# Evaluation of Time Delay Margin for Added Damping of SDOF Systems in Real-Time Dynamic Hybrid Testing (RTDHT) under Seismic Excitation

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**SUMMARY:**
Time delay is known to be an important issue in structural control systems which aim at better dynamic performance of structures during an earthquake. Time delay, which adversely affects the stability and performance of the actively controlled structures, builds up in the system mainly due to sensors, and actuation and communication delays. This paper presents a practical, simple and exact method to compute the delay margin of SDOF systems in Real-Time Dynamic Hybrid Testing (RTDHT) under seismic excitation. Firstly, the transcendental characteristic equation of delayed system is converted into a polynomial without the transcendentality such that its real roots coincide with the imaginary roots of the characteristic equation exactly. A simple criterion to determine the delay-dependency of the system stability is developed using the polynomial without the transcendentality. Afterwards, an expression in terms of system parameters such as mass, *m*, damping, *c* and stiffness, *k* is derived for computing the delay margin. Moreover, the variation pattern of delay margin with respect to these parameters will also be investigated theoretically in order to identify key parameters for stability. Finally, the theoretical delay margin results are verified by using the time domain simulation capabilities of MATLAB program. The outcome of this study will help us to simulate and anticipate the seismic behavior of SDOF systems in real-time dynamic hybrid testing (RTDHT) with time delay better during an earthquake.

*Keywords: Time delay, SDOF systems in real-time dynamic hybrid testing, Delay-dependent Stability*

## 1. INTRODUCTION

The real-time dynamic hybrid testing (RTDHT), which was developed recently [1-4], provides a structural system composed of an experimental substructure and a numerical substructure. Two substructures are coupled such that the target forces, accelerations or displacements are applied to the experimental substructure, and the measured reactions are provided to the numerical substructure in return. Accordingly, seismic response of the entire structure can be evaluated.

In this study, a linear SDOF system, having mass, damping and spring coefficients of *m*, *c* and *k*, respectively, is considered. The SDOF system, which is shown in Figure 1.1, is composed of an experimental substructure and a numerical substructure. The subscripts *e* and *c* will be used in order to define the experimental and the numerical substructures, respectively.

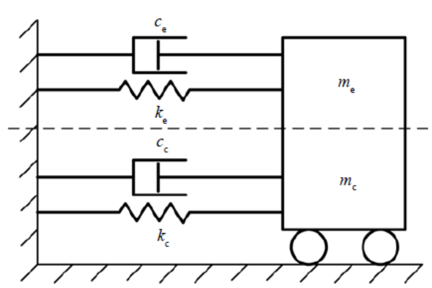


**Figure 1.1.** Schematic representation of the substructured SDOF system model

Mass, damping and spring coefficients are defined as:

$$m_e + m_c = m \, , c_e + c_c = c \, , k_e + k_c = k \tag{1.1}$$

Mass, damping and spring ratios are given as:

$$\mu_m = \frac{m_e}{m_c} \, , \mu_c = \frac{c_e}{c_c} \, , \; \mu_k = \frac{k_e}{k_c} \tag{1.2}$$

Mass, damping and spring proportion factors are given as:

$$\rho_m = \frac{m_e}{m} = \frac{\mu_m}{1+\mu_m} \, , \rho_c = \frac{c_e}{c} = \frac{\mu_c}{1+\mu_c} \, , \; \rho_k = \frac{k_e}{k} = \frac{\mu_k}{1+\mu_k} \tag{1.3}$$

Therefore, the experimental and numerical parts of mass, damping and spring coefficients are:

$$m_c = \frac{m}{1+\mu_m} \quad m_e = \frac{\mu_m m}{1+\mu_m} \tag{1.4a}$$

$$c_c = \frac{c}{1+\mu_c} \quad c_e = \frac{\mu_c c}{1+\mu_c} \tag{1.4b}$$

$$k_c = \frac{k}{1+\mu_k} \quad k_e = \frac{\mu_k k}{1+\mu_k} \tag{1.4c}$$

There is an inevitable time delay in the transfer system response deteriorating the test stability, because of the inherent dynamics of a hydraulic servo system or an actuator.The dynamic equation of the SDOF system considering time delay is defined as [5]:

$$m_c \ddot{x}_c(t) + c_c \dot{x}_c(t) + k_c x_c(t) + m_e \ddot{x}_c(t-\tau) + c_e \dot{x}_c(t-\tau) + k_e x_c(t-\tau) = f(t) \tag{1.5}$$

where displacement, time delay and externally applied force are shown as: *x*, *f(t)* and $\tau$, respectively.

It is well known that time delays can degrade the performance of control systems and can even make closed-loop system unstable. Therefore, in the design of a controller, time delays must be taken into account and methods should be developed to estimate the maximum amount of time delay (delay margin) that the system can tolerate without losing its stability. Such knowledge on the delay margin could also be helpful in the controller design for cases uncertainty in the network-induced delays is unavoidable. There are several methods in the literature to compute delay margins of time-delayed continuous systems. The common starting point of them is the determination of all the imaginary roots of the characteristic equation. The existing procedures can be classified into the following five distinguishable approaches: i) Schur-Cohn (Hermite matrix formation) [6-9]; ii) Elimination of transcendental terms in the characteristic equation [10]; iii) Matrix pencil, Kronecker sum method [6-9]; iv) Kronecker multiplication and elementary transformation [11]; v) Rekasius substitution [12-14]. These methods demand numerical procedures of different complexity and they may result in different precisions in computing imaginary roots. Among these methods, Rekasius substitution known also as pseudo-delay technique has been successfully applied to the stability analysis of SDOF with load-rate independent and dependent restoring forces and the critical time delays that cause instability of the system have been analytically determined [15].

In this paper, we implement the direct method reported in [10] to estimate the delay margin of SDOF systems in Real-Time Dynamic Hybrid Testing (RTDHT) with time delays. The proposed method is an analytically elegant procedure that first converts the transcendental characteristic equation into a polynomial without the transcendentality by eliminating the exponential term of the characteristic equation. This procedure does not use any approximation or transformation to eliminate the transcendentality of the characteristic equation. Therefore, it is exact and the real positive roots of the new polynomial coincide with the imaginary roots of the characteristic equation exactly. The resulting polynomial without the transcendentality also enables us to easily determine the delay-dependency of the system stability and the sensitivities of crossing roots (root tendency) with respect to the time delay. This is a remarkable feature of the proposed method. Then, an easy-to-use formula is derived to compute delay margins in terms of test structure parameters such as mass, damping and stiffness. It must be also stated that the proposed method has been successfully applied into the actively controlled linear SDOF systems under seismic excitation [16], stability analysis of time-delayed electric power systems and generator excitation control system to compute the delay margin for stability [17-18]. The delay margins are computed for a wide range of system parameters and the theoretical delay margin results are verified by using time-domain simulation capabilities of MATLAB/Simulink [19].

## 2. STABILITY ANALYSIS

### 2.1. Problem Formulation

For stability analysis of SDOF test structure shown in Fig. 1.1, the characteristic equation is required. In the absence of an external excitation force, the characteristic equation could be obtained easily by performing Laplace transform of Eqn. 1.5 as follows

$$\Delta(s,\tau) = P(s) + Q(s)e^{-s\tau} = m_c s^2 + c_c s + k_c + \left(m_e s^2 + c_e s + k_e\right)e^{-s\tau} = 0 \tag{2.1}$$

Using Eqn. 1.2 and 1.3, the characteristic equation of Eqn. 2.1 could be written in terms of the mass, damping, and spring proportion factors as:

$$\begin{aligned}\Delta(s,\tau) &= P(s) + Q(s)e^{-s\tau} = p_2 s^2 + p_1 s + p_0 + \left(q_2 s^2 + q_1 s + q_0\right)e^{-s\tau} = 0\\ \Delta(s,\tau) &= (1-\rho_m)s^2 + 2(1-\rho_c)\xi\omega_n s + (1-\rho_k)\omega_n^2 + \left(\rho_m s^2 + 2\rho_c\xi\omega_n s + \rho_k\omega_n^2\right)e^{-s\tau} = 0\end{aligned} \tag{2.2}$$

where $\omega_n$ refers to undamped natural frequency, $\omega_n = \sqrt{\dfrac{k}{m}}$, while $\xi$ refers to damping ratio, $\xi = \dfrac{c}{(2m\omega_n)}$. The coefficients of the polynomials of *P(s)* and *Q(s)* are as follows:

$$\begin{aligned}&p_2 = 1-\rho_m;\;\; p_1 = 2(1-\rho_c)\xi\omega_n;\;\; p_0 = (1-\rho_k)\omega_n^2\\ &q_2 = \rho_m;\;\; q_1 = 2\rho_c\xi\omega_n;\;\; q_0 = \rho_k\omega_n^2\end{aligned} \tag{2.3}$$

The main goal of the stability studies of delayed systems is to determine conditions on the delay for any given set of system parameters that will guarantee the stability of the system. As with the delay-free system (i.e. $\tau = 0$), the stability of the time-delayed SDOF system depends on the locations of the roots of system's characteristic equation defined by Eqn. 2.2. It is obvious that the roots are functions of the time delay, $\tau$. As $\tau$ changes, locations of some of the roots may change. For the system to be asymptotically stable, all the roots of the characteristic equation of Eqn. 2.2, say $s^\tau = \left[s_1^\tau, s_2^\tau, \ldots, s_n^\tau\right]$ must lie in the left half of the complex plane. That is

$$max\left(real\left(s_i^\tau\right)\right) < 0 \text{ for } \forall s_i^\tau \in s^\tau \text{ or } \forall s_i^\tau \in \mathrm{C}^- \tag{2.4}$$

where $C^-$ represents the left half plane of the complex plane.

Depending on system parameters, there are two different possible types of asymptotic stability situations due to the time delay, $\tau$ [7, 10]:

i) *Delay-independent stability*: The characteristic equation Eqn. 2.2 is said to be *delay-independent stable*, if the stability condition of Eqn. 2.4 holds for all positive and finite values of the delay, $\tau \in [0,\infty)$.

ii) *Delay-dependent stability*: The characteristic equation of Eqn. 2.2 is said to be *delay-dependent stable*, if the condition of Eqn. 2.4 holds for some values of delays belonging in the delay interval, $\tau \in [0,\tau^*)$, and is violated for other values of delay $\tau \geq \tau^*$.

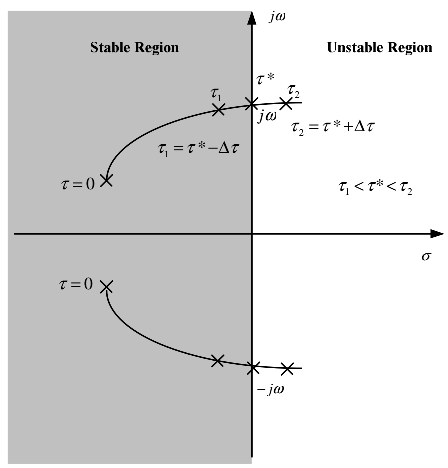


**Figure 2.1.** Illustration of eigenvalue movement with respect to time delay

In the delay-dependent case, the roots of the characteristic equations move as the time delay, $\tau$ increases starting from $\tau = 0$. Fig. 2.1 shows the movement of the roots. Note that the delay–free system ($\tau = 0$) is assumed to be stable. This is a realistic assumption since for the practical values of system parameters, the SDOF test system is stable when the total delay is neglected. Observe that as the time delay, $\tau$ is increased, a pair of complex eigenvalue moves in the left half of the complex plane. For a finite value of $\tau > 0$ they cross the imaginary axis and pass to the right half plane. The time delay value $\tau^*$ at which the characteristic equation has purely imaginary eigenvalues is the upper bound on the delay size, known as delay margin, for which the system will be stable for any given delay less than this bound, $\tau < \tau^*$.

In order to characterize the stability property of Eqn. 2.2 completely, we first need to determine whether the system for any given set of parameters is delay-independent stable or not, and if not, to compute the delay margin $\tau^*$ in terms of system parameters. The stability problem of interest can be stated as follows:

| | |
|---|---|
| ***<u>Given</u>*:** | A time-delay linear system or its characteristic equation of Eqn. 2.1 or 2.2. |
| ***<u>Determine</u>*:** | If it is delay–independent stable or not; if not (the time-delay system is delay–dependent stable); ***<u>find</u>*** the *delay margin*. |

In the following section, we present a practical approach that gives a criterion for evaluating the delay dependency of stability and an analytical formula to compute the delay margin for the delay-dependent case [10].

## 2.2. Solution Method

A necessary and sufficient condition for the system to be asymptotically stable is that all the roots of the characteristic equation of Eqn. 2.2 lie in the left half of the complex plane. In the single delay case, the problem is to find values of $\tau^*$ for which the characteristic equation of Eqn. 2.2 has roots (if any) on the imaginary axis of the *s-plane*. Clearly, $\Delta(s,\tau)=0$ is an implicit function of $s$ and $\tau$ which may, or may not cross the imaginary axis. Assume for simplicity that $\Delta(s,0)=0$ has all its roots in the left half-plane. That is, the delay free system is stable. If for some $\tau$, $\Delta(s,\tau)=0$ has a root on the imaginary axis at $s=j\omega$, so does $\Delta(-s,\tau)=0$, for the same value of $\tau$ and $\omega$. Hence, looking for roots on the imaginary axis reduces to finding values of $\tau$ for which $\Delta(s,\tau)=0$ and $\Delta(-s,\tau)=0$ have a common root. That is,

$$P(s)+Q(s)e^{-s\tau}=0 \quad \text{and} \quad P(-s)+Q(-s)e^{s\tau}=0 \tag{2.5}$$

By eliminating exponential terms in Eqn. 2.5, we get the following polynomial:

$$P(s)P(-s)-Q(s)Q(-s)=0 \tag{2.6}$$

If we replace $s$ by $j\omega$ in Eqn. 2.6, we have the following polynomial in $\omega^2$ [5, 10]:

$$W(\omega^2)=P(j\omega)P(-j\omega)-Q(j\omega)Q(-j\omega)=0 \tag{2.7}$$

Substituting $P(s)=p_2s^2+p_1s+p_0$ and $Q(s)=q_2s^2+q_1s+q_0$ polynomials given in Eqn. 2.3 into Eqn. 2.6, we obtain an augmented characteristic equation as

$$W(\omega^2)=(p_2^2-q_2^2)\omega^4+(p_1^2-q_1^2+2q_0q_2-2p_0p_2)\omega^2+p_0^2-q_0^2=0 \tag{2.8}$$

Please note the augmented polynomial of Eqn. 2.8 is finite in $\omega^2$; it is independent of $\tau$. Moreover, the transcendental characteristic equation with single delay given in Eqn. 2.2 is now converted into a polynomial without transcendentality given by Eqn. 2.8 and its positive real roots coincide with the imaginary roots of Eqn. 2.2 exactly. The roots of this polynomial may easily be determined by standard methods. Depending on the roots of Eqn. 2.8 , the following situation may occur:

i) The polynomial of Eqn. 2.8 does not have any positive real roots, which implies that the characteristic equation of Eqn. 2.2 does not have any roots on the $j\omega$-axis. In that case, the system is stable for all $\tau \geq 0$, indicating that the system is *delay-independent stable*.

ii) The polynomial of Eqn. 2.8 has at least one positive real root, which implies that the characteristic equation of Eqn.  has at least a pair complex eigenvalues on the $j\omega$-axis. In that case, the system is *delay-dependent stable*.

It could be easily shown that the polynomial of Eqn. 2.8 has only one positive real root, say $\omega_k$, which indicates that the SDOF test system is *delay-dependent stable*. Once the value of $\omega_k$ have been found for a given set of parameters, the corresponding value of the delay margin $\tau^*$ can be obtained as follows:

$$\Delta(j\omega_k,\tau) = P(j\omega_k) + Q(j\omega_k)e^{-j\omega_k\tau} = 0$$

$$e^{-j\omega_k\tau} = cos(\omega_k\tau) - jsin(\omega_k\tau) = -\frac{P(j\omega_k)}{Q(j\omega_k)} \tag{2.9}$$

$$cos(\omega_k\tau) = Re\left\{-\frac{P(j\omega_k)}{Q(j\omega_k)}\right\} \quad \text{and} \quad sin(\omega_k\tau) = Im\left\{\frac{P(j\omega_k)}{Q(j\omega_k)}\right\}$$

From Eqn. 2.10, we can determine an analytical formula for the delay margin $\tau^*$ as follows:

$$\tau^* = \frac{1}{\omega_k} Tan^{-1}\left(\frac{Im\left\{\frac{P(j\omega_k)}{Q(j\omega_k)}\right\}}{Re\left\{-\frac{P(j\omega_k)}{Q(j\omega_k)}\right\}}\right) \tag{2.10}$$

An analytical formula for the angle $\tau^*$ and for the upper bound on the delay size for the SDOF structure can easily be obtained by substituting $P(s)$ and $Q(s)$ polynomials given in Eqn. 2.2 and 2.3 into Eqn. 2.10 as follows:

$$\tau^* = \frac{1}{\omega_k} Tan^{-1}\left(\frac{(p_2q_1 - p_1q_2)\omega_k^3 + (p_1q_0 - p_0q_1)\omega_k}{(-p_2q_2)\omega_k^4 + (p_0q_2 + p_2q_0 - p_1q_1)\omega_k^2 - p_0q_0}\right) \tag{2.11}$$

## 3. THEORETICAL AND SIMULATION RESULTS

In this section, the delay margin of SDOF systems in Real-Time Dynamic Hybrid Testing (RTDHT), $\tau^*$ is computed using the expression given by Eqn. 2.11. Theoretical delay margin results are also verified by using Matlab/Simulink. The parameters of the SDOF test structure is as follows [5]:

$\rho_m = 0.25;\ \rho_c = \rho_k = 0.5;\ \xi = 0.05;\ \omega_n = 31.4\ rad/s$

Substituting given parameters into Eqn. 2.3, 2.8, and 2.11 the crossing frequency and the corresponding delay margin are found to be $\omega_k = 6.2781$ rad/s and $\tau^* = 12.6\ ms$. The theoretical delay margin result is verified by using Matlab/Simulink. Fig. 3.1. and 3.2 show the displacement response of the SDOF test structure subjected to the external excitation supposed as a sinusoidal ground motion with an amplitude of 0.1 g and frequency of 4 Hz for three different time delays $\tau = 12.3\ ms$, $\tau = 12.7\ ms$ and $\tau = 13\ ms$ . As can be seen from Fig. 3.1, the response has sustained oscillations indicating the marginal stability of the system for $\tau = 12.7\ ms$. Recall that theoretical delay margin was found to be $\tau^* = 12.6\ ms$. It is clear that the difference between the theoretical delay margin and the one obtained by simulation is just 0.1 ms, which is negligible. Fig. 3.1. also clearly illustrates that the response is stable for a time delay smaller than the delay margin ($\tau = 12.3\ ms \le \tau^* = 12.7\ ms$).

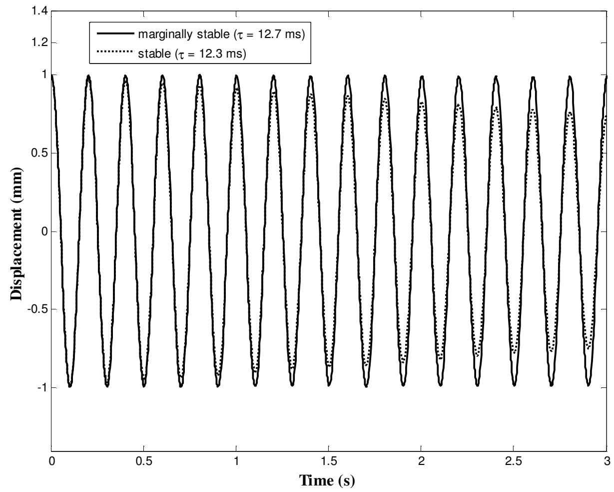


**Figure 3.1.** The response of the displacement for $\tau = 12.3\ ms$ and $\tau = 12.7\ ms$ : Stable and marginally stable cases

Fig. 3.2 gives the unstable case with growing oscillations indicating an unstable condition for a time delay larger than the delay margin ($\tau = 13\ ms \geq \tau^* = 12.7\ ms$). These responses indicate that the proposed method could be effectively used to determine the stability delay margin of SDOF structures.

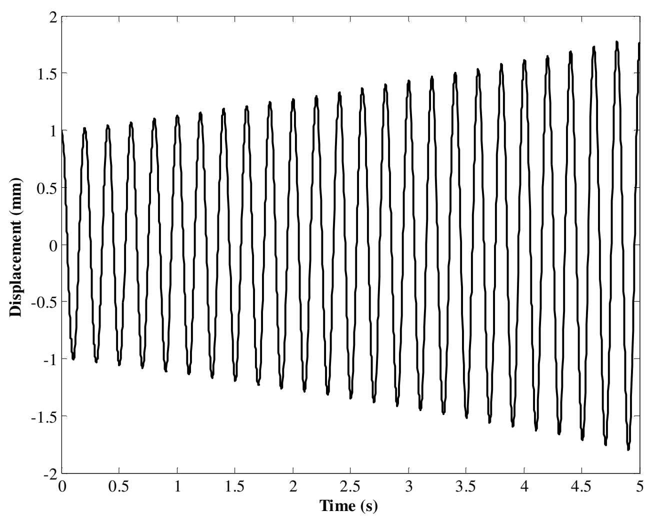


**Figure 3.2.** The response of the displacement for $\tau = 13\ ms$ : Unstable case

Moreover, the effect of the damping proportion, $\rho_m$ on the delay margin is investigated for two different undamped natural frequencies, $\omega_n = 31.4\ \ rad/s$ and $\omega_n = 125.7\ \ rad/s$ . The delay margin results are presented in Table 3.1.

**Table 3.1.** Delay Margin Results for Different Values of Mass Proportion Factor

| Mass proportion factor $\rho_m$ | Delay Margin $\tau^*$ (ms) | |
|---|---|---|
| | $\omega_n = 31.4 \ rad/s$ | $\omega_n = 125.7 \ rad/s$ |
| 0 | 6.35 | 1.58 |
| 0.05 | 7.05 | 1.76 |
| 0.1 | 7.92 | 1.98 |
| 0.15 | 9.04 | 2.26 |
| 0.2 | 10.52 | 2.63 |
| 0.25 | 12.57 | 3.14 |
| 0.3 | 15.60 | 3.89 |
| 0.35 | 20.50 | 5.12 |
| 0.4 | 29.53 | 7.38 |
| 0.45 | 50.02 | 12.50 |
| 0.49 | 87.48 | 21.81 |

Fig. 3.3 shows the variation of the delay margin with respect to the damping proportion, $\rho_m$. It is clear that for a given undamped natural frequency, the delay margin increases as $\rho_m$ is varied in the range of $\rho_m = 0 - 0.49$. Moreover, an increase in the undamped natural frequency $\omega_n$ results in a decrease in the delay margin value for a given mass proportion $\rho_m$.

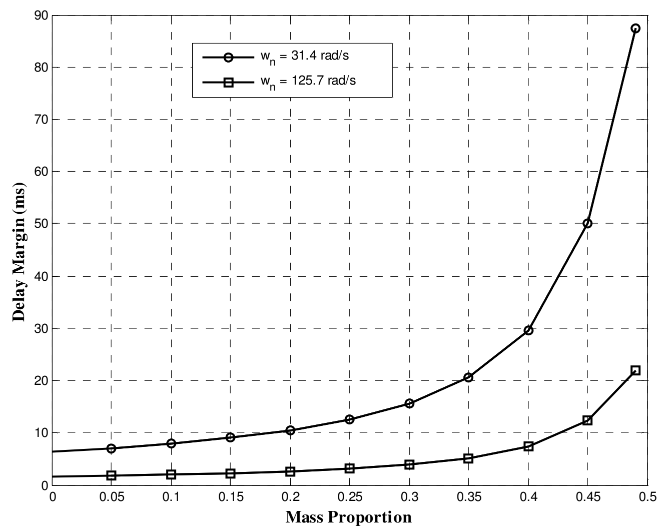


**Figure 3.3.** The variation of the delay margin with respect to the mass proportion for two different values of undamped natural frequency

## 4. CONCLUSIONS

In this paper, an analytical method is presented to compute the delay margin for SDOF systems in Real-Time Dynamic Hybrid Testing (RTDHT) under seismic excitation. To assess the effect of a time delay on the stability of SDOF structure, an analytical expression is developed to compute the delay margin in terms of system parameters. The simulation results are found to be in good agreement with the delay margin obtained from the analytical expression. The ability to predict the delay margin for SDOF structures is of significance. The delay margin information is vital towards determining when a system becomes unstable, to set the requirements for the compensation methods to reduce delay, and thus its adverse impact on the system dynamics.

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