Response spectrum analysis of multi-storey building with accidental eccentricity

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SUMMARY

In the elastic response of building the uncertainties, due to the sources of accidental torsion, are taken into account in building codes introducing the so-called accidental eccentricity. This eccentricity can mathematically be expressed as a modification of the mass matrix.
In the framework of the response spectrum analysis (RSA) each mass modification, due to the accidental eccentricity, requires new analyses that could be numerically cumbersome.
A new combination rule, here called Interval Complete Quadratic Combination (ICQC), is proposed to obtain in closed form the maximum responses of structures with mass modification with a single RSA, without solving any further eigenproblem. This procedure is based on the application to the RSA of the interval perturbation method, recently proposed by the authors for determining the upper and lower bounds of the dynamic response of structures with uncertain-but-bounded mass distribution vibrating under either deterministic or stochastic input.

Keywords: Accidental Eccentricity, Response Spectrum Analysis (RSA), Complete Quadratic Combination (CQC), Interval Analysis.

1. INTRODUCTION

In the elastic response of building subjected to seismic excitation discrepancies between the computed and actual response have been shown (De La Lera and Chopra, 1994). The increase in building response can be due to several sources of accidental torsion: rotational motions of the buildings foundation, uncertainty in the stiffness of structural elements, uncertainty in the location of the centre of mass (CM), uncertainty in mass, stiffness and strength distributions of stories and in other sources of torsion. This is the reason why building codes (EC8, 2004; IBC 2006), to take into account of the effects of accidental torsion, establish of shifting the center of mass (CM) of each floor, from its nominal position, by a distance equal to the so-called *accidental eccentricity*. This latter can be defined by the quantity $d_a = \pm\ \alpha b$, where b is the plan dimension of the building perpendicular to the direction of horizontal component of ground motion considered and $\alpha\ (0.05 \div 0.1)$ is a parameter which define the uncertainty in accidental eccentricity.
In the framework of the seismic analysis of structures, the response spectrum analysis (RSA) is the most utilized procedure by engineers. The RSA requires the following main steps: (1) to solve an eigenproblem to evaluate for the structure the first natural frequencies, modes and their participation factors; (2) to select the damping ratios for each mode; (3) to read the maximum structural response for each mode from the code-specific design spectrum or from the site-specific spectrum; (4) to combine the maximum responses of all modes considered to have the expected maximum response of the structural systems. The last item is generally covered, in seismic codes, by means of the so-called complete quadratic combination (CQC) rule (Wilson et al., 1981).
In the building codes the effects of the accidental eccentricity are taking into account considering all

the possible permutations of the center of mass of the structural system to find the worst condition for the structural elements. This approach requires, for each center mass modification, a new RSA analysis, which needs the four steps before described. It follows that the described code approach is very cumbersome from a numerical point of view.
In this paper a new combination rule to obtain in closed form the maximum responses of structures with accidental eccentricity by RSA without solving any further eigenproblem is proposed. The proposed procedure is based on the application to the RSA of the interval perturbation method, recently proposed by the authors (Cacciola et al., 2011) for determining the upper and lower bounds of the dynamic response of structures with uncertain-but-bounded mass distribution vibrating under either deterministic or stochastic input. Finally, an extension of the classical CQC rule to the analysis of structural systems with uncertain-but-bounded parameter is presented. In particular, for structural systems with accidental eccentricity, the proposed approach allows to directly evaluate the worst condition for the structural elements with a single RSA. This very remarkable result is obtained by adopting a new modal combination rule, here called *Interval CQC* (ICQC).

2. EQUATIONS GOVERNING THE PROBLEM

Let us consider an idealized multi-storey buildings with rigid floor diagrams, where the floor masses are lumped and the lateral resistance is provided by resisting frames in the *x* and *y* directions. The structure subjected to ground motion has three degrees of freedoms (DoF) for each floor. Under the previous assumption the equation of motion of a linear quiescent *n*-floor building subjected to seismic excitation, represented by the accelerogram $\ddot{u}_g(t)$, can be written in the form:

$$\mathbf{M}\,\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\,\mathbf{u}(t)=-\mathbf{M}\,\boldsymbol{\tau}\,\ddot{u}_{\mathrm{g}}(t)\,, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are the $3\mathrm{n}\times 3\mathrm{n}$ inertia, damping, and stiffness matrices of the structure; $\mathbf{u}(t)=\left\{\mathbf{u}_x(t)\quad \mathbf{u}_y(t)\quad \mathbf{u}_\theta(t)\right\}^{\mathrm{T}}$ is the vector of floor displacements relative to the ground collecting the *n* translational and rotational components in the *x* and *y* directions $\left(u_{x,j}(t),u_{y,j}(t)\right)$ and around the vertical axis $\left(u_{\theta,j}(t)\right)$, respectively $\left(j=1,2,\cdots,n\right)$; a dot over a variable and the superscript T denote differentiation with respect to time and the transpose operator, respectively; finally, $\boldsymbol{\tau}\equiv\boldsymbol{\tau}_i$ is the $3\mathrm{n}$ order influence vector evaluated considering the seismic excitation in *i* direction $\left(i=x,y\right)$.
By defining the accidental eccentricity (EC8, 2004; IBC 2006) as $d_{i,s}=\pm\,\alpha\, b_{i,s}$ where α is a positive parameter which define the uncertainty in accidental eccentricity and $b_{i,s}$ is the plan dimension of *s*-th floor of the building, Eq. (1) can be rewritten as

$$\mathbf{M}_i(\alpha)\ddot{\mathbf{u}}_i(\alpha,t)+\mathbf{C}\dot{\mathbf{u}}_i(\alpha,t)+\mathbf{K}\,\mathbf{u}_i(\alpha,t)=-\mathbf{M}_i\left(\alpha\right)\tau_i\;\;\ddot{u}_{\mathrm{g}}(t)\,,\qquad i=x,y \tag{2}$$

where i is the direction of horizontal component of ground motion considered.
Let us definer now, according to the *interval analysis*, the uncertain-but-bounded α such as $\underline{\alpha}\le\alpha\le\bar{\alpha}$, being $\underline{\alpha}$ and $\bar{\alpha}$ its lower and upper bound, respectively. It follows that $\alpha^I\triangleq\left[\underline{\alpha},\bar{\alpha}\right]\in\mathbb{IR}$, where with $\mathbb{IR}$ the set of all closed real interval numbers has been denoted. The interval variable α^I is also defined (Moore, 1966):

$$\alpha^I=\alpha_0+\Delta\alpha\;e^I\,, \tag{3}$$

where e^I denotes the unitary interval variable, whose values lies in the unitary interval $\left[-1,1\right]$; α_0 and $\Delta\alpha$ denote the nominal (or midpoint) value and the deviation amplitude (or radius) of the interval variable α^I, respectively. In this paper it is assumed that α^I posses zero midpoint value. It follows that $\Delta\alpha \equiv \alpha$, consequently the interval variable α^I can be redefined as:

$$\alpha^I = \left[-\alpha, \alpha\right] = \alpha\ e^I \tag{4}$$

For the previous positions the inertia matrix $\mathbf{M}_i(\alpha)$, evaluated considering the seismic excitation in *i* direction $(i = x, y)$, can be expressed by the following relationship:

$$\mathbf{M}_i\left(\alpha\right) = \mathbf{M}_0 + \alpha e^I \mathbf{M}'_i, \qquad i = x, y \tag{5}$$

where

$$\mathbf{M}_0 = \mathbf{M}(0); \quad \mathbf{M}'_i = \frac{\partial}{\partial\alpha}\mathbf{M}_i(\alpha)\bigg|_{\alpha=0}; \quad i = x, y. \tag{6}$$

Notice that the nominal (or midpoint) position of CM is obtained setting $\alpha = 0$ in Eq. (2), so in this case Eqs. (1) and (2) coincide.
In the framework of the traditional modal analysis, the solution of Eq. (2) may be pursued by introducing the following coordinate transformation:

$$\mathbf{u}_i(\alpha,t) = \mathbf{\Phi}_0\, \mathbf{q}_i(\alpha,t), \qquad \alpha \in \alpha^I = \left[-\alpha, \alpha\right], \qquad i = x, y \tag{7}$$

where $\mathbf{q}_i(\alpha,t)$ is the interval vector gathering the first m modal coordinates $q_{i,j}(\alpha,t)$ ($j = 1, 2, \ldots, m \le n$); $\mathbf{\Phi}_0$ is the modal matrix, of order $n \times m$, pertaining to the midpoint or nominal configuration in which $\mathbf{M}_0 = \mathbf{M}(0)$. Specifically, the modal matrix $\mathbf{\Phi}_0$, collecting the first m eigenvectors, normalized with respect to the mass matrix $\mathbf{M}_0$, is evaluated as solution of the following eigenproblem:

$$\mathbf{K\Phi}_0 = \mathbf{M}_0\mathbf{\Phi}_0\mathbf{\Omega}_0^2; \quad \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{M}_0\mathbf{\Phi}_0 = \mathbf{I}_m \tag{8}$$

$\mathbf{\Omega}_0^2$ being a diagonal matrix, of order *m*, listing the squares of the natural circular frequencies $\omega_{0,j}^2$ $(j = 1, 2, ..., m)$ for the nominal value of the uncertain parameter. The differential equations governing the evolution of vector $\mathbf{q}_i\left(\alpha,t\right)$ is obtained by applying the position (7) to equation (2) and pre-multiplying the result by the matrix $\mathbf{\Phi}_0^{\mathrm{T}}\mathbf{M}_0\mathbf{M}_i\left(\alpha\right)^{-1}$:

$$\ddot{\mathbf{q}}_i\left(\alpha,t\right) + \Xi_i\left(\alpha\right)\dot{\mathbf{q}}_i\left(\alpha,t\right) + \mathbf{\Omega}_i\left(\alpha\right)\dot{\mathbf{q}}_i\left(\alpha,t\right) = -\mathbf{p}_{0,i}\,\ddot{u}_{\mathrm{g}}(t), \qquad \alpha \in \alpha^I = \left[-\alpha, \alpha\right], \quad i = x, y. \tag{9}$$

where

$$\Xi_i\left(\alpha\right) = \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{M}_0\mathbf{M}_i\left(\alpha\right)^{-1}\mathbf{C\Phi}_0; \quad \mathbf{\Omega}_i\left(\alpha\right) = \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{M}_0\mathbf{M}_i\left(\alpha\right)^{-1}\mathbf{K\Phi}_0; \quad \mathbf{p}_{0,i} = \mathbf{\Phi}_0^{\mathrm{T}}\mathbf{M}_0\boldsymbol{\tau}_i. \tag{10}$$

In the last equation, for the nominal structure, $\mathbf{p}_{0,i}$ is the vector of participation coefficients for the seismic excitation acting in *i* direction. In structural dynamics, the solution of Eq.(2) is often obtained introducing the state variables. Let the vector of state variables $\mathbf{z}\left(\alpha,t\right)$, of order $2m \times 1$, be written in the form

$$\mathbf{z}_i\left(\alpha,t\right) = \begin{bmatrix} \mathbf{q}_i\left(\alpha,t\right) \\ \dot{\mathbf{q}}_i\left(\alpha,t\right) \end{bmatrix}, \qquad \alpha \in \alpha^I = \left[-\alpha, \alpha\right], \qquad i = x, y \tag{11}$$

Then the equations of motion in modal space state variables can be recast as

$$\dot{\mathbf{z}}_i(\alpha,t)=\mathbf{D}_i(\alpha)\mathbf{z}_i(\alpha,t)-\mathbf{v}_{0,i}\,\ddot{u}_g(t), \qquad \alpha\in\alpha^I=[-\alpha,\alpha] \qquad i=x,y. \tag{12}$$

where

$$\mathbf{D}_i(\alpha)=\begin{bmatrix}\mathbf{O}_m & \mathbf{I}_m\\ -\boldsymbol{\Xi}_i(\alpha) & -\boldsymbol{\Omega}_i(\alpha)\end{bmatrix}, \quad \mathbf{v}_{0,i}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{p}_{0,i}\end{Bmatrix}. \tag{13}$$

According to the mass matrix decomposition in given Eq.(5), the matrix $\mathbf{D}_i(\alpha)$ can be split as:

$$\mathbf{D}_i(\alpha)=\mathbf{D}_0+\alpha e^I\mathbf{D}_i, \qquad i=x,y \tag{14}$$

where

$$\mathbf{D}_0=\mathbf{D}(0)=\begin{bmatrix}\mathbf{O}_m & \mathbf{I}_m\\ -\boldsymbol{\Omega}_0^2 & -\boldsymbol{\Xi}_0\end{bmatrix}; \quad \mathbf{D}_i=\frac{\partial}{\partial\alpha}\mathbf{D}_i(\alpha)\bigg|_{\alpha=0}=\begin{bmatrix}\mathbf{O}_m & \mathbf{O}_m\\ \mathbf{A}_i\boldsymbol{\Omega}_0^2 & \mathbf{A}_i\boldsymbol{\Xi}_0\end{bmatrix}, \quad i=x,y. \tag{15}$$

with $\mathbf{A}_i$ a symmetric matrix of order *m*, having for element $a_{i,jk}$, and defined as:

$$\mathbf{A}_i=\boldsymbol{\Phi}_0^{\mathrm{T}}\mathbf{M}_i'\,\boldsymbol{\Phi}_0, \qquad i=x,y \tag{16}$$

Notice that in Eqs.(15) the modal damping matrix $\boldsymbol{\Xi}_0=\boldsymbol{\Phi}_0^{\mathrm{T}}\mathbf{C}\,\boldsymbol{\Phi}_0$, for classically damped systems, is a diagonal one listing $2\zeta_j\,\omega_{0,j}$, with ζ_j the damping ratio associated to *j*-th mode.
In analogy to the decomposition of the mass matrix defined in Eq.(5), the vector response can be also evaluated as sum of two aliquots: the midpoint or nominal solution, $\mathbf{z}_{0,i}(t)\equiv\mathbf{z}_i(0,t)$, and a deviation, $\Delta\mathbf{z}_i(\alpha,t)$, that is:

$$\mathbf{z}_i(\alpha,t)=\mathbf{z}_{0,i}(t)+\Delta\mathbf{z}_i(\alpha,t), \qquad \alpha\in\alpha^I=[-\alpha,\alpha], \quad i=x,y. \tag{17}$$

The differential equation governing the evolution of the midpoint solution $\mathbf{z}_0(t)$ is obtained by setting $\alpha=0$ in Eq. (12):

$$\dot{\mathbf{z}}_{0,i}(t)=\mathbf{D}_0\,\mathbf{z}_{0,i}(t)-\mathbf{v}_{0,i}\,\ddot{u}_g(t) \tag{18}$$

Similarly, the equations governing the deviation vector $\Delta\mathbf{z}_i(\alpha,t)$ can be obtained substituting the relationship (17) into Eq.(12), and by taking into account Eq.(18), as:

$$\Delta\dot{\mathbf{z}}_i(\alpha,t)=\left[\mathbf{D}_0+\mathbf{D}_i\,\alpha e^I\right]\Delta\mathbf{z}_i(\alpha,t)+\mathbf{D}_i\,\alpha e^I\mathbf{z}_{0,i}(t), \quad i=x,y. \tag{19}$$

Notice that by setting the unitary interval variable e^I at its bounds, the differential equation (19) leads to the following two sets of differential equations:

$$\Delta\dot{\mathbf{z}}_i^-(\alpha,t) = \left[\mathbf{D}_0 - \mathbf{D}_i\,\alpha\right]\Delta\mathbf{z}_i^-(\boldsymbol{\alpha},t) - \mathbf{D}_i\,\alpha\,\mathbf{z}_{0,i}(t);$$
$$\Delta\dot{\mathbf{z}}_i^+(\alpha,t) = \left[\mathbf{D}_0 + \mathbf{D}_i\,\alpha\right]\Delta\mathbf{z}_i^+(\boldsymbol{\alpha},t) + \mathbf{D}_i\,\alpha\,\mathbf{z}_{0,i}(t), \qquad (i = x, y). \tag{20}$$

In the previous equations, the superscripts " $-$ " and " $+$ " mean that the corresponding quantity is evaluated assuming the unitary interval variable e^I equal to its lower and upper bound, respectively.

3. RESPONSE SPECTRUM ANALYSIS OF STRUCTURES WITH UNCERTAIN-BUT-BOUNDED PARAMETERS

3.1 Traditional complete quadratic combination (CQC) rule

Before proceeding with the formulation of the proposed method let us summarize the main steps and assumptions of the conventional Response Spectrum Analysis (RSA). It has to remark that, for the sake of simplicity, in this section, the subscript index *i*, that until now has been used in order to specify the direction of horizontal component of ground motion considered $(i = x, y)$, is neglected.
The RSA by means of the Complete Quadratic Combination (CQC) rule, estimates the maximum displacement of the $\ell -$th Dof, $u_\ell(t)$, as a combination of the maximum modal responses, where each maximum is obtained in terms of the mean response spectrum associated with the corresponding modal frequency and damping ratio, that is:

$$\max\left|u_{0,\ell}(t)\right| \equiv u_{0,\ell,\max} = \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{m}\rho_{0,jk}\,\phi_{0,\ell j}\,\phi_{0,\ell k}\,p_{0,j}\,p_{0,k}\,\frac{S_{\mathrm{pa}}\left(T_{0,j},\zeta_j\right)}{\omega_{0,j}^2}\frac{S_{\mathrm{pa}}\left(T_{0,k},\zeta_k\right)}{\omega_{0,k}^2}} \tag{21}$$

where $\phi_{0,\ell j}$ and $\omega_{0,j}$, are the element of the modal matrix Φ_0 and associated diagonal spectral matrix Ω_0, solutions of the eigenproblem (8), respectively; $p_{0,j} \equiv p_{0,i,j}$ is the j-th participation factor; $S_{\mathrm{pa}}\left(T_{0,j},\zeta_j\right)$ is the j-th ordinate of the response spectrum in terms of pseudo-acceleration. This quantity depends in turn on periods of vibration $T_{0,i} = 2\pi/\omega_{0,i}$ and viscous damping ratios ζ_j of the m modes of vibration. It is emphasized that Eq. (21) is evaluated by making the following assumptions (Der Kiureghian, 1980, 1981): (a) the maximum value of the structural response is considered to be proportional to its standard deviation by means of the so-called peak factor (Vanmarcke, 1975, 1976); (b) the mean value of the modal maximum response is evaluated in terms of the mean response spectrum associated with the corresponding modal frequency and damping ratio (Der Kiureghian, 1980), and (c) the peak factor is considered to be approximately the same for the response of interest $u_\ell(t)$ and for all the modal responses $q_{0,j}(t) \equiv q_{0,i,j}(t)$ (Der Kiureghian, 1981).
In Eq.(21) $\rho_{0,jk}$ is the cross-correlation coefficient, between j-th and k-th modes of vibration, which can be defined as:

$$\rho_{0,jk} = \frac{\sigma_{0,jk}}{\sigma_{0,jj}\,\sigma_{0,kk}} \tag{22}$$

$\sigma_{0,jk}$ is the cross-covariance between j-th and k-th modes of vibrations; and $\sigma_{0,jj}$ is the standard deviation of the *j*-th mode.
Closed form expression of the cross-correlation coefficient can be evaluated under the assumption that the input process is a stationary Gaussian white noise process (Der Kiureghian, 1980; Wilson et al., 1981). In this case it reads:

$$\rho_{0,jk} = \frac{8\sqrt{\zeta_j \zeta_k}\left(\zeta_j \omega_j + \zeta_k \omega_k\right)\left(\omega_j \omega_k\right)^{3/2}}{\kappa_{jk}} \tag{23}$$

with κ_{jk} defined as:

$$\kappa_{jk} = 4\omega_j \omega_k \left(\zeta_k \omega_j + \zeta_j \omega_k\right)\left(\zeta_j \omega_j + \zeta_k \omega_k\right) + \left(\omega_j^2 - \omega_k^2\right)^2 \tag{24}$$

3.2 Interval CQC (ICQC) rule for structural systems with uncertain-but bounded parameters

The aim of this paper is to propose a new CQC rule for structural analysis of buildings with uncertain-but-bounded parameters by adopting the interval algebra. Indeed this algebra leads to a very suitable formulation to evaluate, in closed form expression, a combination rule in presence of accidental eccentricity. The rule here proposed is defined interval CQC (ICQC).
It has been largely recognized that the combination rules can be derived by stochastic analysis assuming the input as a stationary Gaussian process.
Since it is assumed that the system behaves linearly the structural response is a stationary zero-mean Gaussian process too. This implies that the complete probabilistic characterization of the response is ensured by the knowledge of the modal covariance matrix that, for stationary excitation, can be evaluated by solving the *algebraic Lyapunov matrix equation*, which, in the modal subspace, can be written as:

$$\mathbf{D}_i(\alpha)\mathbf{\Sigma}_{\mathbf{z}_i}(\alpha) + \mathbf{\Sigma}_{\mathbf{z}_i}(\alpha)\mathbf{D}_i^{\mathrm{T}}(\alpha) + \mathbf{B}_{\mathbf{z}_i}(\alpha) = \mathbf{O}_{2m}, \qquad \alpha \in \alpha^I = [-\alpha, \alpha] \tag{25}$$

where $\mathbf{B}_{\mathbf{z}_i}(\alpha)$ is a symmetric matrix defined as:

$$\mathbf{B}_{\mathbf{z}_i}(\alpha) = \mathbf{v}_{0,i}\,\mathrm{E}\left\langle \ddot{u}_g(t)\Delta\mathbf{z}_i^{\mathrm{T}}(\alpha,t)\right\rangle + \mathrm{E}\left\langle \Delta\mathbf{z}_i(\alpha,t)\,\ddot{u}_g(t)\right\rangle \mathbf{v}_{0,i}^{\mathrm{T}}, \qquad \alpha \in \alpha^I = [-\alpha, \alpha] \tag{26}$$

in which $\mathrm{E}\langle \bullet \rangle$ is the mathematical expectation operator. According to Eq.(17), the matrix $\mathbf{B}_{\mathbf{z}_i}(\alpha)$ and the covariance matrix of the modal random response can be split as the sum of two aliquots: the midpoint or nominal solution and a deviation, that is:

$$\begin{aligned} &\mathbf{B}_{\mathbf{z}_i}(\alpha) = \mathbf{B}_{\mathbf{z}_{0,i}} + \Delta\mathbf{B}_i(\alpha) \\ &\mathbf{\Sigma}_{\mathbf{z}_i}(\alpha) = \mathbf{\Sigma}_{0,i} + \Delta\mathbf{\Sigma}_i(\alpha), \qquad \alpha \in \alpha^I = [-\alpha, \alpha]; \qquad i = x, y \end{aligned} \tag{27}$$

where $\mathbf{\Sigma}_{0,i}$ denotes the midpoint modal covariance matrix, while $\Delta\mathbf{\Sigma}_i(\alpha)$ is the deviation of the modal covariance matrix due to the uncertain-but-bounded parameter α. By substituting Eqs. (14) and (27) into Eq.(25) the following relationship is obtained:

$$\begin{aligned} &\mathbf{D}_0\left[\mathbf{\Sigma}_{0,i} + \Delta\mathbf{\Sigma}_i(\alpha)\right] + \left[\mathbf{\Sigma}_{0,i} + \Delta\mathbf{\Sigma}_i(\alpha)\right]\mathbf{D}_0^{\mathrm{T}} + \alpha e^I \mathbf{D}_i\left[\mathbf{\Sigma}_{0,i} + \Delta\mathbf{\Sigma}_i(\alpha)\right] \\ &\quad + \alpha e^I\left[\mathbf{\Sigma}_{0,i} + \Delta\mathbf{\Sigma}_i(\alpha)\right]\mathbf{D}_i^{\mathrm{T}} + \mathbf{B}_{\mathbf{z}_{0,i}} + \Delta\mathbf{B}_i(\alpha) = \mathbf{O}_{2m}, \qquad \alpha \in \alpha^I = [-\alpha, \alpha]; \qquad i = x, y. \end{aligned} \tag{28}$$

Collecting the terms associated with the nominal values of the bounded parameters $(\alpha = 0)$, the equations ruling the midpoint modal covariance matrix $\mathbf{\Sigma}_{0,i}$ is given as:

$$\mathbf{D}_0\mathbf{\Sigma}_{0,i} + \mathbf{\Sigma}_{0,i}\mathbf{D}_0^{\mathrm{T}} + \mathbf{B}_{\mathbf{z}_{0,i}} = \mathbf{O}_{2m}. \tag{29}$$

The equations ruling the deviation of the covariance matrix $\Delta\boldsymbol{\Sigma}_i(\alpha)$ can be derived from Eq.(28), as:

$$\begin{aligned}&\left[\mathbf{D}_0+\alpha e^I\mathbf{D}_i\right]\Delta\boldsymbol{\Sigma}_i(\alpha)+\Delta\boldsymbol{\Sigma}_i(\alpha)\left[\mathbf{D}_0^{\mathrm{T}}+\alpha e^I\mathbf{D}_i^{\mathrm{T}}\right]\\&\qquad+\alpha e^I\mathbf{D}_i\boldsymbol{\Sigma}_{0,i}+\alpha e^I\boldsymbol{\Sigma}_{0,i}\mathbf{D}_i^{\mathrm{T}}+\Delta\mathbf{B}_i(\alpha)=\mathbf{O}_{2m},\qquad i=x,y.\end{aligned}\tag{30}$$

Finally setting the variable e^I at its bounds, Eq.(30) leads to the following two sets of $2m$ algebraic equations (Muscolino and Sofi, 2012):

$$\begin{aligned}&\left[\mathbf{D}_0-\alpha\mathbf{D}_i\right]\Delta\boldsymbol{\Sigma}_i^-(\alpha)+\Delta\boldsymbol{\Sigma}_i^-(\alpha)\left[\mathbf{D}_0^{\mathrm{T}}-\alpha\mathbf{D}_i^{\mathrm{T}}\right]-\alpha\mathbf{D}_i\boldsymbol{\Sigma}_{0,i}-\alpha\boldsymbol{\Sigma}_{0,i}\mathbf{D}_i^{\mathrm{T}}-\alpha\Delta\mathbf{B}_i^-=\mathbf{O}_{2m},\\&\hspace{25em} i=x,y\\&\left[\mathbf{D}_0+\alpha\mathbf{D}_i\right]\Delta\boldsymbol{\Sigma}_i^+(\alpha)+\Delta\boldsymbol{\Sigma}_i^+(\alpha)\left[\mathbf{D}_0^{\mathrm{T}}+\alpha\mathbf{D}_i^{\mathrm{T}}\right]+\alpha\mathbf{D}_i\boldsymbol{\Sigma}_{0,i}+\alpha\boldsymbol{\Sigma}_{0,i}\mathbf{D}_i^{\mathrm{T}}+\alpha\Delta\mathbf{B}_i^+=\mathbf{O}_{2m},\end{aligned}\tag{31}$$

In the previous equations, the superscripts "$-$" and "$+$" mean that the corresponding quantity is evaluated assuming the unitary interval variable e^I equal to its lower and upper bound, respectively. Moreover, in Eq.(31), under the assumption of small deviation amplitude of the uncertain-but-bounded parameters α, the terms $\alpha\mathbf{D}_i$, in square bracket, can be reasonably neglected. Under this assumption, it is readily found that $\Delta\boldsymbol{\Sigma}_i^+(\alpha)\equiv-\Delta\boldsymbol{\Sigma}_i^-(\alpha)=\Delta\boldsymbol{\Sigma}_i(\alpha)$ and $\Delta\mathbf{B}_i^+\equiv-\Delta\mathbf{B}_i^-=\Delta\mathbf{B}_i$ so that it is sufficient to solve just one of one set of algebraic equations for each seismic input direction:

$$\mathbf{D}_0\,\Delta\boldsymbol{\Sigma}_i(\alpha)+\Delta\boldsymbol{\Sigma}_i(\alpha)\mathbf{D}_0^{\mathrm{T}}+\alpha\left(\mathbf{D}_i\boldsymbol{\Sigma}_{0,i}+\boldsymbol{\Sigma}_{0,i}\mathbf{D}_i^{\mathrm{T}}+\Delta\mathbf{B}_i\right)=\mathbf{O}_{2m},\qquad i=x,y\tag{32}$$

It has to emphasize that Eq.(32) evidences the linear dependency of the deviation matrix $\Delta\boldsymbol{\Sigma}_i(\alpha)$ on α that is $\Delta\boldsymbol{\Sigma}_i(\alpha)=\alpha\boldsymbol{\Sigma}_i$; where $\boldsymbol{\Sigma}_i$ is the solution of the following algebraic equation, directly derived from Eq. (32):

$$\mathbf{D}_0\boldsymbol{\Sigma}_i+\boldsymbol{\Sigma}_i\mathbf{D}_0^{\mathrm{T}}+\mathbf{D}_i\boldsymbol{\Sigma}_{0,i}+\boldsymbol{\Sigma}_{0,i}\mathbf{D}_i^{\mathrm{T}}+\Delta\mathbf{B}_i=\mathbf{O}_{2m},\qquad i=x,y\tag{33}$$

Once the modal covariance matrices $\boldsymbol{\Sigma}_{0,i}$ and $\boldsymbol{\Sigma}_i$ have been determined as solution of Eqs. (29) and (33), respectively, the corresponding nodal covariance matrices can be computed as follows:

$$\boldsymbol{\Sigma}_{\mathrm{n}_{0,i}}=\boldsymbol{\Pi}_0\boldsymbol{\Sigma}_{0,i}\boldsymbol{\Pi}_0^{\mathrm{T}};\quad\boldsymbol{\Sigma}_{\mathrm{n}_i}=\boldsymbol{\Pi}_0\boldsymbol{\Sigma}_i\boldsymbol{\Pi}_0^{\mathrm{T}},\qquad i=x,y;\qquad\boldsymbol{\Pi}_0=\begin{bmatrix}\boldsymbol{\Phi}_0&\mathbf{0}\\\mathbf{0}&\boldsymbol{\Phi}_0\end{bmatrix}\tag{34}$$

where the matrix $\boldsymbol{\Pi}_0$ of order $2m\times2m$ is defined as a function of the modal matrix $\boldsymbol{\Phi}_0$, solution of the eigenproblem (8).

Finally, according to interval analysis, and for the linear dependency of the deviation matrix $\Delta\boldsymbol{\Sigma}_i(\alpha)$ on α, the nodal interval variable covariance matrix, can be evaluated as:

$$\boldsymbol{\Sigma}_{\mathrm{n},i}(\alpha)=\boldsymbol{\Sigma}_{\mathrm{n}_{0,i}}+\Delta\boldsymbol{\Sigma}_{\mathrm{n},i}(\alpha)=\boldsymbol{\Sigma}_{\mathrm{n}_{0,i}}+\alpha\,e^I\boldsymbol{\Sigma}_{\mathrm{n}_i},\qquad\alpha\in\alpha^I=\left[-\alpha,\alpha\right];\quad i=x,y.\tag{35}$$

where $\Delta\boldsymbol{\Sigma}_{\mathrm{n},i}(\alpha)$ is the total deviation, in nodal or geometric space, of the covariance matrix. It follows that the interval that contains all possible values of the nodal interval covariance matrix $\boldsymbol{\Sigma}_{\mathrm{n},i}^I(\alpha)$ can be defined by the lower and upper bounds, that is:

$$\underline{\boldsymbol{\Sigma}}_{\mathrm{n}_i}=\boldsymbol{\Sigma}_{\mathrm{n}_{0,i}}-\alpha\left|\boldsymbol{\Sigma}_{\mathrm{n}_i}\right|;\quad\overline{\boldsymbol{\Sigma}}_{\mathrm{n}_i}=\boldsymbol{\Sigma}_{\mathrm{n}_{0,i}}+\alpha\left|\boldsymbol{\Sigma}_{\mathrm{n}_i}\right|\quad i=x,y.\tag{36}$$

In this equation the symbol $|\bullet|$ denotes absolute value component wise, while $\underline{\boldsymbol{\Sigma}}_{\mathrm{n}_i}$ and $\overline{\boldsymbol{\Sigma}}_{\mathrm{n}_i}$ define the lower and upper bounds of the nodal interval variable covariance matrix $\boldsymbol{\Sigma}^I_{\mathrm{n},i}(\alpha)$, respectively.
By applying the interval algebra, and according to Eq. (36), the lower and upper bounds of the variance of the $\ell-\mathrm{th}$ nodal displacements can be evaluated as:

$$\underline{\sigma}^2_{u,\ell}(\alpha)=\sigma^2_{u_0,\ell}-\alpha\,\Delta\sigma^2_{u,\ell};\qquad \overline{\sigma}^2_{u,\ell}(\alpha)=\sigma^2_{u_0,\ell}+\alpha\,\Delta\sigma^2_{u,\ell}\qquad \ell=1,2,\cdots,3n. \tag{37}$$

Furthermore, according to the philosophy of the CQC approach, the upper bound of the maximum value of the structural response, $u_{\ell,\max}$, can be considered to be proportional to the upper bound of the corresponding standard deviation, $\overline{\sigma}_{u,\ell}(\alpha)$ multiplied by the so-called peak factor, μ_ℓ (Vanmarcke, 1975, 1976):

$$u_{\ell,\max}(\alpha)=\mu_\ell\overline{\sigma}_{u,\ell}(\alpha)=\sqrt{\mu_\ell^2\sigma^2_{u_0,\ell}+\mu_\ell^2\,\alpha\,\Delta\sigma^2_{u,\ell}};\qquad \ell=1,2,\cdots,3n. \tag{38}$$

The nodal variances, according to Eq.(34), are evaluated as linear combination of modal covariances as follows:

$$\sigma^2_{u_0,\ell}=\sum_{j=1}^{m}\sum_{k=1}^{m}\phi_{0,\ell j}\,\phi_{0,\ell k}\,\sigma_{0,jk};\qquad \Delta\sigma^2_{u,\ell}=\sum_{j=1}^{m}\sum_{k=1}^{m}\phi_{0,\ell j}\,\phi_{0,\ell k}\,\sigma_{jk}=\sum_{j=1}^{m}\sum_{k=1}^{m}\phi_{0,\ell j}\,\phi_{0,\ell k}\,r_{jk}\,\sigma_{0,jj}\,\sigma_{0,kk}\;. \tag{39}$$

where r_{jk} is the following coefficient:

$$r_{jk}=\frac{\sigma_{jk}}{\sigma_{0,jj}\,\sigma_{0,kk}}. \tag{40}$$

while $\sigma^2_{0,pp}$ $(p=j,k=1,2,\cdots,m)$ and σ_{jk} are the elements of modal covariance matrices $\boldsymbol{\Sigma}_{0,i}$ and $\boldsymbol{\Sigma}_i$ $(i=x,y)$, obtained by solving Eqs.(29) and (33), respectively. Moreover, the basic assumptions of the traditional CQC are: i) the peak factor, μ_ℓ, for the nodal response of interest $u_\ell(t)$ is assumed approximately the same as the peak factors, μ_j, of all modal responses $q_{0,j}(t)$ and ii) the median of the maximum peak of the modal response is expressed, in approximate form, as a function of the mean response pseudo-acceleration spectrum $S_{\mathrm{pa}}(T_{0,j},\zeta_j)$: $\mu_j\,\sigma_{0,jj}\approx p_{0,j}\,S_{\mathrm{pa}}(T_{0,j},\zeta_j)$. Then, according to these assumptions it possible to derive the following relationships:

$$\begin{aligned}\mu_\ell^2\sigma^2_{u_0,\ell}&\approx\sum_{j=1}^{m}\sum_{k=1}^{m}\rho_{0,jk}\,\phi_{0,\ell j}\,\phi_{0,\ell k}\;p_{0,j}\;p_{0,k}\,\frac{S_{\mathrm{pa}}(T_{0,j},\zeta_j)}{\omega^2_{0,j}}\frac{S_{\mathrm{pa}}(T_{0,k},\zeta_k)}{\omega^2_{0,k}},\\ \mu_\ell^2\Delta\sigma^2_{u,\ell}&\approx\sum_{j=1}^{m}\sum_{k=1}^{m}r_{jk}\,\phi_{0,\ell j}\,\phi_{0,\ell k}\;p_{0,j}\;p_{0,k}\,\frac{S_{\mathrm{pa}}(T_{0,j},\zeta_j)}{\omega^2_{0,j}}\frac{S_{\mathrm{pa}}(T_{0,k},\zeta_k)}{\omega^2_{0,k}}.\end{aligned} \tag{41}$$

where the cross-correlation coefficient $\rho_{0,jk}$ has been introduced in Eq.(22). Finally, substituting Eqs.(41) into Eq.(38) the ICQC rule for structural systems with uncertain-but-bounded parameter is obtained:

$$u_{\ell,\max}(\alpha)=\sqrt{\left(u_{0,\ell,\max}\right)^2+\alpha\left|\sum_{j=1}^{m}\sum_{k=1}^{m} r_{jk}\,\phi_{0,\ell j}\,\phi_{0,\ell k}\,p_{0,j}\,p_{0,k}\,\frac{S_{\mathrm{pa}}\left(T_{0,j},\zeta_j\right)}{\omega_{0,j}^2}\frac{S_{\mathrm{pa}}\left(T_{0,k},\zeta_k\right)}{\omega_{0,k}^2}\right|}. \tag{42}$$

where $u_{0,\ell,\max}$ is the maximum of the response evaluated by applying the traditional CQC rule (see Eq. (21)). Under the assumption of stationary Gaussian white noise input process this assumption, closed-form of the coefficient r_{jk} can be evaluated as:

$$r_{jk}=\frac{\omega_h\sqrt{\kappa_{jj}\,\kappa_{kk}}}{4p_{0,j}\,p_{0,k}\,\kappa_{jk}\sqrt{\zeta_j\omega_j\zeta_k\omega_k}}\sum_{h=1}^{m}\left[\varepsilon_{hjk}-\delta_{hjk}+\omega_h\left(8\zeta_h\lambda_{hjk}-\gamma_{hjk}\right)\right] \tag{43}$$

where

$$\begin{aligned}
\beta_{jk}&=2\left(\omega_j^2-\omega_k^2-4\zeta_j\omega_j\left(\zeta_j\,\omega_j+\zeta_k\,\omega_k\right)\right)\\
\gamma_{hjk}&=\left(\frac{\beta_{jk}}{\kappa_{jh}}a_{kh}p_{0,j}\left(\zeta_j\,\omega_j+\zeta_h\,\omega_h\right)+\frac{\beta_{kj}}{\kappa_{kh}}a_{jh}\,p_{0,k}\left(\zeta_k\,\omega_k+\zeta_h\,\omega_h\right)\right)p_{0,h}\\
\delta_{hjk}&=2\omega_h\left(\zeta_j\,\omega_j+\zeta_k\,\omega_k\right)\left(\frac{\left(\omega_j^2-\omega_h^2\right)}{\kappa_{jh}}a_{kh}\,p_{0,j}+\frac{\left(\omega_k^2-\omega_h^2\right)}{\kappa_{kh}}a_{jh}p_{0,k}\right)p_{0,h}\\
\varepsilon_{hjk}&=\zeta_h\left(\left(\omega_h^2-\omega_j^2\right)\frac{\beta_{jk}}{\kappa_{jh}}\,a_{kh}\,p_{0,j}+\left(\omega_h^2-\omega_k^2\right)\frac{\beta_{kj}}{\kappa_{kh}}a_{jh}\,p_{0,k}\right)p_{0,h}\\
\lambda_{hjk}&=\left(\zeta_j\,\omega_j+\zeta_k\,\omega_k\right)\left(\frac{\omega_j\left(\zeta_h\omega_j+\zeta_j\omega_h\right)}{\kappa_{jh}}a_{kh}\,p_{0,j}+\frac{\omega_k\left(\zeta_h\omega_k+\zeta_k\omega_h\right)}{\kappa_{kh}}a_{jh}\,p_{0,k}\right)p_{0,h}
\end{aligned} \tag{44}$$

In these equations a_{jk} represents the j,k element of the symmetric matrix $\mathbf{A}_i\ \left(i=x,y\right)$ of order m introduced in Eq.(16), while κ_{jk} has been defined in Eq.(24) and $p_{0,j}$ is the j-th participation factor.

4. NUMERICAL EXAMPLE

With the purpose of offering a robust assessment of the proposed approach, a two-storey building depicted in Figure 4.1 is investigated. The structure has mass $m=36,000\,kg$, uniformly distributed on the floor, length of the side orthogonal to the ground motion acceleration modelled in *y*-direction $b_x=6$ m and the Young modulus is $E_c=3\times10^{10}\ N/\mathrm{m}^2$.
The maximum value of the structural response by applying the classical CQC and the proposed ICQC for the structural system with accidental eccentricity $e_x=\pm0.05b_x$ has been evaluated and compared with the mean value of the structural response evaluated by means of one thousand of Monte Carlo Simulations (MCS), whose consistency with the response spectrum is verified by applying the procedure described in Cacciola et al. (2004). In the simulations a type-A soil, which may correspond to a rock or other rock-like geological formation, according to EC8, has been considered along with a duration of the stationary part of the accelerogram, i.e. $T_{\mathrm{g}}=\ 20\,\mathrm{s}$.
Table 4.1 shows the values of the maximum displacement in *y*-direction of the first and second floor. One can note the good performance of the proposed ICQC that allows to directly evaluate, with a maximum inaccuracy of about 1%, the worst condition ($e_x=+0.05b_x$) for the structural elements with a single RSA, indeed it avoids of calculating new eigenproblems that should be solved every time that accidental eccentricity is imposed to the CM of the building.

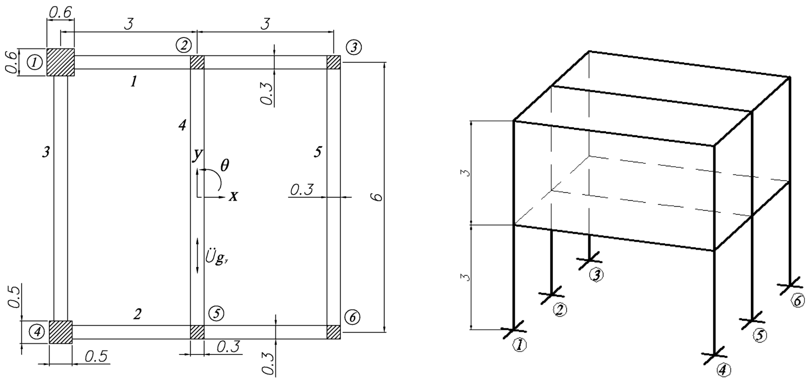

Figure 4.1 Two storeys structure: Finite element model (left-hand side); plan view (right-hand side).

Table 4.1 Maximum values of the displacements in *y*-direction of different methods of seismic analysis.

Accidental eccentricity $e_x = +0.05b_x$	MCS [m]	CQC (EC8) [m]	ICQC (Interval) [m]
$u_{y1,max}$	*0.003330*	*0.003345*	*0.003329*
$u_{y2,max}$	*0.005317*	*0.005400*	*0.005373*
Accidental eccentricity $e_x = -0.05b_x$			
$u_{y1,max}$	*0.003010*	*0.002990*	*0.003329*
$u_{y2,max}$	*0.004800*	*0.004832*	*0.005373*

Given the reduced computational effort (the formulae are known in closed form), the proposed ICQC appears to be a very effective replacement of the traditional procedure recommended by building codes, considering the effects of the accidental eccentricity in the determination of the structural response of buildings.

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