An adaptive physics-based method for the solution of wave motion problem in two dimensions

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SUMMARY:
In this paper, an adaptive physics-based method is developed for solving wave motion problem in two dimensions (i.e., lamb waves). The solution of the problem has two main parts. In the first part, after discretization of the domain, a physics-based method is developed considering the conservation of mass and the balance of momentum. In the second part, adaptive points are determined using wavelet theory. This part is well done using D-D wavelets introduced by Deslauries and Dubuc. Solving the problem, in the first step, the domain of the problem is discretized by the same cells attending loading and characteristics of the structure. After the first trial solution, D-D interpolation shows the lack and redundancy of points in the domain. These points will be added or eliminated for the next solution. This process may be repeated for achieving adaptive mesh for each step. Finally, the results of the proposed method are compared with the results available in the literature. This comparison shows excellent agreement between the obtained results and those already reported.

Keywords: *Physics-based solution, Adaptive solution, Multi-resolution analysis, D-D wavelets*

1. INTRODUCTION

A discrete computation method, such as the cellular automaton (CA) method or the lattice gas automaton method, have been already introduced to analyze some problems in the engineering field. In this paper, CA method is used for solution of wave motion problem in two dimensions, where in-plane wave propagation problem is formulated and implemented. CA method first developed by von Neumann (1966) in Los Alamos national Lab. Different types of CA (for example deterministic or probabilistic, and/or continuous or totalistic) is developed and implemented for various types of problems in science and engineering. CA systems are intrinsically dynamic, so this method can be implemented to the problems which are posed to dynamic behavior. There are many problems in science and engineering which has this characteristic. In the CA method, the analysis domain is divided into similar finite parts called “cells”. The state of each cell is updated according to local rules at every discrete time step. The state of a cell at a given time step depends only on its previous state and that of the neighbor cells. The states of all cells are updated synchronously. Because of such computational characteristics, analysis can be performed for only a desired portion of the total space. Schreckenburg et al. (1995) developed a stochastic CA model for traffic flow. Santos and Coutinho (2001) implemented a CA model for modeling of HIV epidemics. In 2002, a serious evolution occurred in CA's employment for solving various problems after publishing of Wolfram’s book entitled "A new kind of science". Physics-based modeling via CA was then followed for solution of wave motion problem using reflection theory. Since 1986, CA has been employed in many different manners in wave propagation problems (Rothman (1987), Chopard and Droz (1998), Kawamura (2005 and 2006) Leamy (2008) and Hopman and Leamy (2011)). CA paradigm can be distinguished from other numerical methods in different aspects. The mentioned researches emphasized in a specific property of this method. Derivation of neighborhood rules is the most crucial issue in problems being solved using CA. Some researchers derived these rules by considering corresponding differential equation, whereas others tried to make these rules by considering the laws of physics such as mass conservation and momentum balance for elastic wave motions. Common numerical methods (e.g., Finite element method, Finite difference method, Boundary element method and so on) use corresponding differential equation of a problem to formulate and solve the problem. On the other hand, some researchers used CA for similar problems in such a way that neighborhood rules are

derived from related concepts in physics. In fact, a principally distinguishing aspect of cellular automata modeling has an application for physics-based solution. Briefly, in CA method, each part (cell) of the solution domain has a relatively independent role. Behavior of each cell depends on its neighbors, whereas in other mentioned numerical methods, differential equation acts in the whole domain. This difference separates CA from other numerical methods and leads to many computational advantages. Despite these advantages, CA also has some shortcomings. In definition, CA divides the solution domain into similar cells with equal dimensions and properties. Accurate solution of a problem with low computational cost is possible when discretization of the solution domain (or order of solution) is appropriate to complexity of the solution. For this purpose, various methods are employed in recent years. A very common method is D-D interpolating wavelets. In this paper, these two concepts (i.e., physics-based formulation and D-D interpolating wavelets) are employed to solve the wave motion problem in a plate in an adaptive form. In the next section, D-D interpolating wavelets are discussed and in section 3, physics-based solution formulation is presented.

2. D-D INTERPOLATING WAVELETS

Adaptive wavelet grid-based methods were successfully applied in elliptic, parabolic and hyperbolic PDEs by Cruze et al. (2001), Liu et al. (2000), Jameson (1998), Vasilyev and Paolucci (1996), and Holmström (1999). Special characteristics of wavelet concept such as, the complete multi-resolution property, fast algorithms, and data compression ability, posed this method as a fast versatile tool for various purposes. In multi-resolution analysis, each wavelet coefficient is linked to a particular point of underlying grid. Solution of the problem will be started via the same cells. Centroid of each cell is a solution point and of course a sample for multi-resolution analysis. Depending on the gradient of the solution, in some regions, cells should be regenerated in a finer or coarser form. Achieving a good resolution in all parts of the solution domain, this process will be repeated. Assume j be a scale level for solution in a first step, in a coarser level (j-1), D-D interpolating scheme (Deslauriers and Dubuc, 1989) can predict the eliminated values. As noted above, the difference of eliminated value and predicted value in each solution point shows whether each point is needed in level j or not. In each point, this comparison is independent of other points of the solution domain. In some parts of the solution domain, initial resolution may not be appropriate. In these parts new solution points will be added in the middle point of current solution points (level j+1). This process will be continued until the solution value and predicted value (from a coarser level) has acceptable difference. For two dimensional problems, this concept is applied to both directions independently. In addition to noted method, in this research, a new method is presented for two dimensional problems.

3. CA FORMULATION FOR SOLUTION OF WAVE MOTION

In the first step, solution domain is divided to equal parts (cells) as shown schematically in Fig. 1. Each cell has its neighbors; for example, cell (i , j) is in neighborhood with cells are colored below. Formulation of the CA method for solution of this problem has been presented by Hopman and Leamy (2011). Summary of this solution method is presented below. As discussed, solution of a problem via CA method is the derivation of local rules for all possible conditions.

3.1. Local rules

As discussed in section 1, compared with other numerical methods, CA uses local rules to solve a problem, so the derivation of local rules corresponds to the solution of this problem via this method. In the establishment of the equations, the balance of forces is performed on a discrete cell. This leads to global equations called "rules". Solving this problem, local rules have been directed. Assume a cell of a two-dimensional (2D) medium as shown in Fig. 2. X-axis is assumed to be in horizontal direction and Y-axis is normal to the X-axis in the plane. According to relative displacement of the cells, applied strains would be available.

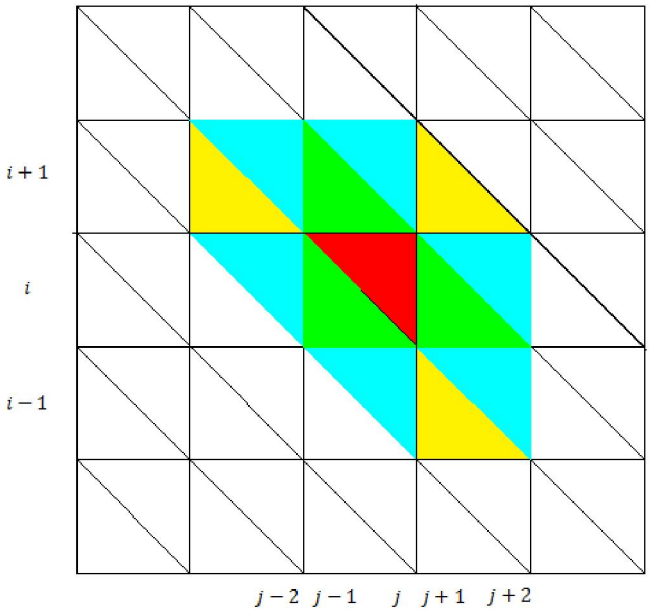

Figure 1. Target cell (red), von Neumann neighborhoods (green), Moore neighborhoods (cyan and yellow)

For calculating the normal and tangential strains, a local coordinate is defined for each face. These strains can be calculated on each face of the cell in tangential and normal directions using derivations in normal and tangential directions. According to Hook's law, corresponding stresses will be available too. For each face of the cell, tangential and normal forces will be available by multiplying the achieved stresses by the dimensions of the face (F_n and F_t in Fig. 2).

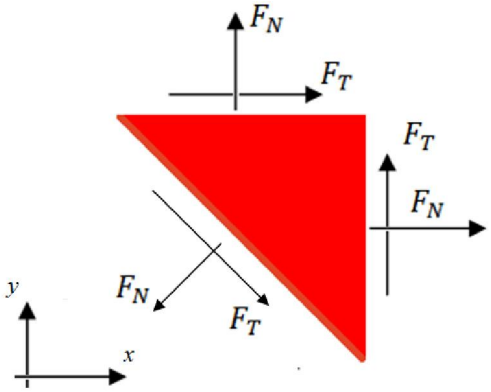

Figure 2. A cell of a 2D medium and applied forces

$$F_t = wl\,\mu\left[\frac{u_t^N - u_t^T}{\Delta n} + \frac{1}{2}\left(\frac{u_n^{N+} - u_n^{N-}}{\Delta s_+^T + \Delta s_-^T} + \frac{u_n^{M+} - u_n^{M-}}{\Delta s_+^N + \Delta s_-^N}\right)\right] \tag{3.1}$$

$$F_n = wl\left[(\lambda + 2\mu)\frac{u_n^N - u_n^T}{\Delta n} + \frac{\lambda}{2}\left(\frac{u_t^{N+} - u_t^{N-}}{\Delta s_+^T + \Delta s_-^T} + \frac{u_t^{M+} - u_t^{M-}}{\Delta s_+^N + \Delta s_-^N}\right)\right] \tag{3.2}$$

in which λ and μ are lame constants, u_t and u_n show the displacement of the cell in tangential and normal directions, w is the thickness of the 2D domain, l denotes the length of the face, and the remaining parameters are displayed in Fig.3. N-, N+, M-, and M+ superscripts are addresses of the neighbor cells around the target cell (marked in red in Fig. 3).

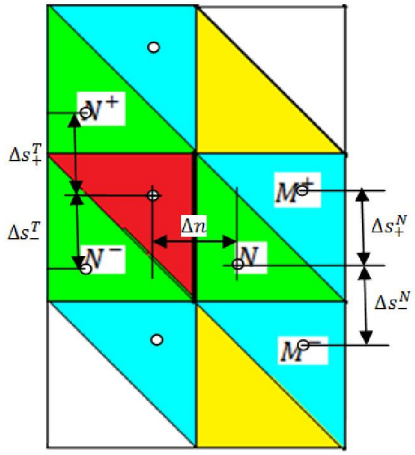

Figure 3. Target cell (red), Neighbors (green, cyan and yellow) and used dimensions

Forces in the x and y directions are obtained by the well-known inverse rotation transformation for each face of the cell as follows

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \mathbf{R}^{Tr} \begin{pmatrix} F_n \\ F_t \end{pmatrix} \tag{3.3}$$

where

$$\mathbf{R} = \begin{pmatrix} Cos\,\theta & Sin\,\theta \\ -Sin\,\theta & Cos\,\theta \end{pmatrix} \tag{3.4}$$

and Tr superscript shows transpose of the **R** matrix. After these calculation for each face of the triangular cell, internal forces would be available in x and y directions. Finally, the force balance for these two directions may be written as

$$F_x^{Total} = F_x^{External} + \sum F_x \tag{3.5}$$

$$F_y^{Total} = F_y^{External} + \sum F_y \tag{3.6}$$

where $F_x^{External}$ and $F_y^{External}$ are external loads in x and y directions, respectively. Knowing the resultant applied force, balance of the momentum may be written as below for both x and y directions

$$v_x^{k+1} = v_x^k + \frac{1}{\rho w A m} F_x^{Total} \tag{3.7}$$

$$v_y^{k+1} = v_y^k + \frac{1}{\rho w A m} F_y^{Total} \tag{3.8}$$

where ρ and A are mass per unit volume of the material, and area of the cell, respectively. m indicates the number of time steps per unit time. Similarly for displacement update, one may write

$$u_x^{k+1} = u_x^k + \frac{1}{m} v_x^{k+1} \tag{3.9}$$

$$u_y^{k+1} = u_y^k + \frac{1}{m} v_y^{k+1} \tag{3.10}$$

3.2. Boundary conditions

Two well-known boundary conditions (Dirichlet and Neumann) are developed for use in this work. For satisfying Dirichlet boundary condition, an imaginary equilateral triangle is added to the domain (See Fig. 4). In this condition, normal and tangential displacements are known. u_t and u_n for the imaginary added cell would be calculated in such a manner that satisfy known displacement values. For satisfying Neumann boundary condition, in addition to the added equilateral cell, two other imaginary equilateral cells are added to the domain. In this condition, normal and tangential stresses are known. u_t and u_n for the imaginary cell would be calculated in such a manner that satisfy known stress values (Hopman and Leamy, 2011).

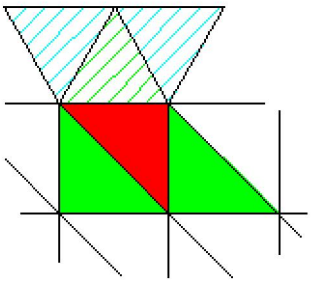

Figure 4. Added imaginary cells for satisfying boundary conditions

4. WAVELET CONCEPT AND ADAPTIVE SOLUTION

Wavelets are usually introduced by defining scaling functions, $\phi_{j,k}$, wavelets, $\psi_{j,k}$, and the associated function spaces V_j (corresponding to scale function) and W_j (corresponding to wavelets). Since we are using an interpolating wavelet transform, it is possible to define the transform in terms of interpolation on dyadic grids, instead. First, we present the interpolating subdivision idea by Deslauriers and Dubuc (1989) and Dubuc (1986). Assume that we have a set of dyadic grids on the real line,

$$V_j = \{x_{j,k} \in \mathbf{R} : x_{j,k} = 2^{-j}k, k \in \mathbf{Z}\} \quad and \quad j \in \mathbf{Z} \tag{4.1}$$

Fig. 5 shows locations of these points on such grid. Given function values on V_j, $\{f_{j,k}\}, k \in \mathbf{z}$, where $f_{j,k} = f(x_{j,k})$ is a function defined on the grid points in V_j, we would like to extend them to all points $f_{j+1,k}$ in V_{j+1}. The interpolating subdivision idea is an algorithm to achieve this goal. The even-numbered grid points $x_{j+1,2k}$ already exist in V_j and the corresponding function values are kept unchanged. Values at the odd-numbered grid points $x_{j+1,2k+1}$ are computed by polynomial interpolation from the values at the even-numbered grid points.

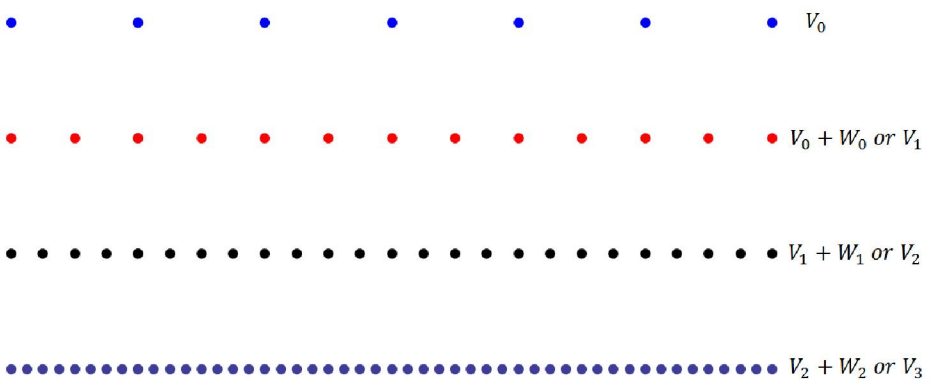

Figure 5. Point positions in dyadic grid

The degree of this interpolating polynomial is p-1; and we say that the interpolation is of order p. The order is chosen to be even to make the interpolation symmetric. As discussed, the interpolating subdivision idea generates function values on a fine grid, given values on a coarse grid. If we wanted to do the reverse, we could just throw away half of the grid points at each level, but we would lose some information. Instead we can, at each level, for odd-numbered grid points, compute the difference between the known function value and the function value predicted by the interpolation from the coarser grid. We call these differences in function values wavelet coefficients, $d_{j,k}$. The computation of a wavelet coefficient is illustrated in Fig. 6 for the cubic curve case (p = 4).

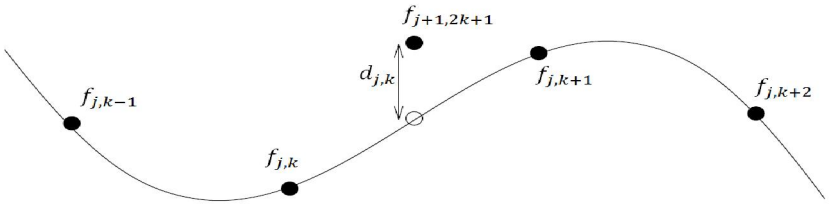

Figure 6. Prediction of a known value from higher resolution via cubic curve and determination of corresponding wavelet coefficient

Repeating this recursively, we have an algorithm for computing the full wavelet representation from function values on a fine grid. In Fig. 7, we have an illustration of such a wavelet representation. This interpolating wavelet transform was introduced, independently, by Donoho (1992) and Harten (1994).

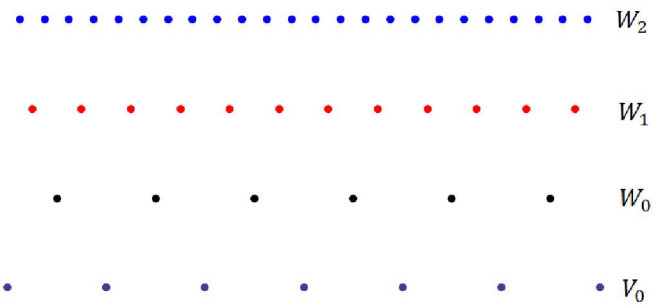

Figure 7. Points in basis space (V_0) and added points corresponding to added spaces ($W_0 - W_2$) corresponding to wavelets

After these manipulations, we have sets of numbers in each level. Depending on the expected accuracy, we can eliminate some values of $d_{j,k}$ smaller than a threshold e. Considering these

elimination, a large number of wavelet coefficient would be eliminated for a case of smooth function. The number of remaining coefficients mainly depends on the variation of the gradient in the original function.

4.1. Generalization and application

When we use cubic curves, we need four points to predict the unknown value (see Fig. 6). Values of points which are in neighbor of boundaries cannot be predicted via this approach. As mentioned before, additional solution points are embedded to satisfy boundary conditions. These points can aid to predict values of points which are in proximity to boundaries. Another method is use of non-symmetric set of points to predict this unknown value. D-D adaptive scheme were used in 2D differential equations by other researchers before. They used this method in two directions independently. In this work, D-D interpolation scheme is generalized for use in a 2D wave propagation problem with triangular cells in a different manner. Solution points are centroid of the cells. A cell would be divided to four cells in a refinement (See Fig. 8). For horizontal and vertical directions, known D-D interpolation is used for prediction of the values in finer cells (See green and red points). For inclined direction, a cubic spline is used. Differences of the known values and predicted values in these three directions are named η_i, η_j and η_k.

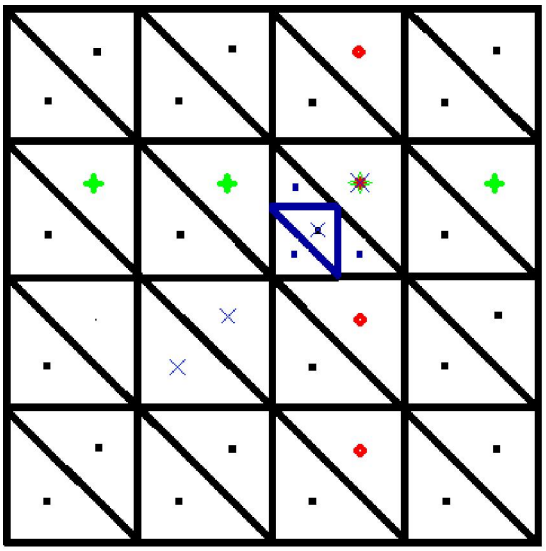

Figure 8. Interpolating points and prediction of the values in a finer cell

In comparison with known D-D scheme, here three parameters are available $(\eta_i, \eta_j$ and $\eta_k)$. Therefore, a new parameter is defined as below

$$\eta = \sqrt{\eta_i^2 + \eta_j^2 + \eta_k^2} \tag{4.2}$$

According to these explanations, adaptive solution of the mentioned problem has the following steps:

a) Select appropriate cells for the first step.
b) Solve the problem for this level and an upper level.
c) Find difference of the predicted values and existing ones in a finer level.
d) Eliminate points for which $\eta < e$.
e) Refine remained mesh for regions where $\eta > e$, and solve the problem again (time step changes considering Courant condition).
f) Go to 4 until points exist with $\eta > e$.
g) Go to 2 to solve the problem in the next time step.

5. NUMERICAL RESULTS AND DISCUSSION

A 2D plane stress problem is considered for comparison of the efficiency of the present method versus those reported recently. A 2D square aluminum plate is considered, whose dimensions and material properties are as follows: length = 1000 mm, width = 1000 mm, thickness = 1 mm, Young's modulus E = 72.9 GPa, Poisson's ratio v = 0.33, and mass density $\rho = 2700 Kg/m^3$. Using these mechanical properties, the primary and secondary wave velocities may be calculated as C_p=5432.3 m/s and C_s=3144.2 m/s . This plate is divided to 128000 similar cells for solution in first step. All four edges of the plate are stress free, while its four corners are fixed in both directions. An excitation signal in the form of a force pulse signal (Fig. 9) has been applied at mid-point of the left face of the plate. Refinement of the cells is limited to 6 levels and threshold value (e) is assumed to be 1e-8.

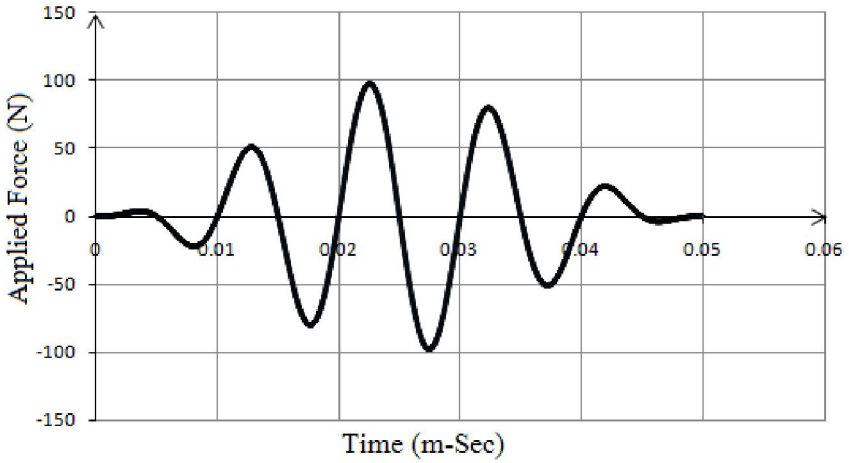

Figure 9. Applied excitation function at the mentioned point

Results of the solution by this method are presented and compared with those achieved using spectral finite element (SFE) method (Khaji and Kazemi Noureini, 2012). Horizontal displacement at the excitation point is presented below.

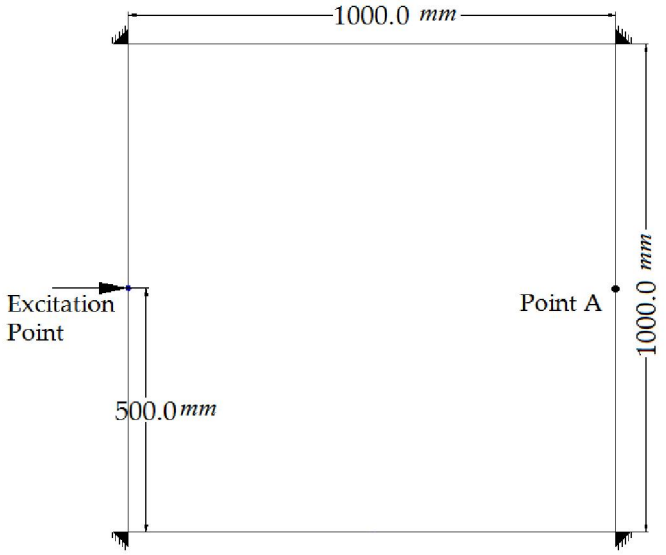

Figure 10. Solution domain and excitation point

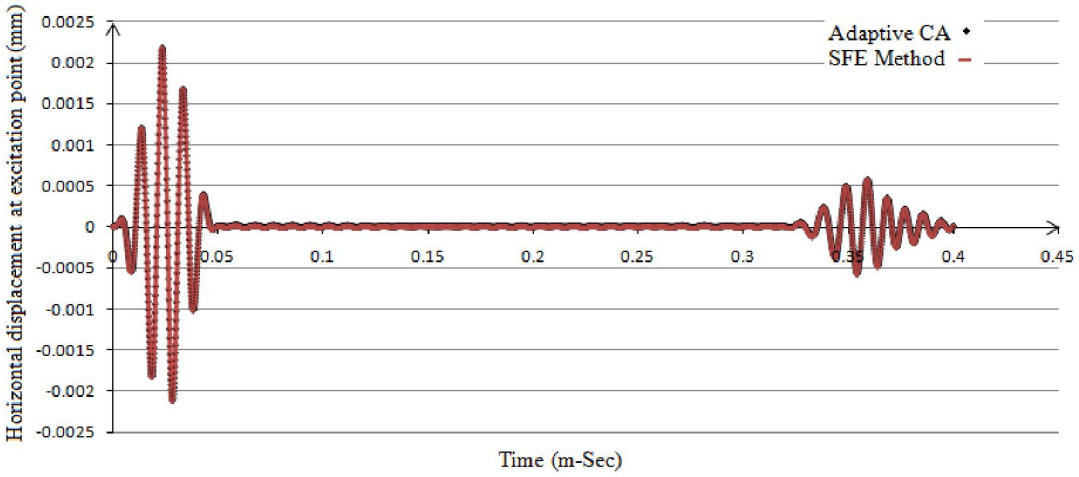

Figure 11. Horizontal displacement at excitation point

In addition to the excitation point, results of the solution (horizontal displacement) are compared with those reported by Khaji and Kazemi Noureini (2012) in point A. Good agreements are observed between the present approach and those referred to.

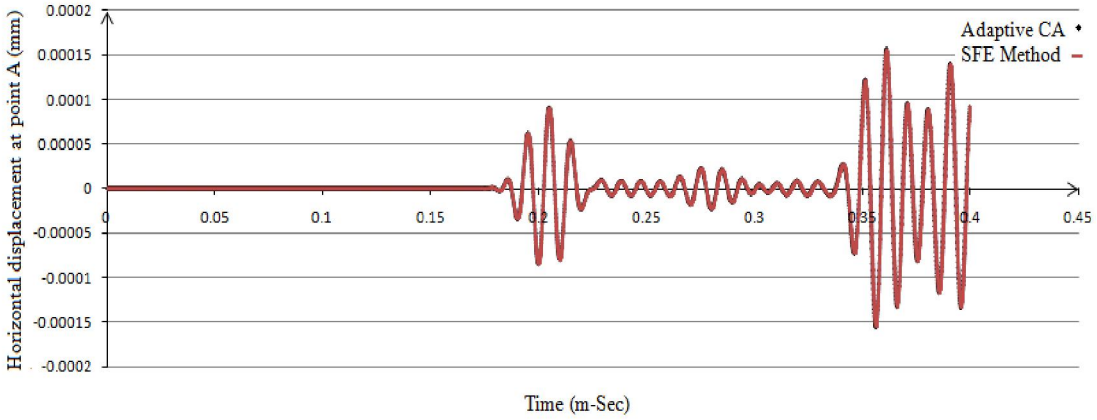

Figure 12. Horizontal displacement at point A

6. CONCLUTION

Advantages of the CA method, as discussed above, can change this method to an efficient method for solution of these types of problems. Results of the present method are in good agreement with those reported in literature. It should be noted that adaptive use of the CA method needs more researches to be used efficiently.

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