Comparison of several variants of the response spectrum method and definition of equivalent static loads from the peak response envelopes

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SUMMARY:
The response spectrum method is based on the definition of a suitable combination of the seismic peak responses of several single-degree-of-freedom oscillators representing the modes of the analyzed structure. Different variants exist, associated with different ways of accounting for the simultaneity of peak responses of physically distinct quantities. Peak simultaneity can be rigorously treated by the (hyper-ellipsoid) peak response envelopes. Firstly, a novel interpretation of these envelopes is proposed, based on the notion of "coefficients of the linear combination of modal peaks". Then, an algorithm allowing the approximation of a hyper-ellipsoid by a polyhedron is analyzed. Moreover, a procedure based on the peak response envelopes is proposed to define static force fields equivalent to the seismic action. In the last part of the article, four different methods, including the hyper-ellipsoid envelopes and the equivalent static force fields, are used to compute and compare the total reinforcement demands of a reinforced concrete building.

Keywords: Hyper-ellipsoid response envelope, polyhedral approximation of ellipsoids, equivalent static loads

1. INTRODUCTION

The response spectrum method is a popular method of seismic structural analysis, based on the assumption of linear elastic structural behavior. The seismic response is computed as a suitable combination of the extreme responses to the given earthquake of several single-degree-of-freedom oscillators, each one associated with one mode of the analyzed structure. The main difficulty in the application of this method is the definition of the combination rules among modal peak responses to get the global response of the whole structure. The well-known Complete Quadratic Combination - CQC (Der Kiureghian, 1979) defines a combination of peak modal responses accounting for the coupling effects due to close modes. Moreover, the superposition of the effects of different seismic directions is usually made by a quadratic combination rule or by the Newmark's combinations approach (Newmark, 1975). Finally, the simultaneous occurrence of peak values of physically different quantities (e.g. a normal effort and a bending moment in a beam section) was studied by Gupta and Singh (1977), Leblond (1980), Menun and Der Kiureghian (2000a,b), among others. These works contributed to the definition of the notion of *hyper-ellipsoid peak response envelope*, which is investigated here.

In the first part of the paper, a novel interpretation of the *peak response envelope* is proposed, based on the notion of "envelope of the coefficients of the linear combination of modal peaks" (Martin, 2004), named here α-envelope. The relationship between the α-envelope and the classical definition of peak response envelopes (Menun and Der Kiureghian, 2000a,b) is discussed (Section 2). A discretization procedure of a hyper-ellipsoid, based on the use of an enveloping polyhedron, will be presented in Section 3: it is an extension of the method presented by Leblond (1980). Actually, this extended discretization procedure was proposed by Vézin et al. (2007) and the present paper discusses in more detail some properties of this algorithm. Notice that other discretization methods are suggested in ASCE (2009). Then, the α-envelope is used to define several pseudo-acceleration fields, where each field defines a probable distribution of accelerations simultaneously acting on each point of a structure during the given earthquake. These accelerations fields are used to define several static load cases equivalent to the given earthquake (Section 4).

The second part of the article (Section 5) regards the application of four variants of the response spectrum method to a reinforced concrete structure: (i) Superposition of the peak modal responses in each earthquake direction using the Complete Quadratic Combination (CQC) method, followed by ”Newmark's combinations” to obtain the global response due to different earthquake directions; (ii) Superposition of the peak modal responses using the CQC method for each earthquake direction, followed by quadratic combinations to consider the effects of different earthquake directions and permutations of generalized forces signs to estimate the most critical envelope; (iii) Hyper-ellipsoid envelope of simultaneous generalized forces; (iv) Static load cases defined by using modal linear combinations of accelerations. The results are compared in terms of the total reinforcement quantity and of estimated peak responses for several shell elements of the structure.

2. A NOVEL INTERPRETATION OF PEAK RESPONSE ENVELOPES

In this Section, a new interpretation of peak response envelopes is proposed, based on linear combinations of modal response peaks. The notion of "envelope of the coefficients of the linear combination of modal peaks" is introduced and compared with the classical definitions.

2.1. Linear combinations of modal response peaks

Consider an N-degree-of-freedom linear and classically damped structure, for which N real eigenmodes can be calculated. For seismic applications, only $n \le N$ modes are usually retained, by guarantying that the sum of effective masses of the *n* modes is high enough or introducing a pseudo-mode. The seismic effects are estimated by considering three earthquakes (represented by pseudo-acceleration spectra), one per each direction ($k = x, y, z$). For an earthquake in direction k, the 3N-components displacement vector $\underline{u}_k(t)$ can be written by a linear combination of modes:

$$\underline{u}_k(t) = \left[u^1_{k,x}, u^2_{k,x}, \ldots, u^N_{k,x}, u^1_{k,y}, u^2_{k,y}, \ldots, u^N_{k,y}, u^1_{k,z}, u^2_{k,z}, \ldots, u^N_{k,z}\right]^T = \sum_i r_{i,k}(t)\, p_{i,k}\, \underline{\phi}_i \tag{2.1}$$

where $\underline{\phi}_i$ is the 3N-components eigenvector for mode *i*, $p_{i,k}$ is the participation factor for mode *i* and the earthquake direction *k* and $u^l_{k,x(y,z)}$ is the displacement in the direction *x(y,z)* of the node *l* due to the earthquake in the direction *k*. Each term $\underline{u}_{i,k}(t) = r_{i,k}(t)\, p_{i,k}\, \underline{\phi}_i$ represents a structural displacement proportional the *i-th* mode shape. The time-function $r_{i,k}(t)$ is the solution of dynamics equation of the single-degree-of-freedom oscillator representing the mode *i*, under the given accelerogram $\ddot{u}_g(t)$:

$$\ddot{r}_{i,k}(t) + 2\xi_i \omega_i \dot{r}_{i,k}(t) + \omega_i^2 r_{i,k}(t) = -\ddot{u}_g(t) \tag{2.2}$$

where ω_i and ξ_i are the pulsation and the damping coefficient of the mode *i*, respectively. When (ω_i, ξ_i) are known for a given mode, Eqn. 2.2 can be solved and the maximum absolute displacement can be computed:

$$R_{i,k} = \max_t \left|r_{i,k}(t)\right| = S_{d,k}(\omega_i, \xi_i) \tag{2.3}$$

Observe that the use of Eqn. 2.3, instead of $\tilde{R}_{i,k} = \max_t \left[r_{i,k}(t)\right]$, leads to slightly conservative results (Menun and Der Kiureghian, 2000a). The pseudo-acceleration becomes $S_{a,k}(\omega_i, \xi_i) = \omega_i^2 S_{d,k}(\omega_i, \xi_i)$.

Using Eqns. 2.1 and 2.3, the displacement vector $\underline{u}_k(t)$ can be rewritten as a linear combination of the modal peak displacement vectors $\underline{U}_{i,k} = R_{i,k}\, p_{i,k}\, \underline{\phi}_i$:

$$\underline{u}_k(t) = \sum_i \alpha_{i,k}(t) R_{i,k}\, p_{i,k}\, \underline{\phi}_i = \sum_i \alpha_{i,k}(t) \underline{U}_{i,k} \qquad \text{with} \qquad -1 \le \alpha_{i,k}(t) = \frac{r_{i,k}(t)}{R_{i,k}} \le 1 \tag{2.4a}$$

where $\alpha_{i,k}(t)$ are named here “coefficients of the linear combination of modal peaks”. The total displacement due to the earthquakes in the three directions reads:

$$\underline{u}(t) = \sum_k \underline{u}_k(t) = \sum_k \sum_i \alpha_{i,k}(t) \underline{U}_{i,k} \tag{2.4b}$$

Let us consider a (generalized) force $f(t)$, e.g. an axial force, a bending moment, etc. in a section or element of the structure. By virtue of structural linearity, one can always find a vector $\underline{d}$ such that:

$$f(t) = \underline{d}^T \underline{u}(t) = \sum_k f_k(t) = \sum_k \sum_i \alpha_{i,k}(t) \underline{d}^T \underline{U}_{i,k} = \sum_k \sum_i \alpha_{i,k}(t) F_{i,k} \tag{2.5}$$

where $F_{i,k} = \underline{d}^T \underline{U}_{i,k}$ is the value of $f(t)$ corresponding to the peak displacement vector $\underline{U}_{i,k}$ for the mode *i* and the direction *k*. Eqns. 2.4a,b show that at any time *t*, the displacement of all points (nodes) of a structure can be determined by a linear combination of the modal peak displacement vectors $\underline{U}_{i,k}$. Likewise, Eqn. 2.5 shows that a (generalized) force $f(t)$ can be determined by a linear combination of $F_{i,k}$ using the same coefficients $\alpha_{i,k}(t)$.

2.2. Hyper-ellipsoid envelope of linear combinations coefficients (α-ellipsoid)

In order to estimate the probable maximum value of the (generalized) force $f_k(t)$ due to an earthquake in the direction *k*, one can use the Complete Quadratic Combination (Der Kiureghian, 1979):

$$f_{k,\max} = \sqrt{\sum_{ij} \rho_{ij} F_{i,k} F_{j,k}} \tag{2.6}$$

where ρ_{ij} is the modal cross-correlation coefficient between the modes *i* and *j* that can be calculated as follows (Der Kiureghian, 1979):

$$\rho_{ij} = \frac{8\sqrt{\xi_i \xi_j \omega_i \omega_j}\,(\omega_i \xi_i + \omega_j \xi_j)\,\omega_i \omega_j}{(\omega_i^2 - \omega_j^2)^2 + 4\xi_i \xi_j \omega_i \omega_j (\omega_i^2 + \omega_j^2) + 4\omega_i^2 \omega_j^2 (\xi_i^2 + \xi_j^2)}$$

Thus, in the sense of probability:

$$f_k(t) = \sum_i \alpha_{i,k}(t) F_{i,k} \le f_{k,\max} \qquad \text{or} \qquad \underline{\alpha}_k^T(t) \underline{F}_k \le \sqrt{\underline{F}_k^T \underline{\underline{H}}\, \underline{F}_k} \tag{2.7a \& 2.7b}$$

where $\underline{\alpha}_k = [\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{n,k}]^T$ is the vector of the (time-dependent) combination coefficients, $\underline{F}_k = [F_{1,k}, F_{2,k}, \ldots, F_{n,k}]^T$ and $\underline{\underline{H}} = [\rho_{ij}]$. The condition in Eqns. 2.7 can be extended to the case of three seismic directions using a quadratic combination of $f_{k,\max}$:

$$f(t)=\sum_k f_k(t)=\sum_k\sum_i \alpha_{i,k}(t)F_{i,k} \le f_{\max}=\sqrt{\sum_k f_{k,\max}^2}=\sqrt{\sum_k\sum_{ij}\rho_{ij}F_{i,k}F_{j,k}} \qquad (2.8a)$$

or

$$\underline{\alpha}(t)^T\underline{F} \le \sqrt{\underline{F}^T\underline{\underline{\tilde{H}}}\underline{F}}\text{, with } \underline{\alpha}=\left[\underline{\alpha}_x^T,\underline{\alpha}_y^T,\underline{\alpha}_z^T\right]^T,\ \underline{F}=\left[\underline{F}_x^T,\underline{F}_y^T,\underline{F}_z^T\right]^T \text{ and } \underline{\underline{\tilde{H}}}=\begin{bmatrix}\underline{\underline{H}} & \underline{\underline{0}} & \underline{\underline{0}}\\ \underline{\underline{0}} & \underline{\underline{H}} & \underline{\underline{0}}\\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{H}}\end{bmatrix} \qquad (2.8b)$$

From Eqns. 2.7b and 2.8b, supposing that the matrix $\underline{\underline{H}}$ is invertible, one can prove that:

$$\underline{\alpha}_k^T\underline{\underline{H}}^{-1}\underline{\alpha}_k \le 1 \quad\text{and}\quad \underline{\alpha}^T\underline{\underline{\tilde{H}}}^{-1}\underline{\alpha} \le 1 \qquad (2.9a\ \&\ 2.9b)$$

Eqns. 2.9a,b give the definition of two hyper-ellipsoids with dimensions n and 3n. We name them *α_k-ellipsoid* and *α-ellipsoid*, respectively. Each point $\underline{\alpha}$ belonging to the *α-ellipsoid* represents a probable linear combination of modal response peaks. As a consequence, the set of points inside the *α-ellipsoid* defines all the probable configurations of the structure during the seismic event. The previous formulas can be easily modified when a “pseudo-mode” is considered (Vézin et al, 2007).

2.3. Hyper-ellipsoid envelope of generalized forces (F-ellipsoid)

In a given part of a structure (e.g. a beam section S), different generalized forces may act simultaneously (e.g. the normal force N and the bending moment M). In this paragraph, the problem of the simultaneity of the peaks of these different forces is dealt with. In detail, we are interested in the definition of the “envelope” of all probable simultaneous generalized forces (e.g. all the probable couples (N,M) in a section S) due to a given earthquake. Recall that the expressions given in the previous Section only concern the case of a single (generalized) force. Let $\underline{x}_k(t)=\left[f_{1,k}(t), f_{2,k}(t),..., f_{p,k}(t)\right]^T$ be a vector of *p* simultaneous (generalized) forces due to an earthquake in direction k, and let $\underline{\underline{R}}_k=\left[\underline{F}_{1,k},\underline{F}_{2,k},...\underline{F}_{p,k}\right]$ be a matrix whose columns $\underline{F}_{m,k}=\left[F_{1,m,k},F_{2,m,k},...,F_{n,m,k}\right]^T$ are the vectors of peak modal values of the force $f_{m,k}(t)$. By virtue of linearity, one can prove that:

$$\underline{x}_k^T\underline{\underline{X}}_k^{-1}\underline{x}_k=\underline{x}_k^T(\underline{\underline{R}}_k^T\underline{\underline{H}}\underline{\underline{R}}_k)^{-1}\underline{x}_k=\underline{\alpha}_k^T\underline{\underline{H}}^{-1}\underline{\alpha}_k \le 1 \qquad (2.10a)$$

where $\underline{\underline{X}}_k=\underline{\underline{R}}_k^T\underline{\underline{H}}\underline{\underline{R}}_k$ and the inequality follows from Eqn. 2.9a. In the case of three seismic directions, one can also prove that:

$$\underline{x}^T\underline{\underline{X}}^{-1}\underline{x}=\underline{\alpha}^T\underline{\underline{\tilde{H}}}^{-1}\underline{\alpha} \le 1 \qquad (2.10b)$$

where $\underline{\underline{X}}=\sum_k\underline{\underline{X}}_k$ and $\underline{x}=\sum_k\underline{x}_k$. Eqns. 2.10a,b, considered as identities, define two hyper-ellipsoids of dimension *p*, that we name *f_k-ellipsoid* and *f-ellipsoid*, respectively. Each point of the *f-ellipsoid* corresponds to a probable combination of *p* simultaneous generalized forces $f_1(t), f_2(t),..., f_p(t)$. Eqn. 2.10b implies that a point $\underline{x}$ of the *f-ellipsoid* corresponds to one and only one point $\underline{\alpha}$ of the *α-ellipsoid* (Eqn. 2.9b).

3. POLYHEDRAL APPROXIMATION OF HYPER-ELLIPSOID ENVELOPES

For practical application purposes, the number of combinations of simultaneous generalized forces (in other words, the number of chosen points on the *f-ellipsoid* surface) should be relatively small. The approach proposed by Leblond (1980) for cases *p=2* and *p=3* consists in replacing the hyper-ellipsoid by a polyhedron enveloping the hyper-ellipsoid. Other discretization approaches are proposed in ASCE (2009).The extension of Leblond's approach to the general case with $p>3$ dimensions can be made according to the following five-step procedure:

Step 1: Diagonalize the matrix $\underline{\underline{X}}$ defining the *f-ellipsoid*, i.e. find the diagonal matrix $\underline{\underline{Y}} = diag(\lambda_1, \lambda_2, ..., \lambda_p)$ and the matrix $\underline{\underline{D}}$ such that $\underline{\underline{X}} = \underline{\underline{D}}\,\underline{\underline{Y}}\,\underline{\underline{D}}^T$. Notice that $\lambda_1, \lambda_2, ..., \lambda_p$ are the eigenvalues of $\underline{\underline{X}}$.

Step 2: Transform the *f-ellipsoid* into a hyper-sphere with unit radius by the affinity $u : \Re^p \mapsto \Re^p$ such that $u : \underline{S} \mapsto \underline{V} = \underline{\underline{Y}}^{-1/2} . \underline{S}$

Step 3: Define a polyhedron having $p \times 2^p$ points enveloping a unit hyper-sphere:

$$\underline{V}_j^{(i)} = [\pm a, \pm a, ..., \pm 1, ..., \pm a]^T, \text{ with } j = 1, ..., 2^p, \quad i = 1, ..., p \quad \text{and} \quad a = \sqrt{2} - 1 \approx 0.42$$

(the component of $\underline{V}_j^{(i)}$ equal to plus or minus 1 is the *i-th*)

Step 4: Transform the polyhedron enveloping the hyper-sphere (Step 3) into a polyhedron enveloping the hyper-ellipsoid in the diagonalization reference:

$$\underline{S}_j^{(i)} = \underline{\underline{Y}}^{1/2} . \underline{V}_j^{(i)} = \left[\pm a\sqrt{\lambda_1}, \pm a\sqrt{\lambda_2}, ..., \pm\sqrt{\lambda_i}, ..., \pm a\sqrt{\lambda_p}\right]^T$$

Step 5: Transform the polyhedron defined at Step 4 into a polyhedron enveloping the hyper-ellipsoid in the original reference: $\underline{x}_j^{(i)} = \underline{\underline{D}}^T \underline{S}_j^{(i)}$.

Actually, this procedure is equal to the one proposed by Leblond (1980) for the case *p=3*. However, in the case *p>3*, it is necessary to prove that no intersection occurs between the unit hyper-sphere and the polyhedron defined at above step 3. This corresponds to prove that each one of the 2^p hyper-planes defined by the p points $\underline{V}_j^{(1)}, \underline{V}_j^{(2)}, ..., \underline{V}_j^{(p)}$ ($j=1,\ldots,2^p$) has a distance from the reference point $\underline{O}$ greater than 1. For the *j-th* hyper-plane, this distance can be calculated as follows:

$$d\left(\underline{O}, \underline{V}_j^{(1)}, \underline{V}_j^{(2)} ..., \underline{V}_j^{(p)}\right) = \frac{\det\left[\underline{V}_j^{(1)}, \left(\underline{V}_j^{(2)} - \underline{V}_j^{(1)}\right), \left(\underline{V}_j^{(3)} - \underline{V}_j^{(1)}\right), ..., \left(\underline{V}_j^{(p)} - \underline{V}_j^{(1)}\right)\right]}{\left\|\left(\underline{V}_j^{(2)} - \underline{V}_j^{(1)}\right) \wedge \left(\underline{V}_j^{(3)} - \underline{V}_j^{(1)}\right) \wedge ... \wedge \left(\underline{V}_j^{(p)} - \underline{V}_j^{(1)}\right)\right\|}$$

Table 1. Distance from the reference point to the hyper-plane defined by the points $\underline{V}_j^{(1)}, \underline{V}_j^{(2)}, ..., \underline{V}_j^{(p)}$

p	2	3	4	5	6	7	8
$d\left(\underline{O}, \underline{V}_j^{(1)}, \underline{V}_j^{(2)} ..., \underline{V}_j^{(p)}\right)$	1	1.06	1.12	1.19	1.25	1.32	1.38

The distances are greater than 1. This means that the polyhedron is always larger than the unit sphere. Hence, this algorithm is conservative. The margin becomes larger when the dimension p increases.

4. EQUIVALENT STATIC LOAD CASES

For some applications, it may be useful to represent the seismic action on a structure by one (or several) equivalent static load(s), usually defined at each structural node as the product between the nodal mass and suitable nodal acceleration(s). In this Section, a procedure is proposed to define such acceleration fields using the *α-ellipsoid* and a particular case of *f-ellipsoid*. From Eqn. 2.4a and Eqn. 2.4b, one can define the displacement $u_x^l(t)$, the pseudo-acceleration $a_x^l(t)$ and the inertia force $p_x^l(t)$ in the direction x associated with a node l of the structure:

$$u_x^l(t) = \sum_k \sum_i \alpha_{i,k}(t) U_{i,k,x}^l \tag{2.11a}$$

$$a_x^l(t) = \sum_k \sum_i \alpha_{i,k}(t) \omega_i^2 U_{i,k,x}^l = \sum_k \sum_i \alpha_{i,k}(t) A_{i,k,x}^l \tag{2.11b}$$

$$p_x^l(t) = \sum_k \sum_i \alpha_{i,k}(t) m^l \omega_i^2 U_{i,k,x}^l = \sum_k \sum_i \alpha_{i,k}(t) P_{i,k,x}^l \tag{2.11c}$$

Analogous expressions can be written for directions y and z, leading to the following nodal force field at the generic time *t*:

$$\underline{p}(t) = \sum_k \sum_i \alpha_{i,k}(t) \underline{P}_{i,k} \tag{2.12}$$

where $\underline{P}_{i,k} = \left[P_{i,k,x}^1, P_{i,k,x}^2, ..., P_{i,k,x}^N, P_{i,k,y}^1, P_{i,k,y}^2, ..., P_{i,k,y}^N, P_{i,k,z}^1, P_{i,k,z}^2, ..., P_{i,k,z}^N\right]^T$ is the vector of the modal peak forces. Usually, the combination coefficients are not known and the problem to hand is the definition of a superposition rule of the modal peak forces $P_{i,k,k}^l$ in each direction. A possible procedure is the use of the Complete Quadratic Combination. Thus, for each earthquake direction k, there is a force field $\underline{p}_k = \left[p_{k,\max}^1, p_{k,\max}^2, ..., p_{k,\max}^N\right]^T$, with $p_{k,\max}^l = \sqrt{\sum_{ij} \rho_{ij} P_{i,k,k}^l P_{j,k,k}^l}$, to be applied to the structure. The Newmark's rule can be used to combine the force fields associated with the three earthquake directions.

A novel procedure to define a static load for seismic analysis is proposed hereinafter, using the notion of modal linear combinations and *α-ellipsoid*. First, observe that at a given time *t*, the vector of forces $\underline{p}(t)$ in Eqn. 2.12 depends on the vector $\underline{\alpha}(t)$ and corresponds to one static load case. Moreover, it is proven in Section 2.2 that the *locus* of probable values of the combination coefficients $\underline{\alpha}(t)$ is the *α-ellipsoid* defined by Eqn. 2.9b. As an equality, for n modes, the *α-ellipsoid* belongs to the space of dimension 3n. Its polyhedral envelope would have $3n \times 2^{3n}$ points. This number is too large for practical calculations, especially for multi-modal structures, whose number n of significant modes can be very important. Actually, instead of finding all the points $\underline{\alpha}$ approximating the *α-ellipsoid*, a preliminary selection of the *most important ones* (according to some engineering criteria) could be performed. For instance, it is possible to look for the 6 points $\underline{\alpha}$ belonging to the α-ellipsoid *and* maximizing the total shear seismic forces $F_x(t), F_y(t), F_z(t)$ and moments $M_{xx}(t), M_{yy}(t), M_{zz}(t)$ at the base of the building (or at another given level of the structure). However, these six cases do not account for coupling effects between these six generalized forces. Actually, *a complete description of probable seismic forces at the base of the building is provided by the corresponding 6D hyper-ellipsoid (named here T-ellipsoid)*: each point $\underline{T} = \left[F_x, F_y, F_z, M_{xx}, M_{yy}, M_{zz}\right]^T$ of this T-ellipsoid represents one probable combination of the total forces and moments at the base. Hence, it is proposed to look for the points $\underline{\alpha}$ fulfilling Eqn. 2.9b *and* such that the corresponding vector of total forces and moments at the base belongs to T-ellipsoid. In practice, the T-ellipsoid can be approximated by a 6D-polyhedron with 384 vertices and the number of $\underline{\alpha}$ points to compute is 384. This is made by the analytical procedure proposed hereafter. First, observe that a point $\underline{T}$ of the T-ellipsoid can be written as a function of $\underline{\alpha}$:

$$\underline{T} = \underline{T}(\underline{\alpha}) = \left[\underline{c}_x^T \underline{\alpha}, \underline{c}_y^T \underline{\alpha}, \underline{c}_z^T \underline{\alpha}, \underline{c}_{xx}^T \underline{\alpha}, \underline{c}_{yy}^T \underline{\alpha}, \underline{c}_{zz}^T \underline{\alpha}\right]^T \tag{2.13}$$

For instance, the components of $\underline{c}_x = \left[c_{1,x,x}, c_{2,x,x}, ..., c_{n,x,x}, c_{1,y,x}, c_{2,y,x}, ..., c_{n,y,x}, c_{1,z,x}, c_{2,z,x}, ..., c_{n,z,x}\right]^T$, which has dimension 3n, are equal to $c_{i,k,x} = \sum_l P^l_{i,k,x}$. The proof is straightforward and is omitted for brevity. This means that $c_{i,k,x}$ is the total horizontal force (for all the nodes, indicated by the index *l*) in the *x* direction, for the mode *i* and due to an earthquake in the direction k.

Let us consider now one of the 384 known vertices of the polyhedron enveloping the T-ellipsoid. We name this vertex $\underline{A} = [a_x, a_y, a_z, a_{xx}, a_{yy}, a_{zz}]^T$. Then, it is possible to prove that the point $\underline{B} = \underline{A} \big/ \sqrt{\underline{A}^T \underline{\underline{X}}^{-1} \underline{A}} = [b_x, b_y, b_z, b_{xx}, b_{yy}, b_{zz}]^T$ lies on the surface of the T-ellipsoid. Moreover, $\underline{B}$ is the intersection between the T-ellipsoid and the segment linking the origin and the point $\underline{A}$. As a consequence, the problem that we have to solve can be written as follows:

$$\text{Find } \underline{\alpha} \text{ such that } \quad \underline{\alpha}^T \underline{\underline{\tilde{H}}}^{-1} \underline{\alpha} = 1 \quad \text{and} \quad \underline{T} = \underline{T}(\underline{\alpha}) = \underline{B} \tag{2.14}$$

This means that $\underline{\alpha}$ must be a point of the α-ellipsoid (of dimension 3n) and $\underline{T}(\underline{\alpha})$ must be a point of the 6D hyper-ellipsoid of the total forces and moments (T-ellipsoid). The solution $\underline{\alpha}$ of the problem (2.14) is also the solution of the following optimization problem:

$$\text{Find } \underline{\alpha} \text{ such that } \quad \underline{\alpha} = ARG\left(\max_{\forall \underline{\alpha} : \underline{\alpha}^T \underline{\underline{\tilde{H}}}^{-1} \underline{\alpha} = 1} \left[(\underline{T}(\underline{\alpha}))^T \underline{\underline{X}}^{-1} \underline{B} \right] \right) \tag{2.15}$$

According to the Lagrange multiplier method, Eqn. 2.15 is equivalent to:

$$\text{Find } \underline{\alpha} \text{ such that } \quad \underline{\alpha} = ARG\left(\max_{\forall \underline{\alpha}} \left[(\underline{T}(\underline{\alpha}))^T \underline{\underline{X}}^{-1} B + \lambda \left(\underline{\alpha}^T \underline{\underline{\tilde{H}}}^{-1} \underline{\alpha} - 1 \right) \right] \right)$$

Pose $\underline{c} = [\underline{c}_x, \underline{c}_y, \underline{c}_z, \underline{c}_{xx}, \underline{c}_{yy}, \underline{c}_{zz}] \underline{\underline{X}}^{-1} \underline{B}$. This leads to

$$\lambda = \pm \frac{1}{2} \sqrt{\underline{c}^T \underline{\underline{H}} \underline{c}} \quad \text{and} \quad \underline{\alpha} = \pm \frac{\underline{\underline{H}} \underline{c}}{\sqrt{\underline{c}^T \underline{\underline{H}} \underline{c}}} \tag{2.16a \& 2.16b}$$

The vector $\underline{\alpha}_A$ corresponding to the point $\underline{A}$ on the polyhedron reads:

$$\underline{\alpha}_A = \underline{\alpha} \sqrt{\underline{A}^T \underline{\underline{X}}^{-1} \underline{A}} \tag{2.17}$$

Each vector $\underline{\alpha}_A$ can be introduced into Eqn. 2.12 in order to define a static load field to be applied to structural nodes. Actually, there are 384 points $\underline{A}$, thus 384 vectors $\underline{\alpha}_A$ and 384 static load cases. All these load cases reproduce some probable combinations of the three total seismic forces and moments at the building basis.

5. APPLICATION

In this part, seismic effects to a building in reinforced concrete will be studied using the following four approaches:

1. Complete Quadratic Combinations of the modal responses in one direction and Quadratic Combination of three directions (CQC-Quadratic Combination),
2. Complete Quadratic Combinations of the modal responses in one direction and Newmark's Combinations of three directions (CQC-Newmark's Combinations),
3. Hyper-ellipsoid envelope of simultaneous shell (or beam) efforts in each element of the model, i.e. $\underline{x}_k = [N_{xx,k}, N_{yy,k}, N_{xy,k}, M_{xx,k}, M_{yy,k}, M_{xy,k}]^T$ according to the notation of Sect. 2.3, where (N_{xx}, N_{yy}, N_{xy}) are membrane efforts and (M_{xx}, M_{yy}, M_{xy}) are bending moments,

4. Static load cases using modal linear combinations and considering 384 probable combinations of three total forces and three total moments at the base of the structure (see Section 4).

Observe that in the 1st and 2nd procedures, the simultaneity of the generalized forces is an approximated way. The 3rd procedure properly takes into accounts simultaneity of generalized forces of each element of the structure. The 4th procedure is expected to consider a large number of probable states of the structure affecting the reinforcement demand. A comparison of these four approaches in terms of total amount of reinforcement demand will be carried out.

5.1. Structure description and modal analysis

Let us consider a reinforced concrete building with the following dimensions: width 16.5m, length 27.5m, height 31.94m (Fig. 1a). The finite element software used for the structural analysis is HERCULE. The number of nodes and (shell or beam) elements is 14400 and 16900, respectively. The soil under the foundation raft is modeled by a set of vertical and horizontal linear elastic springs. After the modal analysis (Table 2), 35 modes plus the pseudo-mode are retained (n=36). A spectrum analysis is then carried out using the spectrum of Fig. 1b. For the earthquake in vertical direction, the spectrum ordinate is reduced by a factor equal to 2/3. The load cases used in this example include the permanent load (G) and the seismic load due to earthquakes in directions x, y and z.

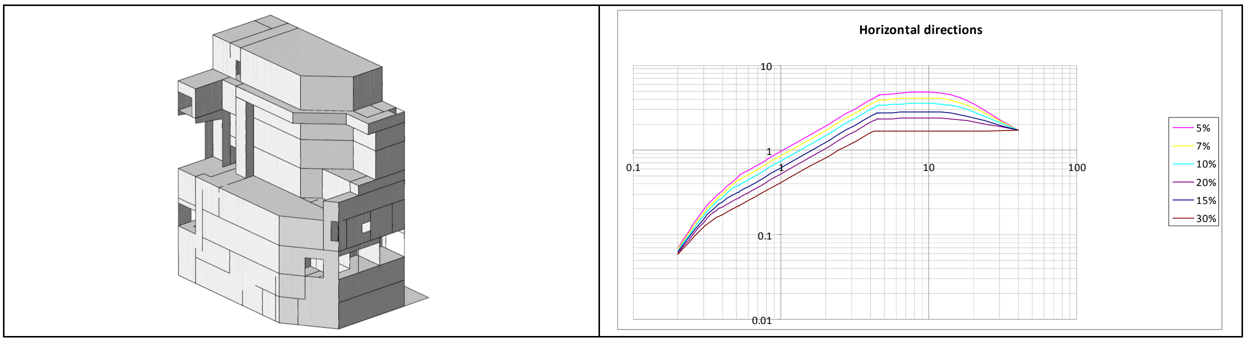


Figure 1. (a) Finite element model. (b) Pseudo-acceleration spectrum in horizontal directions (acceleration (m/s^2) vs. frequency (Hz))

Table 2. Important retained modes

N° mode	Frequency (Hz)	Period (s)	Damping coefficient (%)	Percentage of effective mass (%)		
				X	Y	Z
1	4.41	0.2268	6.6	55.9	0.4	0
2	6.18	0.1618	6.7	0.2	59.7	0
3	9.35	0.107	6.9	1.6	0.3	0.3
4	11.39	0.0878	7.2	19.1	0.2	2.9
5	13.56	0.0738	9.2	1.1	0.2	35.1
6	16.14	0.062	7.4	0.5	17.1	1.4
7	16.54	0.0605	10.2	0.1	1.8	34.1
…	…	…	…	…	…	…
35	39.18	0.0255	7.1	0.2	0.3	1.5
Total of considered modes				82.4	84.4	82.7

5.2. Comparison of four approaches of seismic effort calculation

Once the efforts are known for each element of the model, the reinforcement can be determined. Table 3 gives the ratios between the total reinforcement found by the four approaches, considering the

“CQC-Newmark’s Combinations” as reference method. One can observe that the result obtained using the hyper-ellipsoid is very close to the reference one. The reinforcement amount obtained by the static load cases (T-ellipsoid, Section 4) is more important. The difference between this approach and the hyper-ellipsoid envelope (third case in Table 3) can be explained by the fact that the 6-dimension T-ellipsoid is discretized by a polyhedral envelope which is larger than the original T-ellipsoid. As expected, the “CQC-Quadratic Combination” gives the maximum reinforcement demand.

Table 3. Comparison of the four methods of seismic effort calculation in terms of reinforcement quantity

	CQC – Newmark combinations	CQC – quadratic combination	Hyper-ellipsoid envelope	Equivalent static load case
Total reinforcement ratio	1	1.50	0.99	1.14

Hereinafter, the difference between the four approaches will be illustrated by plotting the points representing the combinations of the six efforts $N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}$ in two shell elements of the structure. A 6D space should be considered. For the finite elements indicated in Fig. 2, the projections of these efforts in the planes N_{xx}, N_{yy} and N_{xx}, M_{yy} are shown in Figs. 3 and 4.

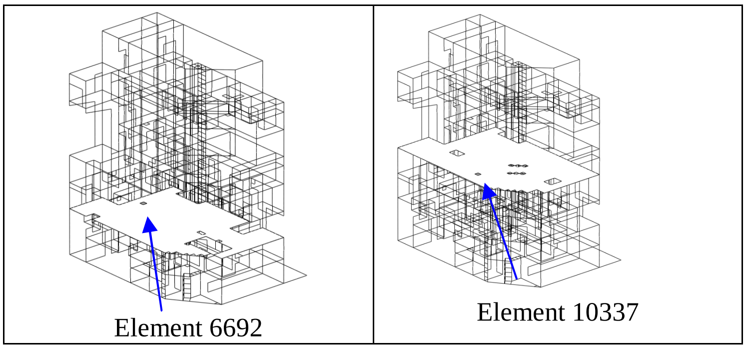


Figure 2. Elements considered

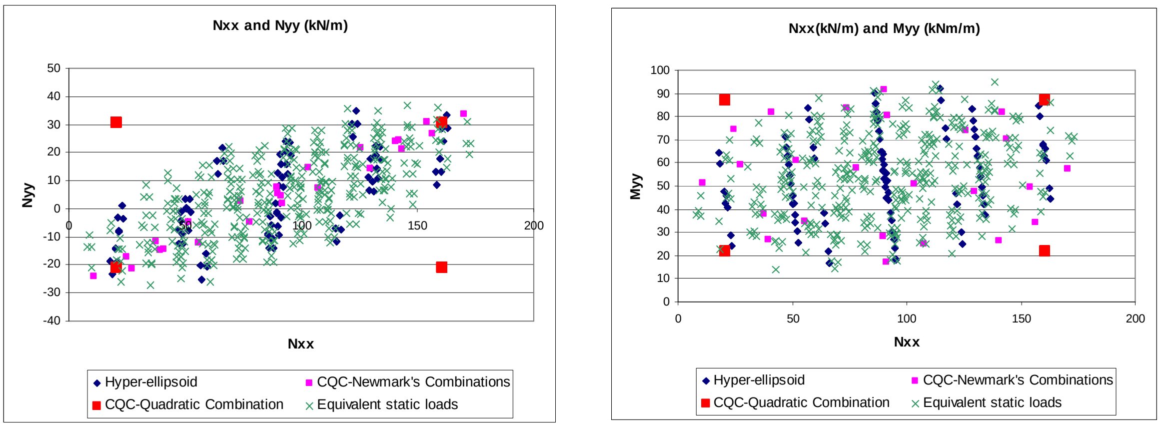
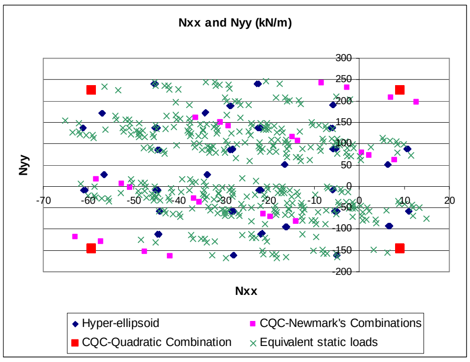


Figure 3. Distribution of forces – element 6692

Figs. 3 and 4 show that (i) there are points obtained by the “equivalent static load” procedure of Section 4 which “envelope” both the points of the hyper-ellipsoid envelope of shell efforts and the CQC-Newmark’s points. That explains why the reinforcement demand found by the equivalent static load approach is more important than ones found by the approaches 2 and 3; (ii) the reinforcement quantity obtained by the “CQC-Quadratic Combinations” is the most important. The efforts are strongly overestimated especially when an important correlation between shell efforts exists; (iii) it is possible to reduce the reinforcement demand found by the equivalent static load approach by a different definition of the 6-dimension T-ellipsoid envelope. Work is in progress on this subject.



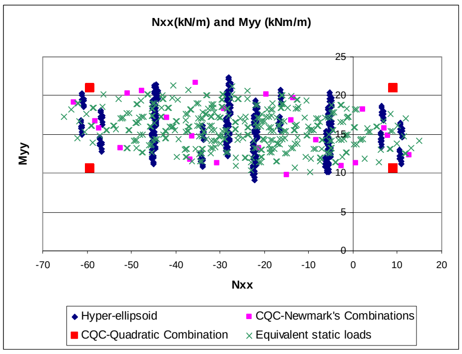


Figure 4. Distribution of forces – element 10337

6. CONCLUSION

In the first part of the paper, a novel interpretation of the so-called (elliptical) peak response envelopes (Menun and Der Kiureghian, 2000a) has been developed, using the notion of "envelope of the coefficients of the linear combination of modal peaks" (Martin, 2004). Moreover, the properties of an algorithm for the discretization of a peak response envelope with a generic dimension have been discussed. Finally, using the peak response envelope approach, a procedure has been proposed to define several static load cases equivalent to the seismic action. In the second part of the paper, the reinforcement demand for a reinforced concrete building is computed using four different procedures based on the modal spectrum method. The first two approaches, the "CQC-Quadratic Combination" and the "CQC-Newmark's Combinations", are classical methods of combining the peak modal responses coming from the response spectrum method. The third procedure is the peak response envelope method (Menun and Der Kiureghian, 2000a) applied on the membrane and bending efforts of each shell element of the finite element model. The last procedure is based on the equivalent static load fields defined according the new method proposed in the paper. As expected, the approach called "CQC-Quadratic Combination" gives the largest reinforcement demand. For the example considered here, the classical "CQC-Newmark's Combinations" and the peak response envelope method give almost the same total reinforcement demand. The equivalent static load fields procedure leads to a reinforcement demand higher than the "CQC-Newmark's Combinations". Work is in progress to improve the definition of the discretization algorithm used to approximate the hyper-ellipsoids.

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