$

With Green's theorem

$$\int_v (\phi\nabla^2\psi - \psi\nabla^2\phi) d^3x = \oint_s [\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}] da \tag{2.13}$$

where ϕ and ψ are arbitrary scalar fields, and the volume V is bounded by the closed surface S, let $\phi = \Phi$, and $\psi = G(x,x')$, the generalized $\Phi(x)$ can be expressed as

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_v \rho(x')G(x,x')d^3x' + \frac{1}{4\pi}\oint_S [G(x,x')\frac{\partial\Phi}{\partial n'} - \Phi(x')\frac{\partial G(x,x')}{\partial n'}] da' \tag{2.14}$$

Equation (2.14) states that if the value or normal derivative is known on a bounding surface, then the value of the function inside the volume is known everywhere. Now the problem becomes equivalent to determining Green's function $G(x,x')$ to satisfy simply the boundary conditions.

The case of method of images is a physical equivalent of the determination of the appropriate $F(x,x')$ to satisfy the boundary conditions. Figure 1 shows a point charge located in front of an infinite plane conductor at zero potential. The original potential problem is on the left, the equivalent-image problem on the right.

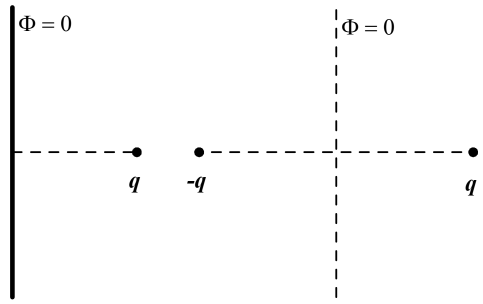

Figure 1. Method of images

3. FINITE ELEMENT ANALYSIS

Consider the Poisson equation (2.6) in a two-dimensional region *R*, with Dirichlet boundary conditions on the boundary curve *C*. The vanishing integral

$$\int_R (\phi\nabla^2\Phi + \rho/\varepsilon_0\ \phi)dxdy = 0 \tag{3.1}$$

where $\phi(x, y)$ is a test function specified as piecewise continuous in *R* and vanishing on *C*. Application of Green's first identity on Equation (3.1) leads to

$$\int_R (\nabla\phi \cdot \nabla\Phi - \rho/\varepsilon_0 \cdot \phi)dxdy = 0 \tag{3.2}$$

The surface integral vanishes since $\phi(x, y)$ vanishes on boundary curve *C*. Approximating the desired solution $\Phi(x, y)$ by a finite expansion in terms of a set of localized linearly independent functions, $\phi_{ij}(x, y)$, with support in a finite neighbourhood of $x = x_i$ and $y = y_j$. With the triangle as the basic unit element *e*(1, 2, 3) (Figure 2) assumed to be small enough that the field variable changes little over the element and can be approximated in a linear form in each direction.

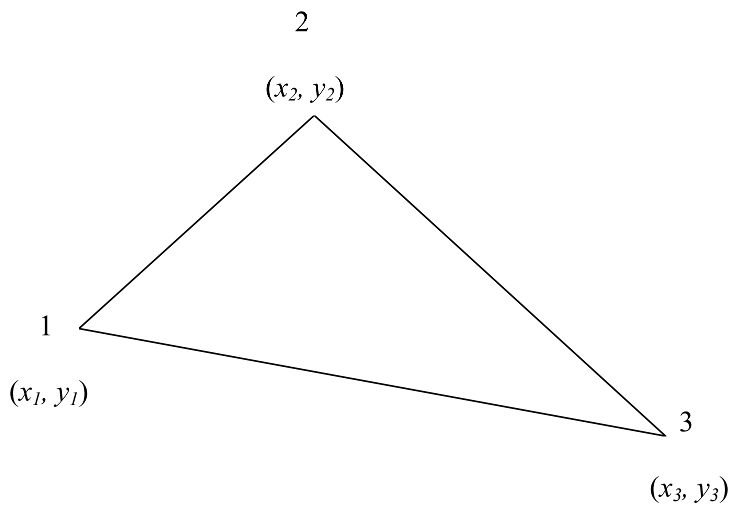

Figure 2. Basic triangular element *e*(1, 2, 3) with area *Se* in 2-dimensions

In this region, the field variable can be approximated as

$$\Phi(x,y) \approx \Phi_e(x,y) = A + Bx + Cy \tag{3.3}$$

The three values (Φ_1, Φ_2, Φ_3) at the nodes or vertices determine the coefficients (*A, B, C*). Shape functions $N_j^{(e)}(x,y)$ can be defined, one for each vertex, such that $N_j^{(e)} = 1$ when $x = x_j$, $y = y_j$, and $N_j^{(e)} = 0$ at other vertices. Thus, the field variable $\Phi(x,y)$ can be expressed as

$$\Phi(x,y) \approx \sum_{f,j} \Psi_j^{(f)} N_j^{(f)}(x,y) \tag{3.4}$$

where the sum goes over all the triangles *f* and over the vertices of each triangle, and the $\Psi_j^{(f)}$ are the constants desired for the filed at the vertices. Let the test function being $\phi(x,y) = N_i^e(x,y)$ for element *e* and vertex *I*, the integral with the inhomogeneity transferred to the right-side, is

$$\sum_{j=1}^{3} \Psi_j^{(e)} \int_e \nabla N_i^e \bullet \nabla N_j^e dxdy = \int_e g N_i^{(e)} dxdy \tag{3.5}$$

$g_e \equiv g(\bar{x}_e, \bar{y}_e)$ at the center of gravity of the triangle if *g* changes little over the element *e*. With the right–hand integral being factored out, the remaining integral is

$$\int_e N_i^{(e)} dxdy = S_e(a_i + b_i \bar{x}_e + c_i \bar{y}_e) = \frac{1}{3} S_e \tag{3.6}$$

For the left-hand side, the linearity of the shape functions means that the integrand is a constant.

$$k_{ij}^{(e)} = S_e(b_i b_j + c_i c_j) \tag{3.7}$$

Thus Equation (3.5) becomes

$$\sum_{j=1}^{3} k_{ij}^{(e)} \Psi_j^{(e)} = \frac{1}{3} S_e g_e \quad (i = 1, 2, 3) \tag{3.8}$$

The three coupled equations for each element except when the side(s) of the triangle form part of the boundary, can be written in matrix form,

$$k^{(e)} \Psi^{(e)} = G^{(e)} \tag{3.9}$$

The generalized equation is the matrix form

$$K\Psi = G \tag{3.10}$$

where $K = (k_{ij})$ with $k_{ii} = \sum_T k_{ii}^{(e)}$ and $k_{ij} = \sum_E k_{ij}^{(e)}$ $i \neq j$ and $G_i = \frac{1}{3} \sum_T S_e g_e - \sum_{j=N+1}^{N_0} k_{ij}^{(e)} \psi_j^{(e)}$

The summation over *T* means over all the triangles connected to the internal node *i*; the summation over *E* means a sum over all the triangles with a side from internal node *i* to internal node *j*; and the final sum in G_i contains the known boundary values of Ψ there and the corresponding $k_{ij}^{(e)}$ values.

4. APPLICATION

The rather involved theoretical background can be better understood through the application in a centrifuge test case. Li et al. (2008) proposed a closed form solution for monitoring movement of buried objects using an electrode switching system which was described more in detail in Versteeg et al. (2005). The case of boundary conditions was simplified through the application of method of images, a unique form of the Green's function in rectangular coordinates. For illustration, here the test model is reproduced in Figure 3.

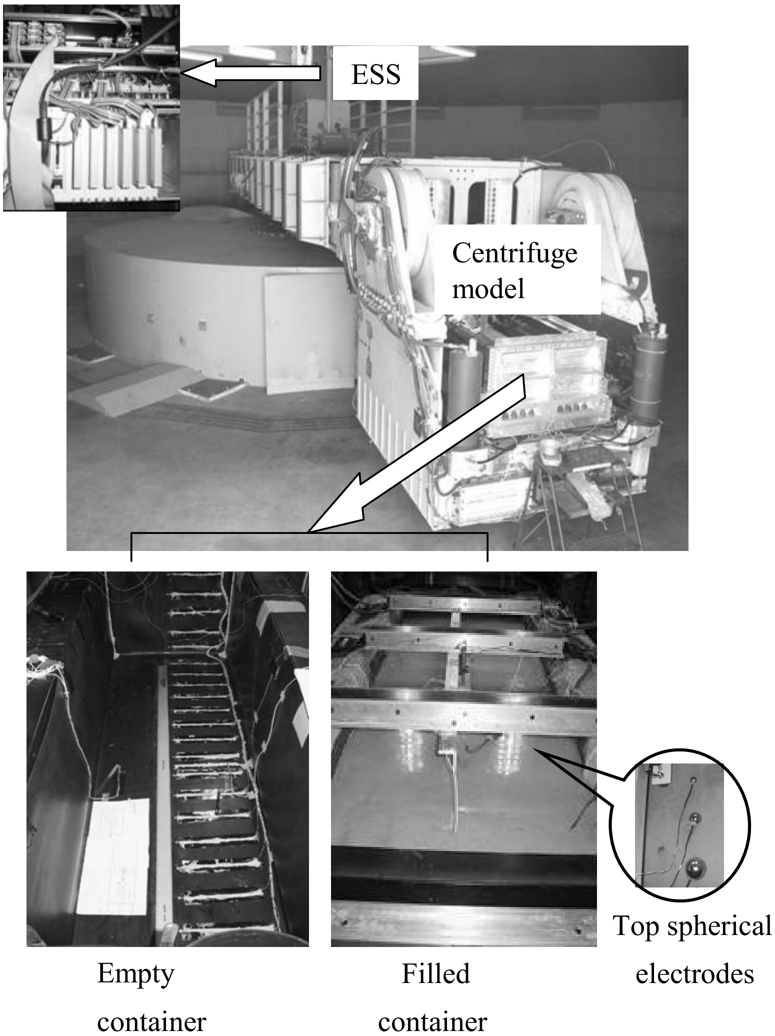

Figure 3. Geotechnical centrifuge at UC Davis with the soil model placed at the end of the arm and the Electrode Switching System (ESS) mounted at the center of the arm. Also shown is the empty container with boundary electrodes attached to the liner, the filled container with soil model, and target spherical electrodes on the soil surface (Li et al. 2008)

The rectangular centrifuge box filled with saturated soil with embedded target objects to be studied during earthquake shakings can be viewed as an ideal model for method of images. Electromagnetic fields were established in the conductive soil medium through injecting low-frequency alternating currents through pairs of boundary electrodes mounted at the sides of the centrifuge model and the movements of the embedded objects during earthquake shakings were monitored through measuring the electrical potentials of the electrodes attached to the objects. In this case, the boundary conditions were simplified through applying the method of images, an equivalent application of the appropriate Green's function to satisfy the boundary conditions. Subsequently, Li (2010) described a possible way of evaluating soil strains using electrical methods involving boundary conditions. For a mesh of electrodes embedded in saturated soil model illustrated in Figure 4, the evolution and distribution of soil strains during earthquakes in a geotechnical centrifuge model can be monitored through taking measurements of the electrical potentials associated with the electrodes in an electromagnetic fields introduced. The finite element analysis described above can be applied in solving the numerical problems involved. The generalized equation in the matrix form $K\Psi = G$ (Equation 3.10) can be simplified as a symmetric matrix in this case.

To solve the three-dimensional problem in real life as illustrated in Figure 3, another vertex is added to make a tetrahedron the basic element of volume. Now four shape functions, $N_j^{(e)}(x, y, z)$, are used to give an approximation of the field variable within the tetrahedron. To solve the rather involved algebra, finite element tools POISSON (Lawrence Berkeley National Laboratory jointly with Livermore National Laboratory) and TOSCA and CARMEN (Rutherford-Appleton Laboratory) can be used.

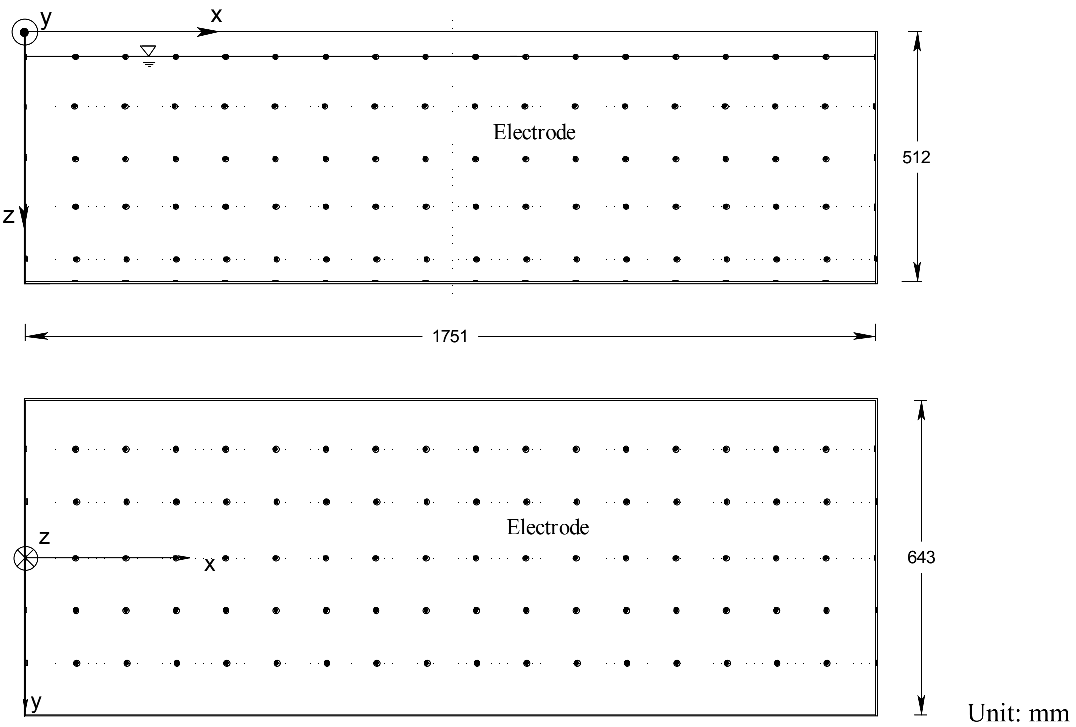

Figure 4. Finite element analysis mesh of electrodes in a centrifuge model. Top: side view; bottom: top view

5. CONCLUSIONS

Green's function has the amazing power of solving inhomogeneous differential equations subject to specific boundary conditions. This paper presents a novel method of applying Green's function with classic electrodynamics in evaluating inhomogeneous soil spatial variability. Through solving Maxwell's equations using Green's function for the boundary conditions, soil's spatial variability can be obtained through measuring electric potential distribution changes during earthquake shakings.

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