Buckling Analysis of Non-Prismatic Columns Using Slope-Deflection Method

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SUMMARY:
Different types of single story rigid steel frames for various purposes have been manufactured specially for factories. To achieve an optimal design leading to more efficient use of structural steel, these frames are normally composed of built- up tapered I-Sections. The effective length coefficient approach for column stability and design has long played an important role in method of buckling analysis. Exact buckling load for some special cases of non-uniform columns were derived in the past. In this paper by using the slope-deflection equations, a new analytical method for tapered columns is presented. Corresponding critical load and subsequently effective length coefficient are obtained regarding some examples for practical use. This method is exact and has fast converges with any desired accuracy in comparison with approximate and finite element methods. This method can also be further extended to treat free vibration of tapered columns with axially variable material and cross section properties.

Keywords: Buckling Analysis, Steel Frames, Slope-Deflection Method, Critical Load, Tapered Columns.

1. INTRODUCTION

Extensive theoretical and experimental research has been conducted into elastic stability or buckling, in which, buckling of non-prismatic members is of major importance. The first solutions presented to deal with the calculation of the critical load in elastic buckling of tapered columns, approximated by step-column, include approximated solutions of Timoshenko (1908), Morley (1917) and Dink (1929,1932). By solving the differential equation of the deflected curve of an ideal column on the verge of elastic buckling, Gere and Carter (1963) have obtained dimensionless charts for various cross-sections and different shape factors and boundary conditions. These charts are applicable in designing of single columns. Ketter et al. (1979) investigated the use of energy methods in the non-exact solution of the buckling of a member. According to their research, based on the energy method, for an object to be stable, the change in the total potential energy of the system must equal zero. In this method, according to Rayleigh-Ritz theory, the critical load of buckling is obtained by minimizing energy.

Other researchers investigate the exact solution of a non-prismatic column using power series method (Li and Li, 2002; Al-Sadder, 2004). They solve the fourth-order differential equation with variable coefficients of a non-prismatic column using power series and extract accurate functions of elastic stability for any generic non-prismatic column and gabled frame. Bazeos and Karabalis (2005), using stability analysis and matrix method for solving stability function of a non-prismatic column, have plotted charts for obtaining the critical load of a non-prismatic single column.

2. MOMENT OF INERTIA FOR I-SHAPED MEMBERS WITH CONSTANT WINGS

Fig. 1 shows an I-column with variable cross-section. Along the length of the member, the dimensions of the flange remain constant whereas the height of the web varies linearly. The extensions of the two flanges' lines intersect at point O.

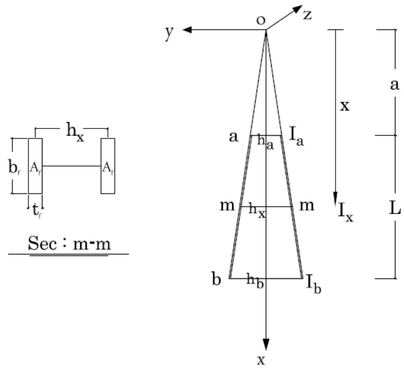


Figure 1. I-shaped member with variable cross section

In Fig.1, h_a, h_x and h_b are the heights of the section at x = a, x = x and x = a+L, respectively, measured from point O. Also A_f, t_f and b_f are the area, thickness and width of the I-section flange, respectively. By ignoring the negligible effect of the moment of inertia for the web, the moment of inertia for the member about its strong axis at section x=a equals:

$$I_a = 2[\frac{1}{12}b_f t_f^3 + A_f(\frac{h_a}{2})^2] \tag{2.1}$$

Due to the small amount of t_f, the $\frac{b_f t_f^3}{12}$ term can be ignored:

$$I_a = 2A_f(\frac{h_a}{2})^2 \tag{2.2}$$

Similarly, I_x, the moment of inertia of the member at section m-m, located at distance x with respect to point O follows this equation:

$$I_x = 2A_f(\frac{h}{2})^2 \tag{2.3}$$

On the other hand, as the change in dimensions of the member is linear, the similarity between the two triangles in Fig.1 results as follows:

$$\frac{x}{a} = \frac{h}{h_a} \tag{2.4}$$

Thus, the equation for the variable moment of inertia along the member can be obtained using Eqn. 2.2, 2.3 and 2.4:

$$I_x = I_a(\frac{x}{a})^2 \tag{2.5}$$

In the above equation, I_a is the moment of inertia of the member at the beginning and with a distance of a with respect to O, and I_x is the moment of inertia at a section with a distance of x with respect to point O.

3. DEVELOPING THE SLOPE-DEFLECTION EQUATIONS FOR MEMBERS WITH VARIABLE MOMENT OF INERTIA WITHOUT CONSIDERATION OF THE EFFECT OF AXIAL LOAD

Fig. 2 illustrates the deflection of a member with variable moment of inertia under its end moments and without the critical load:

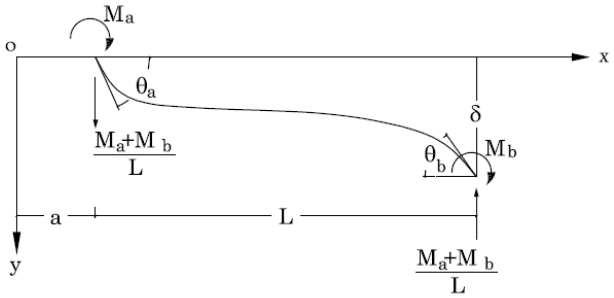


Figure 2. Node forces and deflections of variable cross section member without axial load effects

The differential equation resulted from the equilibrium of moments at a section with a distance of x from the intersection point of the two flanges' lines (Fig. 1) is written as follows:

$$\mathrm{EI}_x = \frac{d^2y}{\mathrm{dx}^2} = -M(x) = M_a - \left(\frac{M_a+M_b}{L}\right)(x-a) \tag{3.1}$$

By substituting the Eqn. 2.5 in the above differential equation and solving it and then applying the boundary conditions, the slope-deflection equations of members with variable moments of inertia without the effect of axial load can be obtained:

$$\begin{cases} M_a = \frac{2\mathrm{EI}_a}{L}\left[\beta_1\theta_a + \beta_2\theta_b - (\beta_1+\beta_2)\frac{\delta}{L}\right] \\ M_b = \frac{2\mathrm{EI}_a}{L}\left[\beta_2\theta_a + \beta_3\theta_b - (\beta_2+\beta_3)\frac{\delta}{L}\right] \end{cases} \tag{3.2}$$

In the above equation, M_a and M_b are the end moments at the beginning and the end of member, θ_a and θ_b are the rotation at the beginning and the end of member and δ is the relative displacement of the two ends with respect to the initial condition. β_1, β_2 and β_3 parameters are calculated as follows:

$$\begin{cases} \beta_1 = \frac{-f_2}{2(f_1^2-f_2f_3)} \\ \beta_2 = \frac{-f_1}{2(f_1^2-f_2f_3)} \\ \beta_3 = \frac{-f_3}{2(f_1^2-f_2f_3)} \end{cases} \tag{3.3}$$

The f_i parameters in the above equations can be obtained from Eqn. 3.4:

$$\begin{cases} f_1 = 2\lambda^2 - \lambda^2(2\lambda + 1)\ln(1 + \frac{1}{\lambda}) \\ f_2 = \lambda^2(\frac{1+2\lambda}{1+\lambda}) - 2\lambda^3\ln(1 + \frac{1}{\lambda}) \\ f_3 = \lambda + 2\lambda^2 - 2\lambda^2(1 + \lambda)\ln(1 + \frac{1}{\lambda}) \end{cases} \tag{3.4}$$

$$\lambda = \frac{1}{\eta} \tag{3.5}$$

$$\eta = \frac{L}{a} = \sqrt{\frac{I_b}{I_a}} - 1 \tag{3.6}$$

In Eqn. 3.6, I_a and I_b are the moment of inertia at the beginning and the end of the member, respectively. The η parameter is defined as the section constant.

4. DEVELOPING THE SLOPE-DEFLECTION EQUATIONS FOR MEMBERS WITH VARIABLE MOMENT OF INERTIA CONSIDERING AXIAL LOAD EFFECTS

Fig.3 shows a member with variable moment of inertia and its end moments:

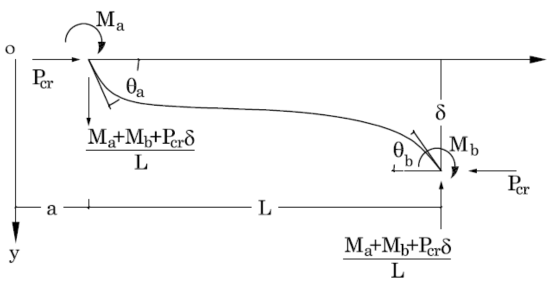


Figure 3. Node forces and deflections of variable cross section member with axial load effects

As Fig. 4 illustrates, equilibrium of moments in a section with a distance of x from point O (Fig. 1) is written as follows:

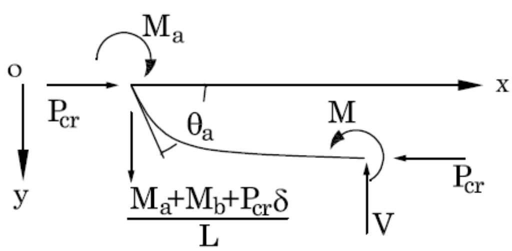


Figure 4. Moment equilibrium in x cross section from point O

$$M = M_a + \mathrm{Py} - (\frac{M_a+M_b+P_{cr}\delta}{L})(x - a) \tag{4.1}$$

The differential equation resulted from the equilibrium of moments at a section distanced x from point O (Fig. 1) is written as follows:

$$\mathrm{EI}_x \frac{d^2y}{\mathrm{dx}^2} = -M = -M_a - \mathrm{Py} + (\frac{M_a + M_b + P_{\mathrm{cr}}\delta}{L})(x - a) \tag{4.2}$$

Substitution of Eqn. 2.5 in the Eqn. 4.2 and solving it and then applying the boundary conditions, the slope-deflection equations of members with variable moments of inertia with the effect of axial load can be obtained:

$$\begin{cases} M_a = \frac{2\mathrm{EI}_a}{L}[\alpha_1\theta_a + \alpha_2\theta_b - (\alpha_1 + \alpha_2)\frac{\delta}{L}] \\ M_b = \frac{2\mathrm{EI}_a}{L}[\alpha_2\theta_a + \alpha_3\theta_b - (\alpha_2 + \alpha_3)\frac{\delta}{L}] \end{cases} \tag{4.3}$$

α_1, α_2 and α_3 are calculated as follows:

$$\begin{cases} \alpha_1 = \frac{-e_3(\mu^2+\frac{1}{4})\eta}{(e_2^2 - e_1e_3)} \\ \alpha_2 = \frac{-e_2(\mu^2+\frac{1}{4})\eta}{(e_2^2 - e_1e_3)} \\ \alpha_3 = \frac{-e_1(\mu^2+\frac{1}{4})\eta}{(e_2^2 - e_1e_3)} \end{cases} \tag{4.4}$$

The e_i parameters in the above equations can be obtained from Eqn. 4.5:

$$\begin{cases} e_1 = \frac{2}{\eta} + 1 - \frac{2\mu}{\tan(\beta)} \\ e_2 = \frac{2}{\eta} - \frac{2\mu}{\sqrt{\eta+1}}\frac{1}{\mathrm{Sin}(\beta)} \\ e_3 = \frac{2}{\eta} - \frac{1}{1+\eta} - \frac{2\mu}{\eta+1}\frac{1}{\tan(\beta)} \end{cases} \tag{4.5}$$

$$\beta = \mu\ln(\eta + 1) \tag{4.6}$$

μ, in the above equations, is defined as the load constant and is calculated as follows:

$$\mu^2 = \frac{P_{\mathrm{cr}}a^2}{\mathrm{EI}_a} - \frac{1}{4} \tag{4.7}$$

The critical load, based on Eqn. 4.7, can be expressed in terms of μ:

$$P_{\mathrm{cr}} = (\mu^2 + \frac{1}{4})\frac{\mathrm{EI}_0\eta^2}{L^2} \tag{4.8}$$

The critical load equation is defined in terms of the efficient length factor as follows:

$$P_{\mathrm{cr}} = \frac{\mathrm{EI}_0}{(k_\gamma L)^2} \tag{4.9}$$

Using Eqn. 4.8 & 4.9, the efficient length factor of members with variable cross-section is obtained:

$$k_\gamma = \frac{1}{\eta\sqrt{\mu^2+\frac{1}{4}}} \tag{4.10}$$

5. THE EFFICIENT LENGTH FACTOR OF SLOPED GABLED FRAMES

Fig. 5 shows the cross-section of abcdf sloped frame, its deformation during buckling and its end slopes and forces:

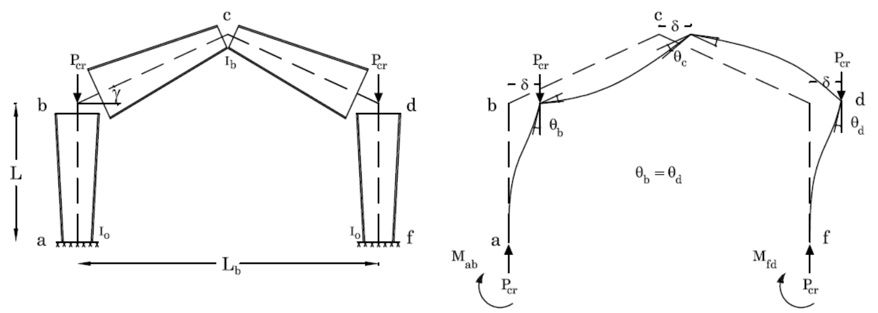


Figure 5. Sloped gabled frame with sway and fixed supports

In Fig. 5, I_o denotes the moment of inertia of frame columns at their beginnings and I_b denotes the moment of inertia of frame beams at their beginnings. In slope-deflection equations, the section constant, η, is assumed to be the same for all members.

With the help of slope-deflection Eqn. 4.3 for ab member, we have:

$$\begin{cases} M_{\mathrm{ab}} = \frac{2\mathrm{EI}_0}{L}\left[\alpha_2\theta_b - (\alpha_1 + \alpha_2)\frac{\delta}{L}\right] \\ M_{\mathrm{ba}} = \frac{2\mathrm{EI}_0}{L}\left[\alpha_3\theta_b - (\alpha_2 + \alpha_3)\frac{\delta}{L}\right] \end{cases} \tag{5.1}$$

Using slope-deflection Eqn. 3.2 for bc and cd members, we also conclude that:

$$\begin{cases} M_{\mathrm{bc}} = \frac{2\mathrm{EI}_a}{L_b} \times 2 \times \cos(\gamma)[\beta_2\theta_c + \beta_3\theta_b] \\ M_{\mathrm{cb}} = \frac{2\mathrm{EI}_b}{L_b} \times 2 \times \cos(\gamma)[\beta_1\theta_c + \beta_2\theta_b] \end{cases} \tag{5.2}$$

$$M_{\mathrm{cd}} = \frac{2\mathrm{EI}_a}{L_b} \times 2 \times \cos(\gamma)[\beta_1\theta_c + \beta_2\theta_d] \tag{5.3}$$

By implementing the equilibrium equation in b and c joints and the help of the free diagram of ab member, the buckling characteristic equation of frame is obtained:

$$
\begin{cases}
M_{ba} + M_{bc} = 0 \\
M_{cb} + M_{cd} = 0 \\
M_{ab} + M_{ba} + P_{cr}\delta = 0
\end{cases}
\tag{5.4}
$$

By simplifying the above equations, it is concluded that the determinant of the coefficient matrix must equal zero, which is the buckling characteristic equation of frame itself:

$$
\begin{vmatrix}
\frac{G_T}{2\times\cos(\gamma)}\alpha_3 + \beta_3 - \frac{\beta_2^2}{\beta_1} & \frac{-G_T}{2\times\cos(\gamma)}(\alpha_2 + \alpha_3) \\
(\alpha_2 + \alpha_3) & -(\alpha_1 + 2\times\alpha_2 + \alpha_3) + (\mu^2 + \frac{1}{4})\frac{\eta^2}{2}
\end{vmatrix} = 0
\tag{5.5}
$$

In Eqn. 5.5, the section constant, η, and the fixed modifying factor, G_T, are calculated as follows:

$$G_T = \frac{L_b I_0}{\mathrm{L}\mathrm{I}_b} \tag{5.6}$$

$$\eta = 0,0.5,1,1.5,2,3,4,5,6 \tag{5.7}$$

By the use of a computer code for solving Eqn. 5.5 numerically, μ, the load constant, is obtained, which then, through Eqn. 4.10, leads us to obtain the efficient length factor of the frame in Fig. 5. Shown in Fig. 6, is the efficient length factor of the frame in Fig. 5 plotted for different values of GT, γ and η.

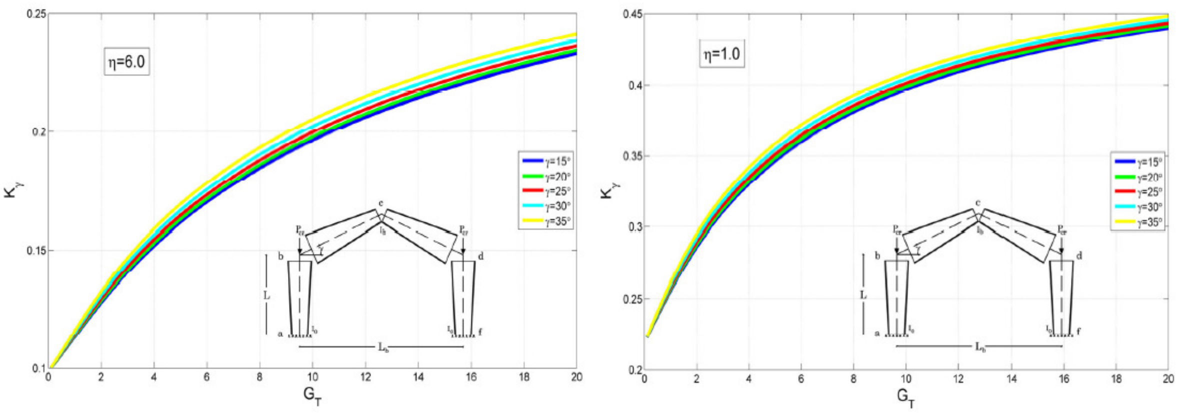


Figure 6. Effective length factor of sloped gabled frame with sway and fixed supports (η=1, 6)

Also, the buckling characteristic equation of this frame with non-sway and fixed support conditions is obtained in a similar manner:

$$\frac{G_T}{2\times\cos(\gamma)}\alpha_3 + \beta_3 - \frac{\beta_2^2}{\beta_1} = 0 \tag{5.8}$$

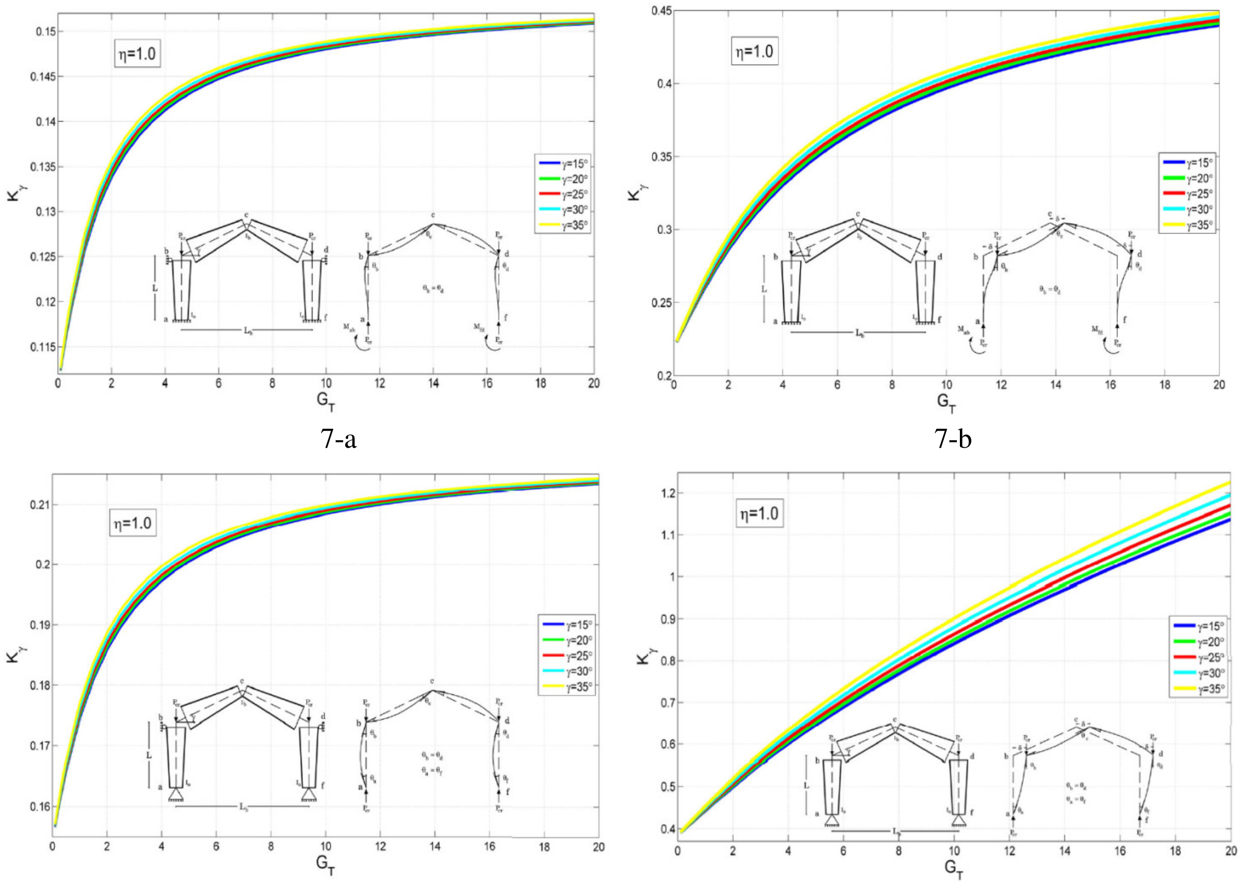
And the buckling Eigen value equation of this frame with non-sway and hinged support conditions is similarly obtained:

$$\frac{G_T}{2\times\cos(\gamma)}\left(\alpha_3 - \frac{\alpha_2^2}{\alpha_1}\right) + \beta_3 - \frac{\beta_2^2}{\beta_1} = 0 \tag{5.9}$$

Finally, the buckling Eigen value equation of this frame with sway and hinged support conditions is likewise obtained:

$$\begin{vmatrix} \frac{G_T}{2\times\cos(\gamma)}(\alpha_3-\frac{\alpha_2^2}{\alpha_1})+\beta_3-\frac{\beta_2^2}{\beta_1} & \frac{-G_T}{2\times\cos(\gamma)}(\alpha_3-\frac{\alpha_2^2}{\alpha_1}) \\ (\alpha_3-\frac{\alpha_2^2}{\alpha_1}) & -(\alpha_3-\frac{\alpha_2^2}{\alpha_1})+(\mu^2+\frac{1}{4})\frac{\eta^2}{2} \end{vmatrix}=0 \tag{5.10}$$

Fig. 7 demonstrates the efficient length factor of four sloped frames with section constant of one, μ=1, and different support conditions:



7-a 7-b

7-c 7-d

Figure 7. Effective length factor of sloped gabled frames
a: Non-sway and fixed support, b: Sway and fixed support, c: Non-sway and simple support, d: Sway and simple support (η=1)

5. NUMERCAL STUDIES

To verify the results of the presented method SAP2000 software is used. Frames of Fig. 7 are used for this verification. It is assumed that the moment of inertia of the members at the beginning is

I_a=I_b=3.671*10^{-5} m^4, the section constant is μ=1, the elastic module of members is E=2*10^6 $\frac{KN}{m^2}$, the height of the column is L= 10 m, the frame span is L_b=20 m and the frame slope is γ = 30 degrees. Using Eqn. 5.3, the modifying fixed factor can be obtained as G_T= 2.

Considering Fig. 7 for G_T= 2 and γ = 30, the effective length factors of A, B, C and D frames are as follows:

$$K_{\gamma a} = 0.1351 \quad , \quad K_{\gamma b} = 0.2924$$
$$K_{\gamma c} = 0.1877 \quad , \quad K_{\gamma b} = 0.5114$$

Eqn. 5.6 gives the critical loads of the frames. These results are written down in Table 1 and are compared with those of SAP2000.

Table 1. Comparison of critical loads for frames in Fig. 7 with SAP2000 results.

Frame Type	Critical Load	Slope-Deflection Method		SAP 2000	
A	P_{cr}	4022.57	KN	4025.93	KN
B	P_{cr}	858.74	KN	858.64	KN
C	P_{cr}	2083.94	KN	2084.44	KN
D	P_{cr}	280.73	KN	280.70	KN

CONCLUSIONS

The studies carried out in the field of efficient length factors of members with variable moment of inertia are mostly based on approximate methods. In this paper, by developing the slope-deflection equations of members with variable moment of inertia, the buckling characteristic equation of columns in sloped frames has been obtained. Since solving these equations is not desirable for engineering use, dimensionless charts have been presented. One of the benefits of these charts is the ease of access to the critical load of columns as opposed to other researches. Also results are verified by comparing them with the results of finite element software. The numerical studies in this paper demonstrate the high level of accuracy and the applicability of the charts.

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