Analytical Solution for Nonlinear Shallow-water Wave Equations

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SUMMARY:
Majority of the hodograph transform solutions of the one-dimensional nonlinear shallow-water wave equations are obtained through integral transform techniques. This approach, however, might involve evaluation of elliptic integrals, which are highly singular. Here, we couple the hodograph transform approach with the classical eigenfunction expansion method rather than integral transform techniques and present a new analytical model for nonlinear long wave propagation over a plane beach. In contrast to classical initial or boundary value problem solutions, an initial-boundary value problem solution is formulated. In general, initial wave profile with nonzero initial velocity distribution is assumed and the flow variables are given in the form of Fourier-Bessel series. The spatial and temporal variation of the flow quantities, i.e., free-surface height and depth-averaged velocity, are estimated accurately through the developed method with much less computational effort compared to the existing integral transform techniques.

Keywords: shallow-water wave equations, hodograph transformation, eigenfunction expansion

1. INTRODUCTION

The shallow-water wave theory has proved to be a key instrument in analytical modeling of propagation and runup of long waves such as tsunamis. Among the analytical models for the nonlinear shallow-water wave (NSW) equations, the solution method of Carrier and Greenspan (1958) remains quite significant as they introduced the state-of-the-art *hodograph transformation*. This transformation has two main advantages. First, it maps the instantaneous shoreline position in the physical space onto a fixed point in the hodograph space. Second, with this transformation, the nonlinear problem can be reduced to a linear one, so that the solution in the transform space can be pursued utilizing the standard methods and can easily be inverted for the nonlinear solution in terms of physical variables. In spite of these advantages, however, the solution method of Carrier and Greenspan (1958) could not been benefited for a long time due to the difficulty of presenting results for geophysically meaningful initial wave profiles. While this is resolved by Synolakis (1987) formulating the solution as boundary value problem (BVP), the complete initial value problem (IVP) solution in one space dimension for realistic initial waveforms could only be achieved after late contributions (Carrier et al., 2003; Kânoğlu, 2004).

The common property of the previous studies is the use of integral transform techniques in obtaining the solutions. To be more specific, Carrier and Greenspan (1958) used the Hankel integral transform method in their work and thus the solution appeared in the form of elliptic integrals, which are singular; hence, in order to proceed Carrier and Greenspan (1958) had to impose *regular* initial wave profiles. Carrier et al. (2003) improved the Carrier and Greenspan (1958) solution in terms of initial wave profiles, but unfortunately they could not avoid elliptic integrals. This forced them to use a linearization of the initial velocity profile in their calculations. Kânoğlu (2004) adopted the same integral transform technique, but treated the IVP differently and linearized the hodograph transformation in order to derive the initial conditions in the transform space which led simpler solution integrals. Later, Kânoğlu and Synolakis (2006) showed how to incorporate the exact nonlinear initial velocity condition into the hodograph transform technique.

An alternate analytical solution for the NSW equations is provided by Aydın and Kânoğlu (2007). Considering a long and shallow bay connected to an infinite-depth ocean, they first derived an explicit analytical expression for the surface profile in presence of a continuous wind blowing in seaward direction using hodograph transformation for the spatial variable. This is called *wind set-down*. Then, Aydın and Kânoğlu (2007) modeled analytically the subsequent wave oscillations in the bay after the wind stopped blowing, which is called the *relaxation phase*. Their formulation for the relaxation phase led an initial-boundary value problem (IBVP) for the NSW equations and was suitable for a solution in terms of a Fourier-Bessel series.

In this study, we follow similar approach and solve the NSW equations, subject to appropriate initial and boundary conditions, with the eigenfunction expansion method. We consider nonzero initial velocity distribution in general. We test the suggested method with a wide class of initial wave profiles and we expose that the shoreline dynamics, i.e., the temporal variations of the shoreline position and the shoreline velocity, are accurately estimated, requiring much less computational effort compared to integral transform techniques.

2. MATHEMATICAL FORMULATION

The NSW equations can be written in nondimensional form as

$$\eta_t + [(h+\eta)u]_x = 0, \tag{2.1a}$$

$$u_t + uu_x + \eta_x = 0, \tag{2.1b}$$

where subscripts denote derivative with respect to the variable. In the equations above $\eta(x,t)$ and $u(x,t)$ represent the free surface elevation and the depth-averaged wave velocity, respectively. The undisturbed water depth is $h(x) = x$ where x $(X_S(t) \le x \le X_L)$ represents the distance from the initial shoreline and t $(0 \le t)$ represents the temporal variable. $X_S(t)$ is the location of the shoreline tip. The nondimensional variables appearing in Eqns. 2.1 are defined to be

$$x = \frac{\tilde{x}}{l}, \quad (h,\eta) = \frac{(\tilde{h},\tilde{\eta})}{l\tan\beta}, \quad u = \frac{\tilde{u}}{\sqrt{g\,l\tan\beta}}, \quad t = \frac{\tilde{t}}{\sqrt{l/(g\tan\beta)}},$$

where l, g and β are the characteristic length scale, the gravitational acceleration, and the beach angle with horizontal, respectively (Fig. 2.1).

We will seek solution for Eqns. 2.1 through the hodograph transformation defined by (Carrier and Greenspan, 1958)

$$\sigma = \sqrt{x+\eta}, \tag{2.2a}$$

$$\lambda = t - u, \tag{2.2b}$$

which will replace the independent variables (x,t) with respective auxiliary variables (σ,λ). Here, it is worth noting that the hodograph transformation maps through Eqn. 2.2a

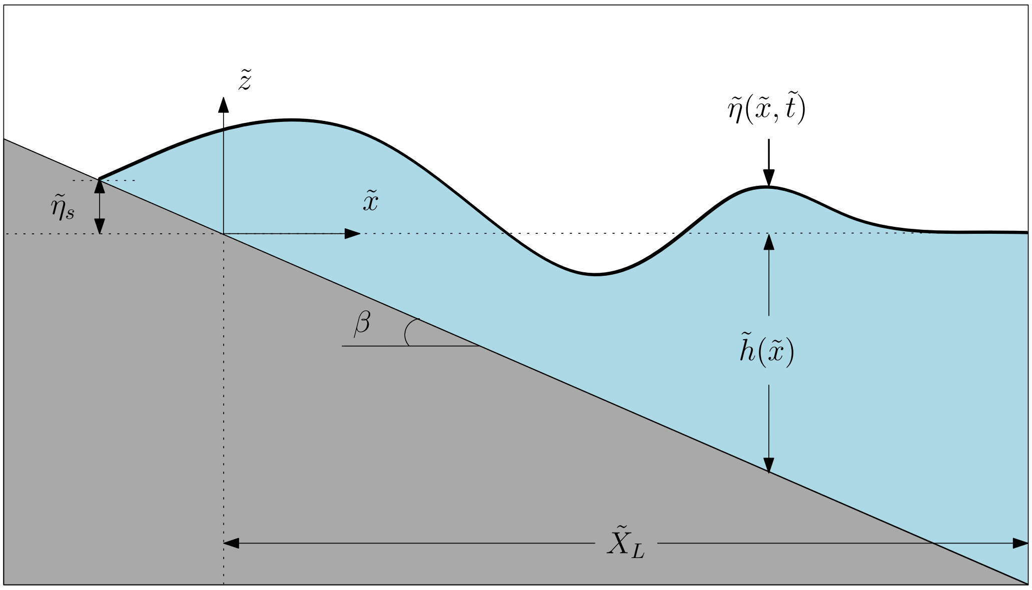

Figure 2.1. Definition sketch (not to scale). Note that the dimensional undisturbed water depth $\tilde{h}(\tilde{x}) = \tilde{x}\tan\beta$ becomes $h(x) = x$ in nondimensional form.

the moving shoreline tip ($x = -\eta$) in the physical coordinates onto a fixed point ($\sigma = 0$) in the hodograph plane. The relations defined by Eqns. 2.2 transform Eqns. 2.1 into

$$(\sigma^2 u)_\sigma + 2\sigma(\eta + \frac{u^2}{2})_\lambda = 0, \tag{2.3a}$$

$$2\sigma\, u_\lambda + (\eta + \frac{u^2}{2})_\sigma = 0, \tag{2.3b}$$

in the hodograph (σ, λ)-plane. Further defining the function

$$\varphi = \eta + \frac{u^2}{2}, \tag{2.4}$$

often called the *potential* of the transformation, it becomes straightforward to combine Eqns. 2.3 into a single second-order linear differential equation expressed in terms of the potential function φ,

$$4\,\varphi_{\lambda\lambda} - \frac{1}{\sigma}(\sigma\varphi_\sigma)_\sigma = 0. \tag{2.5}$$

We will define an IBVP governed by Eqn. 2.5 and we will seek solution through separation of variables method, as in Aydın (2011). This method requires two initial and two boundary conditions. In the most general case, the initial conditions consist of a prescribed initial wave height distribution, $\eta(x, t = 0) = \eta_0(x)$, and a corresponding velocity profile, $u(x, t = 0) = u_0(x) \neq 0$. These conditions are translated into the hodograph plane by directly substituting the linearized form of Eqn. 2.2a, i.e., $x \approx \sigma^2$ (Kânoğlu, 2004). Then, the initial conditions in the hodograph plane can be expressed as $\eta(\sigma, \lambda = \lambda_0) = \eta_0(\sigma)$ and $u(\sigma, \lambda = \lambda_0) = u_0(\sigma)$, or more suitably for the present formulation as

$$\varphi(\sigma, \lambda = \lambda_0) = \eta_0(\sigma) + \frac{u_0^2(\sigma)}{2} \equiv P(\sigma), \tag{2.6a}$$

$$\varphi_\lambda(\sigma, \lambda = \lambda_0) = -u_0(\sigma) + \frac{\sigma u_0'(\sigma)}{2} \equiv F(\sigma), \tag{2.6b}$$

following from Eqn. 2.4 and Eqn. 2.3a, respectively. Here, $\lambda_0 = -u_0(\sigma) \neq 0$ following Eqn. 2.2b written at $t = 0$, since $u(x, t = 0) = u_0(x) \neq 0$. We remark that $t = 0$ in the physical space corresponds to $\lambda_0 = 0$ in the hodograph space in the absence of the initial velocity.

The boundary condition at the shoreline tip is straightforward: we require the solution to be bounded everywhere, including the shoreline. For the condition at the seaward boundary, we adopt $\varphi_\sigma(\sigma = \sqrt{X_L}, \lambda) = 0$ (Given $x \approx \sigma^2$, $x = X_L$ in the physical space corresponds to $\sigma = \sqrt{X_L} = L$ in the hodograph space). Although this condition results in wave reflection from the seaward boundary, with appropriate choice of the parameter X_L it will suffice to provide solutions for the shoreline quantities. So, the boundary conditions in the hodograph plane can be written as

$$\varphi(\sigma = 0, \lambda) = \text{finite}, \tag{2.7a}$$

$$\varphi_\sigma(\sigma = L, \lambda) = 0. \tag{2.7b}$$

The IBVP formulated above is now suitable for a solution by means of separation of variables. We assume φ in the form of $\varphi(\sigma, \lambda) = S(\sigma)T(\lambda)$. Substituting this into Eqn. 2.5 and applying the condition given in Eqn. 2.7a, the series expansion for the potential function becomes

$$\varphi(\sigma, \lambda) = A_0 + \sum_{n=1}^{\infty} J_0(2\alpha_n \sigma)\,[A_n\, \cos(\alpha_n \lambda) + B_n\, \sin(\alpha_n \lambda)], \tag{2.8}$$

denoting $\alpha_n = z_n/2L$ for ease of notation. In the Fourier-Bessel series above $J_0(\bullet)$ is the Bessel function of the first kind of order zero and z_n are the zeros of the function $zJ_0'(z) = 0$, or $zJ_1(z) = 0$, following from the boundary condition at $\sigma = L$, Eqn. 2.7b. Since the equation $zJ_1(z) = 0$ has a double root at $z = 0$, Dini's expansion of order zero yields an initial constant term in the series above (Bowman, 1958), given by

$$A_0 = 2 \int_0^L \sigma\, P(\sigma)\, d\sigma. \tag{2.9}$$

To calculate the unknown coefficients (A_n, B_n) for $n \geq 1$ we apply the initial conditions given in Eqns. 2.6 and solve the resulting matrix equations. The final formulas for A_n and B_n become

$$\left\{ \begin{array}{c} A_n \\ B_n \end{array} \right\} = \frac{\alpha_n}{L^2\, J_0^2(z_n)} [\left\{ \begin{array}{c} \alpha_n P_n \\ F_n \end{array} \right\} \cos(\alpha_n \lambda_0) + \left\{ \begin{array}{c} -F_n \\ \alpha_n P_n \end{array} \right\} \sin(\alpha_n \lambda_0)], \tag{2.10}$$

in which

$$\left\{ \begin{array}{c} P_n \\ F_n \end{array} \right\} = \int_0^L \sigma \left\{ \begin{array}{c} P(\sigma) \\ F(\sigma) \end{array} \right\} J_0(2\alpha_n \sigma)\, d\sigma. \tag{2.11}$$

This completes the solution Eqn. 2.8 of the IBVP defined by Eqns. 2.5-2.7 in the hodograph plane.

The depth-averaged velocity $u(\sigma, \lambda)$ follows from Eqn. 2.3b as

$$u(\sigma, \lambda) = -\frac{2}{\sigma} \sum_{n=1}^{\infty} J_1(2\alpha_n \sigma)\,[A_n\, \sin(\alpha_n \lambda) - B_n\, \cos(\alpha_n \lambda)]. \tag{2.12}$$

The singularity of Eqn. 2.12 at the shoreline ($\sigma = 0$) can easily be handled with the use of $\lim_{z\to 0} J_1(\xi z)/z = \xi/2$, and the shoreline velocity becomes

$$u_s(\lambda) = -2\sum_{n=1}^{\infty} \alpha_n \left[A_n \, \sin(\alpha_n \lambda) - B_n \, \cos(\alpha_n \lambda)\right]. \tag{2.13}$$

Now that $\varphi(\sigma, \lambda)$ and $u(\sigma, \lambda)$ are calculated in the hodograph plane, we have the necessary data to invert the hodograph transformation and express the solution in the physical variables, i.e., (x, t)-plane. The inversion algorithm is as follows,

$$\eta = \varphi - \frac{u^2}{2}, \tag{2.14a}$$

$$x = \sigma^2 - \eta, \tag{2.14b}$$

$$t = \lambda + u. \tag{2.14c}$$

Spatial variation at any time $t = t^*$ or temporal variation at any distance $x = x^*$ can be evaluated through the Newton-Raphson iteration scheme (Synolakis, 1987; Kânoğlu, 2004). Moreover, calculation of the temporal variation of the shoreline location is as easy as substituting $\sigma = 0$ in Eqns. 2.14b and 2.14c,

$$x_s(\lambda) = \frac{u_s^2(\lambda)}{2} - \varphi(0, \lambda), \tag{2.15a}$$

$$t_s(\lambda) = \lambda + u_s(\lambda). \tag{2.15b}$$

3. INITIAL CONDITIONS

In this section, the solution method described above is applied to Gaussian, solitary, and N-wave type initial wave profiles having nonzero initial velocity in general.

3.1. Gaussian initial waves

A widely used profile in analytical benchmarking of long waves is the so-called Gaussian wave (single hump). It is described by

$$\eta_0(x) = h_1 \, \mathrm{e}^{-c_1(x-x_1)^2}, \tag{3.16}$$

where the parameters h_1, x_1, and c_1 determine the height, the location, and the steepness of the initial profile, respectively. An N-wave (plus-minus source) can also be constructed by combining two Gaussian waves, i.e.,

$$\eta_0(x) = h_1 \, \mathrm{e}^{-c_1(x-x_1)^2} - h_2 \, \mathrm{e}^{-c_2(x-x_2)^2}. \tag{3.17}$$

Carrier et al. (2003) computed the time evolution of the shoreline position and the shoreline velocity for initial waves given in Eqns. 3.16 and 3.17 by using the Hankel integral transform method. Kânoğlu (2004) obtained the same results without having to evaluate elliptic integrals. Here, we take the initial wave parameters from Carrier et al. (2003), see Fig. 3.1, and we evaluate the shoreline wave height $\eta_s(t)$ and the shoreline velocity $u_s(t)$ for cases with and without initial velocity (Fig. 3.2). The initial velocity profile is defined by (Kânoğlu and Synolakis, 2006)

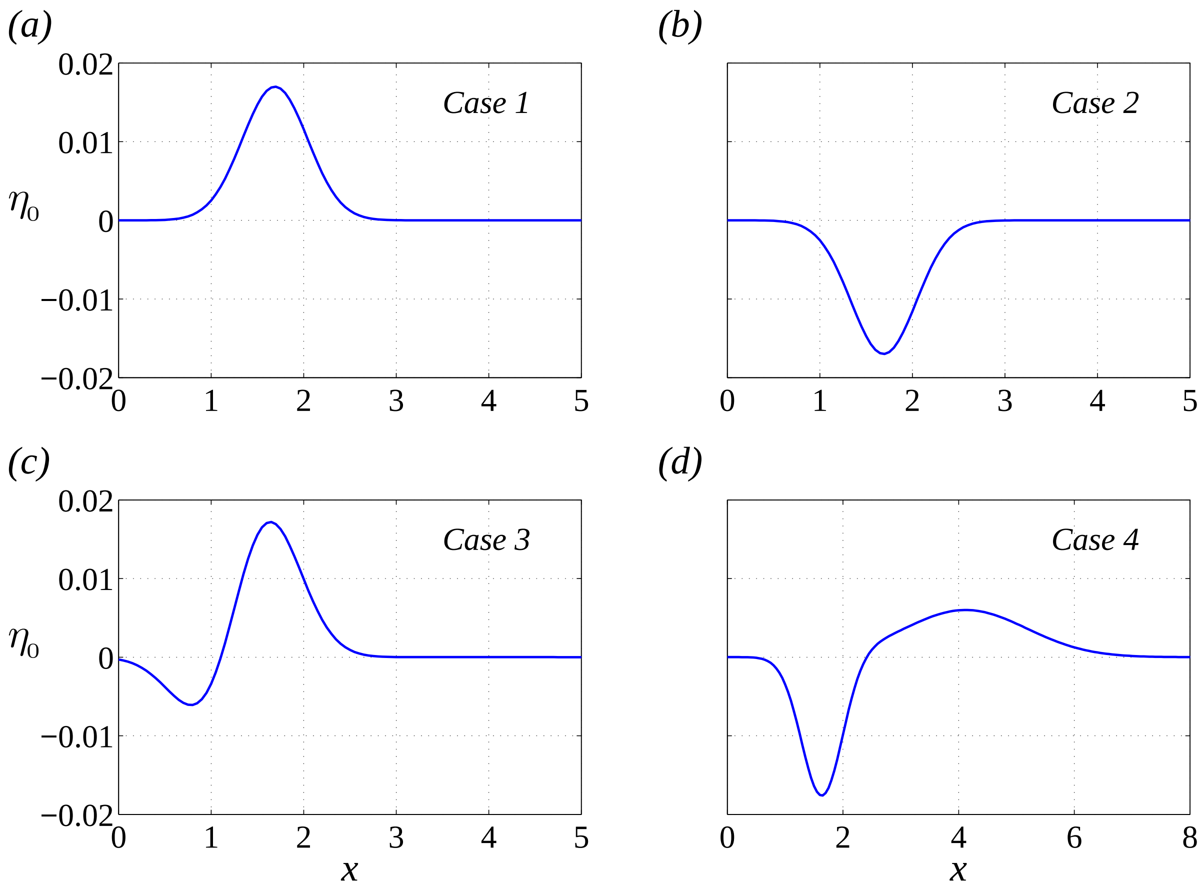

Figure 3.1. Gaussian initial waves used by Carrier et al. (2003). The initial wave parameters in each case are as follows; **(a)** Case 1: $h_1 = 0.017$, $c_1 = 4.0$, $x_1 = 1.69$; **(b)** Case 2: $h_1 = -0.017$, $c_1 = 4.0$, $x_1 = 1.69$; **(c)** Case 3: $h_1 = 0.02$, $c_1 = 3.5$, $x_1 = 1.5625$; $h_2 = 0.01$, $c_2 = 3.5$, $x_2 = 1.0$; **(d)** Case 4: $h_1 = 0.006$, $c_1 = 0.4444$, $x_1 = 4.1209$; $h_2 = 0.018$, $c_2 = 4.0$, $x_2 = 1.6384$.

$$u_0(x) = 2\sqrt{x} - 2\sqrt{x + \eta_0(x)}. \tag{3.18}$$

We also included results of Kânoğlu (2004) solution for comparison purposes. It should be noted that the present method required little computational effort even under the nonlinear initial velocity condition.

3.2. Solitary initial waves

The solitary wave profile is defined by

$$\eta_0(x) = H\,\mathrm{sech}^2\,\gamma(x - x_1), \tag{3.19}$$

with $\gamma = \sqrt{3H/4}$. H is the initial wave amplitude and x_1 is the initial center location of the wave. The shoreline dynamics for a solitary wave are presented in Fig. 3.3 for zero and nonzero initial velocity cases.

3.3. *N*-wave type initial waves

Tadepalli and Synolakis (1994) introduced two dipolar waveforms as more realistic initial profiles for the surface response of seafloor displacements. The isosceles N-wave profile is

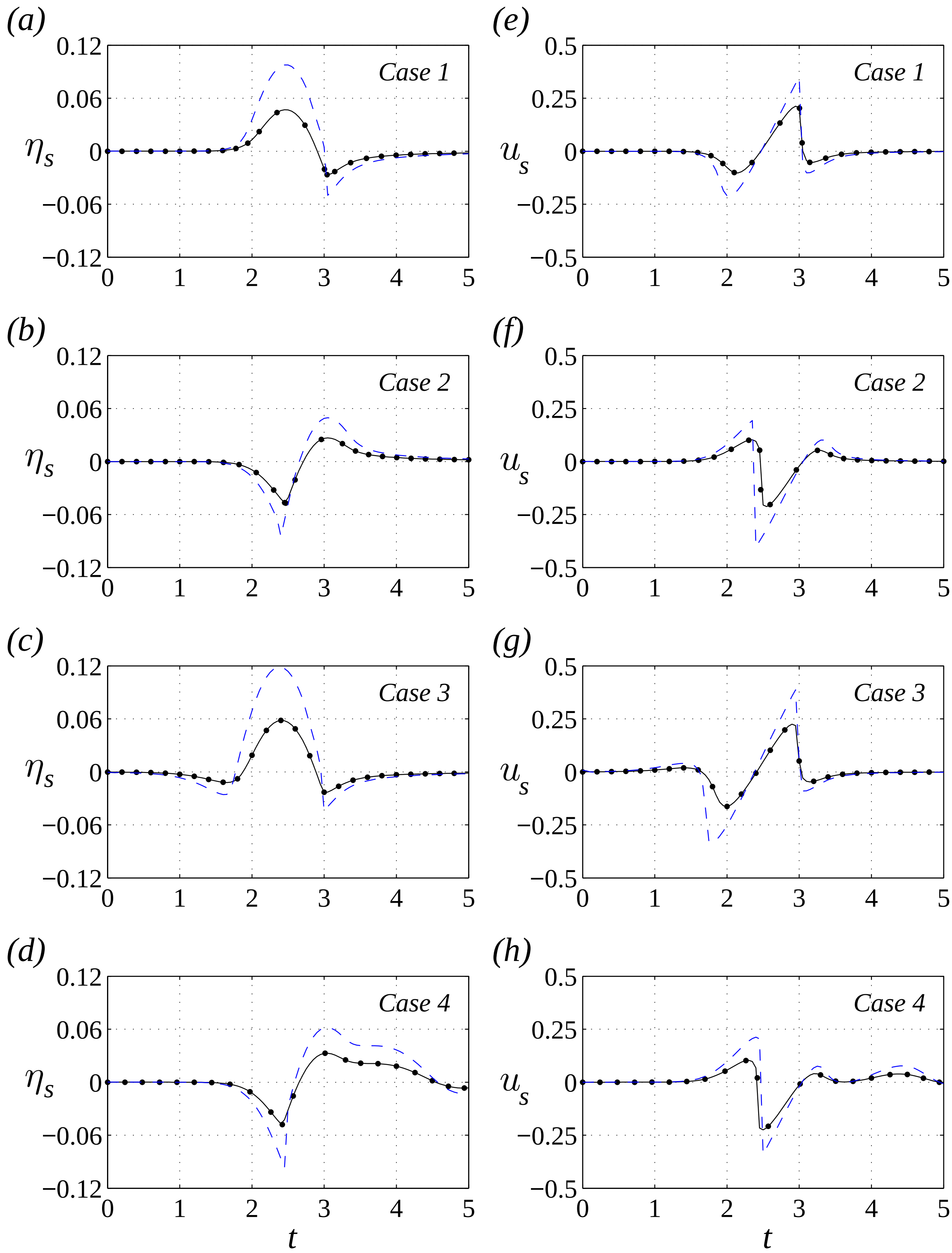

Figure 3.2. Time variations of the shoreline position, $\eta_s(t)$ **(left panel)** and the shoreline velocity, $u_s(t)$ **(right panel)** for the Gaussian initial waves given by Eqns. 3.16 and 3.17. Dashed and solid lines represent the present solution with and without initial velocity while dots are the zero initial velocity solution belonging to Kânoğlu (2004). The initial wave parameters for each case are listed in the caption of Fig. 3.1.

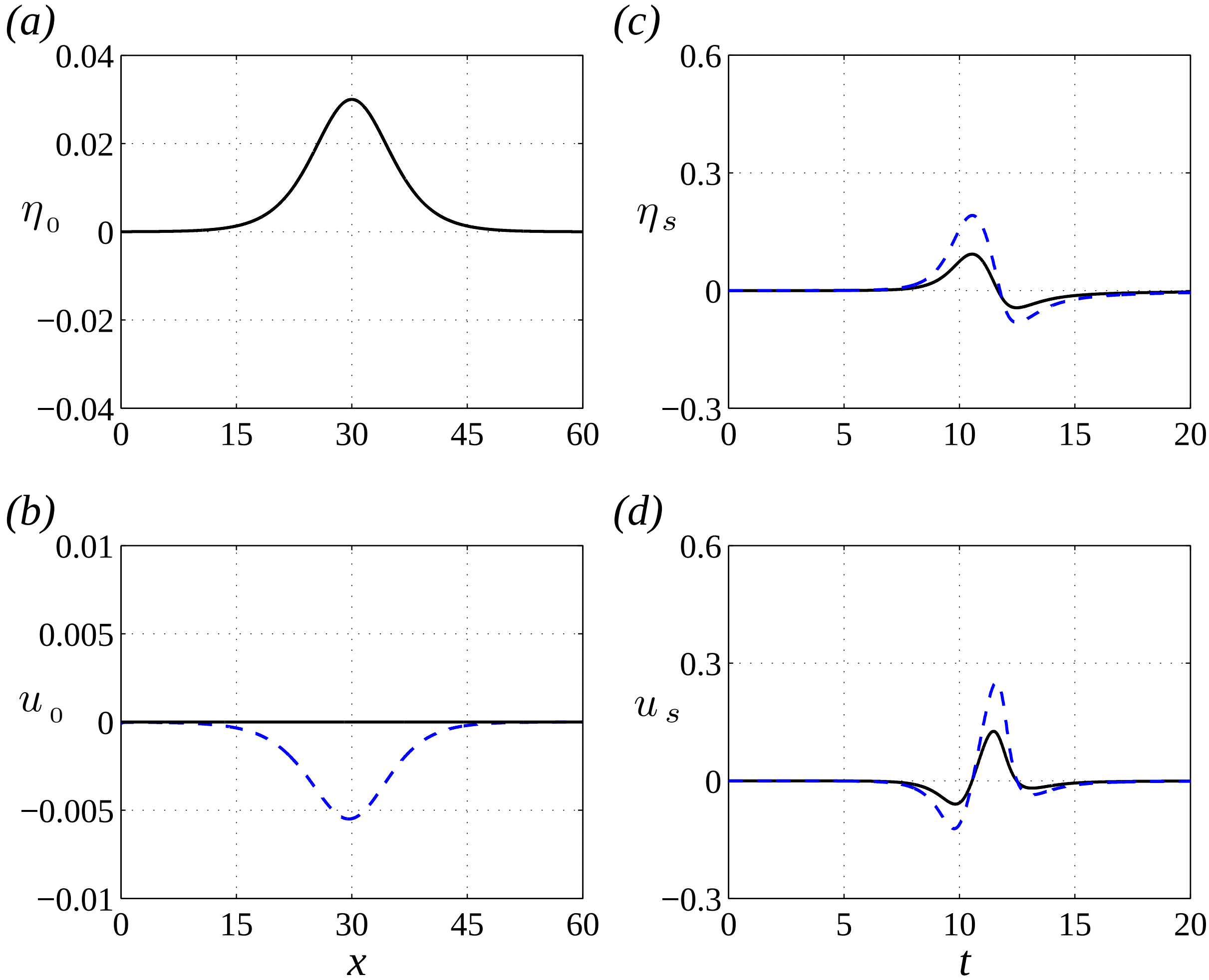

Figure 3.3. **(a)** Solitary initial wave defined in Eqn. 3.19 with parameters $H = 0.03$ and $x_1 = 30$. **(b)** The initial velocity distribution given by Eqn. 3.18 (dashed line) versus the zero initial velocity everywhere (solid line). Time variations of **(c)** the shoreline position and **(d)** the shoreline velocity are plotted for cases with (dashed lines) and without (solid lines) initial velocity.

defined as

$$\eta_0(x) = \frac{3\sqrt{3}}{2} H \operatorname{sech}^2 \gamma(x - x_1) \tanh \gamma(x - x_1), \tag{3.20}$$

with $\gamma = (3/2)\sqrt{H\sqrt{3/4}}$. This profile produces identical depression and elevation heights (Fig. 3.4a). Another dipolar profile which produces uneven positive and negative disturbances is the so-called generalized N-wave defined by

$$\eta_0(x) = \varepsilon\, H\, (x - x_1) \operatorname{sech}^2 \gamma(x - x_2), \tag{3.21}$$

with $\gamma = \sqrt{3H/4}$. The scaling parameter ε ensures that the initial wave amplitude is H.

The shoreline wave height and the shoreline velocity variations for waveforms defined in Eqns. 3.20 and 3.21 are presented in Fig. 3.4. Nonzero initial velocity case is compared with zero initial velocity case. The results suggest that, of the waves having the same initial amplitude, the generalized N-wave produces higher shoreline wave height compared to the solitary wave (Compare Fig. 3.4g with Fig. 3.3c). Among the isosceles and the

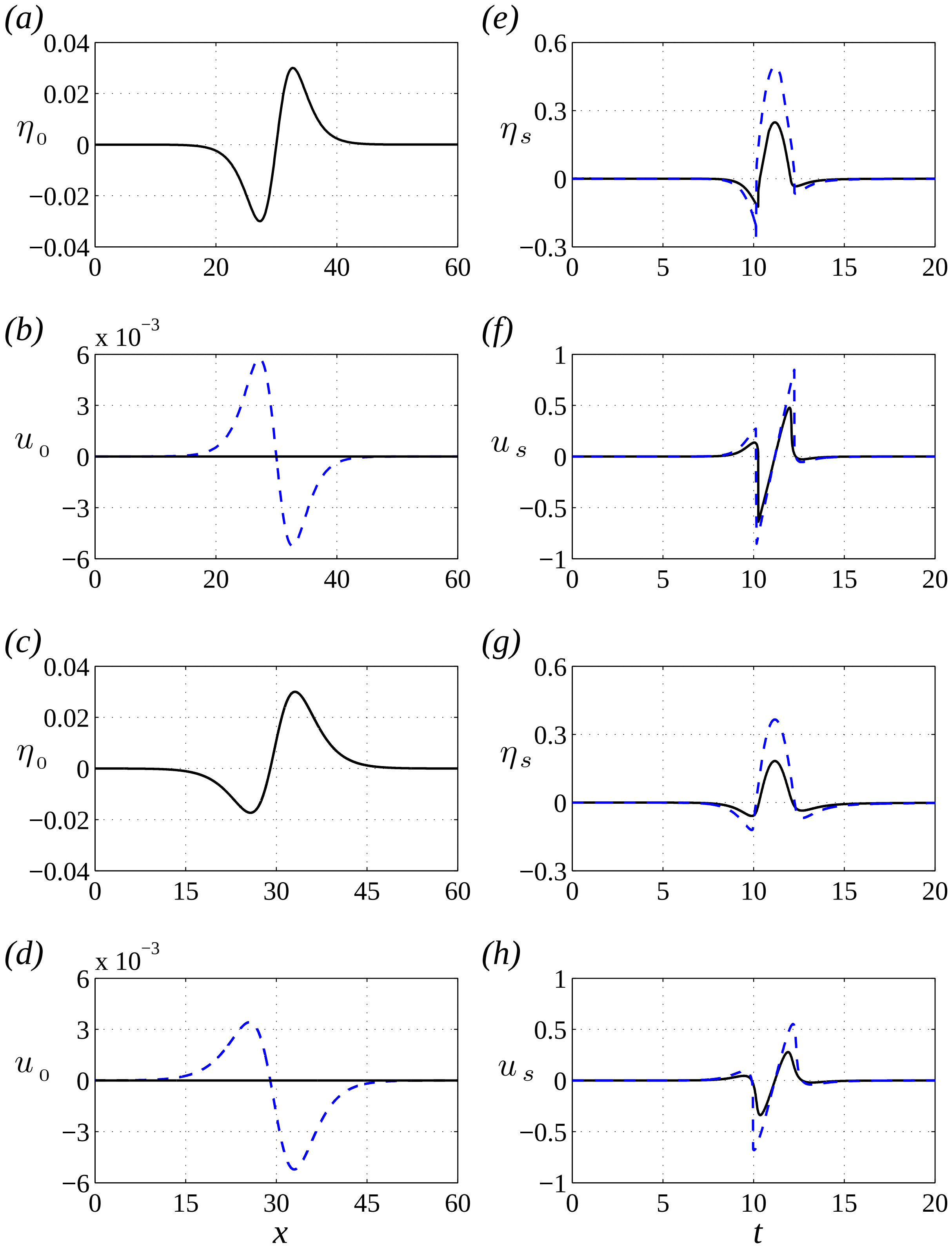

Figure 3.4. **(a, c)** The initial surface profiles and **(b, d)** the initial velocity distributions for the N-waves defined in Eqns. 3.20 and 3.21, respectively. The parameters for the isosceles N-wave, inset **(a)**, are: $H = 0.03$, $x_1 = 30$. The parameters for the generalized N-wave, inset **(c)**, are: $H = 0.06$, $x_1 = 30$, $x_2 = 29$, for which $\varepsilon = 0.1827$. **(e-h)** Time variations of the shoreline position and the shoreline velocity for corresponding initial waveforms. Dashed and solid lines represent the present solution with and without initial velocity, respectively.

generalized N-waves of the same initial elevation amplitude, the former produces higher shoreline wave height compared to the latter, comparing Fig. 3.4e and 3.4g. This is consistent with the results of Tadepalli and Synolakis (1994).

4. CONCLUSIONS

Nonlinear propagation of long waves climbing a linearly sloping beach is modeled by combining a fundamental analytical tool, namely the separation of variables method with the state-of-the-art hodograph transformation. The proposed model handles general initial-boundary value problem, i.e., a prescribed wave profile with nonzero initial velocity. Accurate estimations for time variations of shoreline wave height and velocity distributions are obtained for a variety of initial waveforms; hence, compared to integral transform methods, the proposed model is more flexible in terms of initial conditions. The computational efficiency of the new method is another advantage over the integral transform solutions. Therefore, it is hoped that the proposed solution can serve as a simple analytical benchmark solution for long wave numerical models.

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REFERENCES

Aydın, B. (2011). *Analytical solutions of shallow-water wave equations.* PhD thesis, Middle East Technical University, Ankara, Turkey.

Aydın, B. and Kânoğlu, U. (2007). Wind set-down relaxation. *CMES: Computer Modeling in Engineering and Sciences*, **21(2)**:149–155.

Bowman, F. (1958). *Introduction to Bessel Functions.* Dover Publications Inc.

Carrier, G. F. and Greenspan, H. P. (1958). Water waves of finite amplitude on a sloping beach. *Journal of Fluid Mechanics*, **4**:97–109.

Carrier, G. F., Wu, T. T., and Yeh, H. (2003). Tsunami run-up and draw-down on a plane beach. *Journal of Fluid Mechanics*, **475**:79–99.

Kânoğlu, U. (2004). Nonlinear evolution and runup-rundown of long waves over a sloping beach. *Journal of Fluid Mechanics*, **513**:363–372.

Kânoğlu, U. and Synolakis, C. E. (2006). Initial value problem solution of nonlinear shallow water-wave equations. *Physical Review Letters*, **97**. article no. 148501.

Synolakis, C. E. (1987). The runup of solitary waves. *Journal of Fluid Mechanics*, **185**:523–545.

Tadepalli, S. and Synolakis, C. E. (1994). The run-up of N-waves on sloping beaches. *Proceedings of the Royal Society of London A*, **445**:99–112.