Discrete Analytic Signal Wavelet Decomposition for Phase Localized in Time-Frequency Domain for Generation of Stochastic Signal with Phase Uncertainty

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SUMMARY:
Fourier transform has been widely used in dynamic analysis so far. The stochastic ground motions can be generated by altering the Fourier phase in original ground motion. However, the time-frequency information is only obtained by wavelet transform. But usually, when the phase of a wavelet basis function is altered, the total power of real part is affected and thus the generation of artificial stochastic ground motions is not possible. In this context, we propose to use analytic signal hardy wavelet analysis that describes the time-frequency characteristics and allows consideration of phase uncertainty for generation of stochastic ground motions. This paper verifies the performance of presented scheme with numerical simulations showing more localized disturbance in generated artificial ground motions compared to the conventional scheme. It is also verified that the same transmitting function as in Fourier transform can be used for evaluating the response of a linear structural system.

Keywords: Analytic signal, Uncertainty in Phase, Stochastic ground motions, Time-frequency domain.

1. INTRODUCTION

The dynamic analysis of structures is essential for modern seismic design, such as performance-based design. Since the powerful computing environment is available these days, we are able to perform complicated numerical computations for seismic design and the importance of design input ground motion is emphasized.

Fourier transform has been widely used in dynamic analysis. It analyses an input signal defining both phase and amplitude at various frequencies. It uses sinusoidal base function ($e^{i\omega t}$) that has shift-invariance and orthogonal properties. The shift-invariance property assures the conservation of total power of the base function after the phase is altered, while the orthogonal property assures the unique decomposition and perfect reconstruction of signal. And because of these two properties, the generation of stochastic ground motions considering phase uncertainties is possible using Fourier transform.

Fourier transform also has inconvenient property. Firstly, the temporal variation cannot be explained explicitly since the sinusoidal base function extends from -∞ to +∞ and the exact time cannot be estimated. When the behaviour of a non-linear system is discussed, the time-frequency information of the input signal is very important because once the non-linearity is reached during the vibration, the behavior of structure might be different after every cycle of loading and unloading depending upon the variation of natural frequency of structure with time. Secondly, the power is distributed throughout the time history of the generated ground motion and the desired localized disturbance is not possible.

Recently, wavelet transform has been adopted for time-frequency analysis. In particular, orthogonal wavelet transforms are widely used since they allow inverse transforms and therefore are suitable for wave synthesis. Since the wavelet transform provides explicit time-frequency representation, concept of phase is not important for the sole purpose of temporal representation. Phase can be manifested in wavelet coefficients, but in usual cases, as the phase is altered, the total power of real part is also changed and thus the uncertainties in phase cannot be considered for the purpose of generation of stochastic ground motions.

In this paper, we present an analytic signal hardy wavelet as a base function (as shown in Figures 1.1 and 1.2). Since it is an analytic signal, the shift-invariance property is attained and hardy wavelet inherently has an orthogonal property. These properties in presented scheme enable the explicit definition of both `phase' and `amplitude' with the time-frequency representation. The analytic wavelet coefficients can be expressed into the phase and amplitude like the Fourier coefficients. Then the artificial ground motions can be generated by considering uncertainties in phase spectrum and also the localized disturbance can be achieved in the generated ground motion unlike in Fourier transform.

This paper is organized as follows. The analytic signal is defined and the analytic wavelet transform is explained in section 2, signifying the importance of shift-invariance and orthogonal properties. It is highlighted that analytic Hardy wavelet signal enables phase localized in time-frequency domain in the same section. Section 3 explains the representation of phase uncertainty with the comparison among Fourier transform and analytic wavelet transform with numerical simulations. Section 4 shows the derivation of relation between Fourier coefficients of wavelet coefficients of an input signal and its response. The relation is also verified by the numerical simulations and the procedure is explained in the same section. Section 4 also highlights the convenience maintained in wavelet transform due to the conservation of transmitting function. Finally, section 5 concludes the paper.

2. ANALYTIC WAVELET TRANSFORM

In this paper, Wavelet transform that uses analytic Hardy wavelet signal as a base function, is proposed as a tool for localized time-frequency analysis that allows the consideration of uncertainties due to phase changes.

2.1. Analytic Signal

An analytic signal is a complex signal that has no negative frequency components. A real signal $\psi(t)$ is converted into an analytic signal $\psi_A(t)$ by integrating twice its Fourier transform over the positive frequency range as

$$\psi_A(t) = 2\int_0^\infty \psi(\omega)\, e^{i\omega t} \tag{2.1}$$

so that the total power of real signal is maintained in the analytic signal. Here, $\psi(\omega)$ denotes discrete Fourier transform of a real signal. For discrete analysis, $\omega = k\Delta\omega$ where $\Delta\omega = 2\pi/N\Delta t$ denotes interval of angular frequency, $k = 0,1,2,\ldots\ldots(N-1)$ being the frequency index, while Δt denotes time interval, and, time $t = n\Delta t$, where $n = 0,1,2,\ldots\ldots(N-1)$ being the time index, and N being the total number of discrete data.

2.2. Discrete Analytic Wavelet Transform

Analytic Hardy wavelet is a complex function that has sinc function as its real and imaginary parts as shown in Figure 1.1. For numerical simulation, discretized analytic Hardy wavelet signal is reconstructed by inverse wavelet transform using the filter banks. In case of Fourier transform, the sinusoidal base function extends uniformly from -∞ to +∞ in time domain, but a wavelet function has its amplitude localized in certain time duration and rest of the period have almost zero amplitude. Figure 1.2 shows the same wavelet function in frequency domain and it has no negative frequency components since it is an analytic signal. This feature helps this tool to retain the shift-invariance property, i.e., the phase change doesn't affect the total power of real part. It is also observed that the amplitude of the Hardy wavelet signal is bounded in a certain frequency range. It equipped the Hardy wavelet with an orthogonal property, i.e., the inner product of wavelet functions with different scale or shift generate zero. Orthogonality assures the unique decomposition of an original input signal and perfect reconstruction of the signal by inverse transform.

The discrete analytic wavelet transform of the time series *x(t)* is given as

$$\tilde{X}(\tau, s) = \sum_{n=0}^{N-1} x(n\Delta t)\, \psi_A\left(\frac{n\Delta t-\tau}{s}\right) \tag{2.2}$$

where s denotes scale and τ denotes time-shift. It decomposes a time series signal into wavelet coefficients at different scales (or frequencies) and at various time-shifts as shown in Fig.2. And, the inverse wavelet transform is given as

$$x(t) = \sum_\tau \sum_s \tilde{X}(\tau, s)\, \psi_A\left(\frac{n\Delta t-\tau}{s}\right) \tag{2.3}$$

2.3. Phase and Amplitude localized in time-frequency domain

In Fourier transform, phase and amplitude are defined in frequency domain. In case of wavelet transform, amplitude is defined in time-frequency domain but in usual cases phase is not explicitly defined and we cannot utilize the concept of phase uncertainty in ground motion simulation. So our aim is to present a tool that enables phase localized in time-frequency domain. As is shown above, an analytic Hardy wavelet function is able to define phase in wavelet function as: $\sum \psi_A(t)\, e^{i\theta}$. The formulation is almost identical with that of Fourier transform and it also retains shift-invariance and orthogonal properties. Because of the localized property of wavelets, the localized disturbance is possible and ground motion simulation is more efficient compared to that by Fourier transform. It is observed in Section 3.

In Fourier transform, the response of a linear structure can be evaluated conveniently by the product of input signal and transmitting function in frequency domain. Section 4 verifies that the convenience is maintained in wavelet analysis since the same transmitting function can be used for the evaluation of response in time-frequency domain.

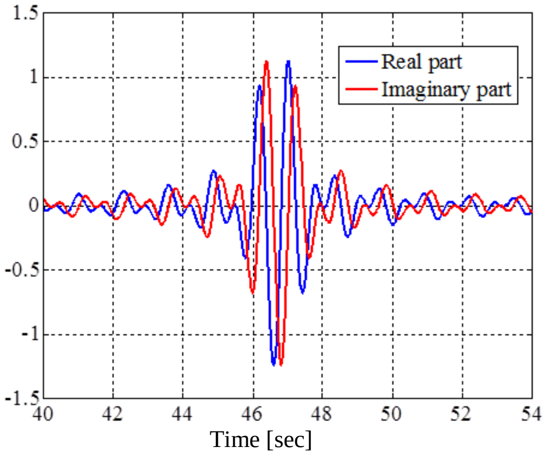

Figure 1.1. Time history of an analytic Hardy wavelet signal

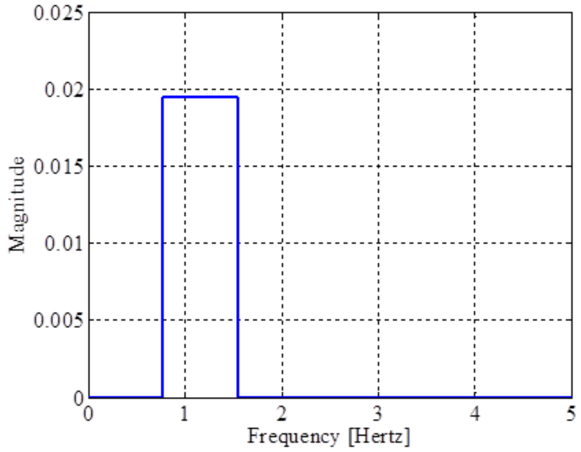

Figure 1.2. Fourier transform of analytic Hardy wavelet signal

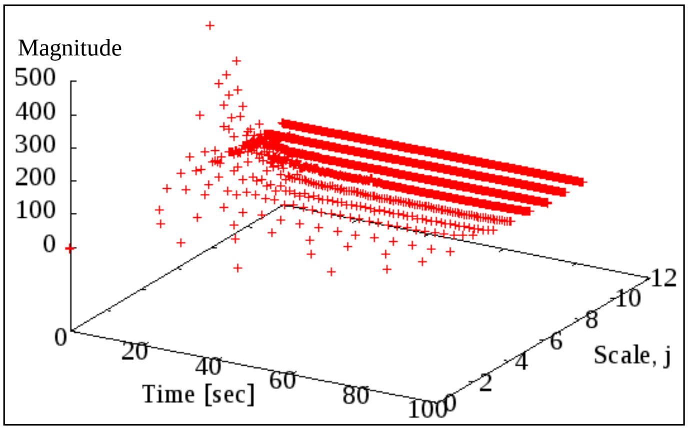

Figure 2. Discrete Analytic Hardy wavelet coefficients representing time-frequency characteristics of a time series signal.

3. UNCERTAINTY IN PHASE

A simple and powerful method for simulating ground motions is to combine parametric or functional descriptions of the ground motion's amplitude spectrum with a random phase spectrum modified such that the motion is distributed over a duration related to the earthquake magnitude and to the distance from the source (Boore 2003). This is known as the stochastic method in which a number of stochastic ground motions are generated from the original ground motion by giving random phase perturbation and keeping the total power of signal constant. Phase is nothing but the relative position of signal and is expressed as an angle from 0 to 2π radian. The response of a structure is likely to be different to each of these generated stochastic ground motions. The random phase changes can represent the uncertainty of ground motion. The representation of phase uncertainty for simulation of input ground motion is discussed in this section with comparison among Fourier transform, usual wavelet transform and proposed analytic Hardy wavelet transform.

3.1. Phase in Wavelet Transform

When a real wavelet signal is used, the wavelet coefficient $\tilde{X}(\tau, s)$ of Equations (2.2) or (2.3) has no phase. In analogy to the Fourier transform, it is possible to introduce phase θ in the form such as: $|\tilde{X}(\tau, s)|e^{i\theta}$. The definition of this phase is almost identical with the phase of Fourier transform and therefore it is expected that we can make use of conventional methods based on Fourier transform without significant difficulty. However, phase defined in this manner does not satisfy the property required for the manipulation of time series signal when real wavelet function is utilized. For example, if the phase value θ is changed to $\pi/2$, the real part disappears. It means that fluctuation of the phase can cause change in the total power of the signal and it is not suitable for our purpose of generating stochastic ground motions.

We use a complex analytic Hardy wavelet signal that is accompanied by the imaginary part as shown in Figure 1.1. This analytic Hardy wavelet function defines phase, retains shift-invariance and orthogonal properties like in Fourier transform and thus it makes a tool that is suitable for our purpose.

3.2. Numerical Simulations

When a phase uncertainty ξ is introduced in the Fourier coefficient of a signal, the total power of real part is unchanged although the configuration of signal may be totally changed, which can be understood mathematically as follows:

$$x_\xi(t) = \sum_{k=0}^{(N-1)} |\hat{X}(k\Delta\omega)| e^{i(\theta+\xi)} e^{i(k\Delta\omega)t} \quad (3.1)$$

where $\left|\hat{X}(k\Delta\omega)\right|$ denotes Fourier amplitude of $x(t)$ and θ denotes Fourier phase. In case of wavelet transform, we can assume almost identical equation as

$$x_{\xi}(t) = \sum_{k=0}^{(N-1)} \left|\tilde{X}(k\Delta\omega)\right| \psi\left(\frac{t-\tau}{s}\right) e^{i\xi} \tag{3.2}$$

The NS component of a strong ground motion data observed at Kobe observatory during the 1995 Hyogoken Nanbu Earthquake is chosen for numerical simulation. The total number of Fourier coefficients is 2048 and out of them, around 200 coefficients are dominant that contribute to most of the total power. Then, 50 out of these 200 coefficients are chosen randomly to be disturbed by the phase noise ξ. The ξ value is fluctuated from 0 to π as represented in equation (3.1). Figure 3.1 shows the comparison of time series of original wave and the Fourier phase-disturbed wave.

In the similar manner, we apply the perturbation to wavelet phase of the same input signal. The total number of analytic wavelet coefficients is 2048. Then, 40 out of around 200 dominant wavelet coefficients are chosen randomly and disturbed by the phase change uncertainties ξ varying from 0 to π as shown in equation (3.2). Figure 3.2 shows the comparison of time series of original wave and the wavelet phase-disturbed wave.

In both cases of Figures 3.1 and 3.2, it is verified that the total power (root mean square) of the wave is unchanged after the phase disturbance. The normalized total power of original signal is 6397.2096 and it is 6397.1740 and 6397.2042 after Fourier phase changes and analytic wavelet phase changes respectively. It verifies that the shift-invariance property is retained in case of analytic Hardy wavelet analysis like in Fourier analysis.

Let us compare the difference of the time history of the original and generated waves in Figures 3.1 and 3.2. In case of Figure 3.1, the unwanted ripples appearing towards the latter part of time history implies that the power is distributed throughout the time history and disturbance is created throughout. On the other hand, the ripples can be hardly seen in those areas in case of wavelet function in Figure 3.2. This is because the wavelet function is localized in time domain and the influence of the phase disturbance. This special feature allows us to consider the uncertainty in phase property without losing the time characteristics of the original wave. Since we want to focus mainly on dominant part of the ground motion and we do not want to reduce the severity in dominant part in the generated waves, this feature in wavelet case proves to be very useful.

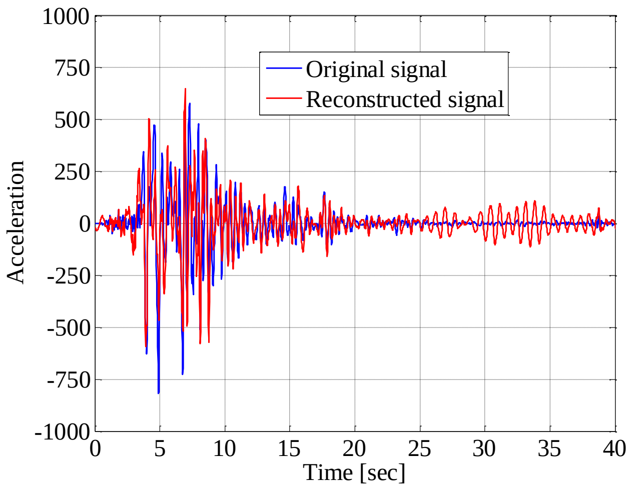

Figure 3.1. Comparison of time series of an original and reconstructed signal after the Fourier phase of 2 percent of the total or 25 percent of the dominant Fourier coefficients are contaminated by noise. Reconstructed wave is accompanied by the ripple which did not exist in the original signal.

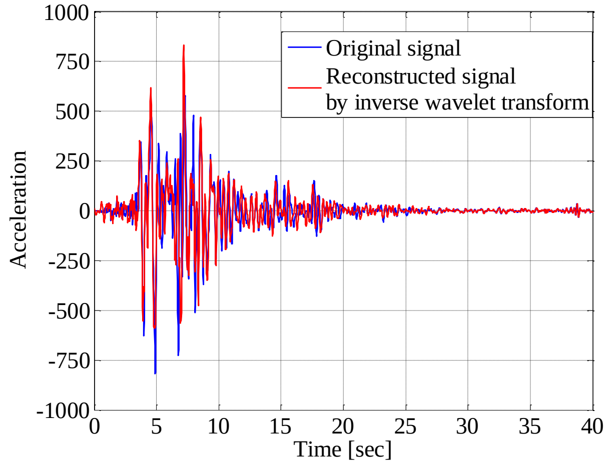

Figure 3.2. Comparison of time series of the original and reconstructed signal after the phase of 2 percent of total or 20 percent of the dominant analytic Hardy wavelet coefficients are contaminated by noise. Reconstructed signal is more affected locally compared to the reconstructed signal in Figure 3.1.

4. CONSERVATION OF TRANSMITTING FUNCTION

The generation of artificial ground motions is followed by the computation of response of a structural system. Let us consider a linear structural system. The response is given as the convolution of impulse response function of the linear system and the input motion. It is well known that Fourier transform of the output response is given as the product of Fourier transform of impulse response function (transmitting function) and that of the input motion as shown in equation (4.5). It allows us to evaluate the response of the structural system easily and therefore very useful for practical purpose.

In this paper, we show that similar relationship can be presented with wavelet coefficients and that the response of a linear structural system can be calculated from wavelet coefficients of input motion and the same transmitting function used in Fourier transform.

4.1. Calculation of Response Using Wavelet Coefficients & Transmitting Function

The analytical relation between response of a linear single degree of freedom system, the transmitting function and the input signal is determined as follows. Let us consider

$x(t)$: An input signal in time domain

$y(t)$: Response of linear system in time domain for the input signal $x(t)$

$h(t)$: Impulse response of the linear system in time domain

$\hat{X}(\omega_k)$, $\hat{Y}(\omega_k)$ & $\hat{H}(\omega_k)$: Discrete Fourier transform of $x(t)$, $y(t)$ and $h(t)$ respectively, where $\hat{H}(\omega_k)$ is also known as a transmitting function, the $\frown$ sign representing discrete Fourier transform

$\tilde{X}(\tau, s_j)$: Discrete Analytic Wavelet transform of $x(t)$, the $\sim$ sign representing wavelet transform

$\hat{\tilde{X}}(\omega_k, s_j)$, $\hat{\tilde{Y}}(\omega_k, s_j)$: Discrete Fourier transform of discrete wavelet transform of $x(t)$, $y(t)$, respectively

The discrete analytic wavelet transform of $x(t)$ is given by its convolution with analytic Hardy wavelet function and is represented as

$$\tilde{X}(\tau, s_j) = \sum_{n=0}^{N-1} x(n\Delta t)\, \psi_A\left(\frac{n\Delta t - \tau}{s_j}\right) \tag{4.1}$$

Let us define the wavelet function corresponding to the *j*-th scale and *n*-th shift as

$$\psi_A^{j,n}(\tau) = \psi_A\left(\frac{n\Delta t-\tau}{s_j}\right) \tag{4.2}$$

The wavelet coefficients are distributed in time-scale (time-frequency) domain. It should be noted that the number of coefficients ($N_j = 2^{j-1}$) is different at each scale s_j. So the discrete Fourier transform of discrete wavelet coefficients $\tilde{X}(\tau, s_j)$ in terms of τ is calculated separately for each scale and is represented as

$$\hat{\tilde{X}}(\omega_k, s_j) = \hat{X}(\omega_k)\,\hat{\psi}_A^{j,n}(\omega_k) \tag{4.3}$$

Similarly, the discrete analytic wavelet transform of response is represented as

$$\hat{\tilde{Y}}(\omega_k, s_j) = \hat{Y}(\omega_k)\,\hat{\psi}_A^{j,n}(\omega_k) \tag{4.4}$$

Utilizing the conventional relationship among $\hat{X}(\omega_k)$, $\hat{Y}(\omega_k)$ & $\hat{H}(\omega_k)$:

$$\hat{Y}(\omega_k) = \hat{H}(\omega_k)\hat{X}(\omega_k) \tag{4.5}$$

where $\omega_k = 2\pi k/N\Delta t$ denotes angular frequency, $k = 0,1,2,\ldots\ldots(N-1)$ is the frequency index, $j = 1,2,3,\ldots\ldots,m$ denotes scale index, $N = 2^m = \sum_{j=1}^{m} N_j$ represents total number of discrete time series data, $N_j = 2^{j-1}$ represents total number of coefficients at each scale index j, and, Δt denotes time interval.

From Equations (4.4) and (4.5) we get

$$\hat{\tilde{Y}}(\omega_k, s_j) = \hat{H}(\omega_k)\,\hat{X}(\omega_k)\,\hat{\psi}_A^{j,n}(\omega_k) \tag{4.6}$$

From Equations (4.3) and (4.6), it leads to

$$\hat{\tilde{Y}}(\omega_k, s_j) = \hat{H}(\omega_k)\,\hat{\tilde{X}}(\omega_k, s_j) \tag{4.7}$$

This equation (4.7) shows that the discrete Fourier coefficients of discrete wavelet coefficients of response can be evaluated by the product of discrete Fourier coefficients of discrete wavelet coefficients of input signal and transmitting function of the structural system. During the computation, some constant terms might be introduced in this equation (4.7) depending upon the definition of Fourier transform of impulse response function, such as

$$\hat{H}(\omega_k) = \frac{1}{N}\sum_{n=0}^{N-1} h(n\Delta t)\, e^{-i\omega_k(n\Delta t)} \tag{4.8}$$

From equations (4.7) and (4.8) we get

$$\hat{\tilde{Y}}(\omega_k, s_j) = N\Delta t\, \hat{H}(\omega_k)\,\hat{\tilde{X}}(\omega_k, s_j) \tag{4.9}$$

In order to reconstruct the signal from the coefficients of response obtained in equation (4.9) two steps are followed. Firstly, the inverse Fourier transform are taken separately for each scale:

$$\tilde{Y}(\tau, s_j) = \sum_{n=0}^{N-1} \hat{\tilde{Y}}(\omega_k, s_j)\, e^{i\omega_k(n\Delta t)} \tag{4.10}$$

Secondly, the inverse wavelet transform is applied to discrete wavelet coefficients obtained in equation (4.10) to obtain the response in time series:

$$y(t) = \sum_{\tau}\sum_{s_j} \tilde{Y}(\tau, s_j)\,\psi_A\left(\frac{n\Delta t-\tau}{s_j}\right) \tag{4.11}$$

4.2. Numerical Simulation

The relation in equation (4.9) is verified by the numerical computation by using strong motion record data as an input signal. We consider a single-degree-of-freedom system exposed to an earthquake motion. The mass and stiffness are set so that natural time period is given as t_n = 1 second. Damping factor (ζ = 0.02) is also added. The strong motion record (NS component) obtained at the Kobe observatory in Kobe during the 1995 Hyogoken Nanbu Earthquake is used as an input signal. The same input signal is used for numerical simulation in section 3 and is shown as original signal in Figures 3.1 or Figure 3.2. Impulse response of the linear single degree of freedom system to the input signal is calculated in time-domain by Newmark-β method. Both the input signal and impulse response are converted into analytic signals, following the definition of analytic signal in equation (2.1). Then the response of structure is obtained by following the procedure explained in section 4.1.

As a reference, the response of the linear structural system is also obtained by using time domain Newmark-β method. Time histories of the response evaluated from wavelet transform and the one obtained by time integration are compared in Figure 4. The two time histories show good agreement with each other, which verifies the relation obtained in equation (4.9).

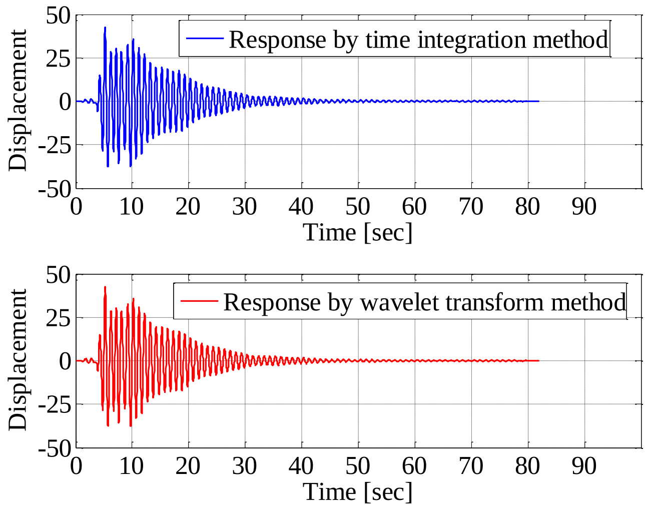

Figure 4. Comparison of time histories of responses of a linear system calculated by the presented wavelet scheme and the time-integration scheme. They show a very good agreement.

5. CONCLUSIONS

The representation of uncertainties is important for dynamic analysis of structures and synthesis of design input ground motion. A number of uncertain stochastic ground motions can be generated by disturbing the Fourier phase spectrum randomly and maintaining the frequency characteristics and total power of an original ground motion. In short, Fourier transform considers the temporal change in ground motion characteristics by using the concept of 'phase'. However, the Fourier phase changes disturb the time-frequency characteristics and localized disturbances at desired time intervals are not possible. For consideration of localized time-frequency characteristics, wavelet analysis is more widely accepted. But unlike in Fourier phase, usually we cannot utilize various conventional methodologies to generate artificial uncertain ground motions considering the wavelet phase. Considering such problems, this paper proposes a scheme to use discrete wavelet transform using analytic Hardy wavelet function. This tool enables phase localized in time-frequency domain by retaining shift-invariance and orthogonal properties like in Fourier analysis and allows performing ground motion simulation considering uncertainties in wavelet phase. Thus the presented scheme incorporates the advantages of both Fourier and wavelet transform by expressing the signal using the phase and amplitude in the similar manner as in Fourier transform and representing localized time-frequency characteristics of a signal as in wavelet transform. Numerical simulation shows that the presented scheme is not only able to generate ground motions with noise in phase but also able to produce localized disturbance in time history of the synthesized signal. Also it is verified with the numerical simulation that the dynamic response of a linear structural system can be evaluated by wavelet transform method using the same transmitting function defined by Fourier transform.

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