# Dynamic response of footings on stratified soil using the Precise Integration Method

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### SUMMARY:

The Precise Integration Method (PIM) is applied for the evaluation of dynamic impedance of rigid strip footing bonded to stratified soil. As formulated in the Hamilton system, the governing equation of layered semi-infinite soil is represented by ordinary differential equations of mixed-boundary value problems in the frequency-wavenumber domain which can be computed accurately by using PIM. At last, based on the domain transformation, the dynamic impedance of rigid strip footing in the frequency-spatial domain is obtained numerically. Various numerical examples and comparison with existing classical results demonstrate the accuracy of the proposed method and its correct implementation. By using the PIM, index overflow problems as encountered when using transfer matrices method can be avoided, and it poses no limitation to the layered soil thickness.

Key words: Precise Integration Method, frequency-wavenumber domain, stratified soil, dynamic impedance.

## **1. GENERAL INSTRUCTIONS**

The dynamic force-displacement relationship for a rigid footing bonded to a layered semi-infinite soil plays an important role in the study of machine vibrations as well as in the study of soil-structure interaction. There has been increasing interest in the dynamic response of soil-footing interaction. However, the dynamic response of the layered soil remains a complicated problem to be solved. In fact, most large civil construction and oceanic structures are built on the complex layered soil. And it is well-known that layering of the soil can significantly influence the dynamic response of the structure and should be taken into account appropriately. For this issue, the major challenge is the correct description of radiation damping in the unbounded soil domain.

Analytical solutions of wave motion in layered soil are difficult to obtain, so in recent decades, various numerical methods have been suggested, such as finite element method, boundary element method (BEM), matrix propagator method and thin layer method. For a comprehensive overview of existing alternative methods and their limitations and advantages the reader is referred to Gazetas (1983). The scaled boundary element method (SBFEM) is a newly developed promising semi-analytical approach to continuum analysis (Wolf J.P. 2003). But it is unsuited for the layered soil rests on a homogeneous semi-infinite space.

In this paper, an efficient approach for the dynamic impedance of rigid strip footing bounded to layered half-space is presented. The problem of forced oscillations of a rigid footing on a layered half-space is a mixed-boundary-value problem in which the displacement is prescribed under the footing while the rest of the surface of the half space is traction free. PIM is a precise numerical method for solving first-order linear ordinary differential equations with species two-point boundary value conditions for space domain problem. And its precision is limited only by the precision of the computer used. The problem considered in this paper is the steady-state harmonic vibrations of a massless rigid strip resting on the surface of a layered system and being excited by forces. The soil is

considered as a series of linearly viscoelastic, homogeneous and isotropic layers resting on the elastic half-space or rigid bedrock. In this paper, the PIM is applied to evaluate the relationship between the surface traction and the displacement in the frequency-wavenumber domain. Then the dynamic impedance of a rigid strip footing bonded to layered soil can be evaluated. The accuracy of the novel method is verified by comparing the solution obtained by the proposed method to numerical reference solutions.

#### 2. STATEMENT OF THE PROBLEM



Figure 2.1. Diagram of footing and coordinate system. (a) problem setup (b) applied forces

A rigid rectangular massless strip footing of infinite length and width 2b resting on the surface of a stratified soil is considered in the following. The footing is excited by harmonic line forces and moments (as shown in Fig. 2.1). Two kinds of stratified soil are considered. For the first kind, the layered soil rests on a homogeneous semi-infinite space (Fig. 2.2(a)). And for the second kind, the layered soil rests on rigid bedrock (Fig. 2.2(b)). In both two cases, assume that there are l layers with different material constants and thickness. Every layer soil is characterized by two Lame constants  $\lambda_r$ , and  $\mu_r$ , a Poisson's ratio  $v_r$ , a mass density  $\rho_r$  and a damping ratio of the hysteretic type  $\xi_r$  (r = 1, 2, ..., l). If damping ratio  $\xi_r$  is not equal to zero, the shear modulus  $\mu_r$  will be replaced by  $\mu_r (1+2i\xi_r)$  according to analogue method (thereinto,  $i = \sqrt{-1}$ ). The materials are assumed to be transversely isotropic, i.e., the parameters of each layer are independent of direction. Therefore, the stratified soils can be simplified as a two-dimensional problem.



**Figure 2.2.** Abridged general view of layered soil (a: Resting on a half space b: Built-in at its base)

#### 3. THE GREEN'S INFLUENCE FUNCTIONS IN FREQUENCY-WAVE-NUMBER DOMAIN

It is important to point out that all variables are independent of the co-ordinate y, so only the displacements  $\tilde{u}(x, z, \omega)$  and  $\tilde{w}(x, z, \omega)$  along the x and z axes respectively corresponds to the compressional-vertically polarized shear waves are considered. The Fourier transform and inverse Fourier transform for the wave equation are defined respectively as

$$\begin{cases} g(\kappa,0,\omega) = \int_{-\infty}^{+\infty} \tilde{g}(x,0,\omega) \exp(-i\kappa x) dx \\ \tilde{g}(x,0,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\kappa,0,\omega) \exp(i\kappa x) d\kappa \end{cases}$$
(3.1)

where  $\kappa$  is the wavenumber along the *x* direction,  $\omega$  is excitation frequency,  $i = \sqrt{-1}$ . *g* and  $\tilde{g}$  represent displacement, strain, tress, etc. in the frequency-wavenumber domain and frequency-spatial domain respectively.

By using Eq. (3.1) we can obtain the general plane wave motion in the Lame Equations form as

$$[J_{22}]\{q\}^{'} + ([J_{21}] - [J_{12}])\{q\}^{'} - ([J_{11}] - \rho\omega^{2}[I])\{q\} = 0$$
(3.2)

where  $\{q\}$  is the displacement vector and  $\{q\} = [u, w]^T$ ,  $(X) = \partial(X)/\partial z$ , *I* is a 2×2 unit matrix.  $\rho$  is the density of the layer which is currently calculated. [*J*] is the coefficient matrix which is defined by the layer soil material constants.

Now introduce the dual vector  $\{p\} = -\{\tau_{xz}, \sigma_z\}^T$ , then Eq. (3.2) can be written in the state space as

$$\{V'\} = [H]\{V\}, \quad [H] = \begin{bmatrix} [H_{11}] & [H_{12}] \\ [H_{21}] & [H_{22}] \end{bmatrix}, \quad \{V\} = \begin{cases} \{q\} \\ \{p\} \end{cases}$$
(3.3)

where

$$[H_{11}] = -[J_{22}]^{-1}[J_{21}], \quad [H_{21}] = -([J_{11}] - [J_{12}][J_{22}]^{-1}[J_{21}] - \rho\omega^{2}[I])$$

$$[H_{22}] = -[H_{11}]^{H}, \quad [H_{12}] = -[J_{22}]^{-1}$$

$$(3.4)$$

The boundary condition at the free surface  $(z = z_0)$  is  $\{p(z_0)\} = \{0\}$ . And For the stratified soil, the continuity conditions at the interfaces are

$$\{q_r^+\} = \{q_r^-\}, \quad \{p_r^+\} = \{p_r^-\}, \quad (r = 1, 2, ..., l - 1, l)$$
(3.5)

The general solution of the state Eq. (3.3) is an exponential function. Select a typical interval  $[z_a, z_b]$  ( $z_a < z_b$ ) within a layer is addressed. Let  $\{q_a\}$ ,  $\{p_a\}$  and  $\{q_b\}$ ,  $\{p_b\}$  are the displacement and force vectors at the two ends  $z_a$  and  $z_b$ , respectively. For linear systems, the following relations stand classically:

$$\{q_b\} = [M_F]\{q_a\} - [M_G]\{p_b\}, \quad \{p_a\} = [M_Q]\{q_a\} + [M_E]\{p_b\}$$
(3.6)

where  $[M_F]$ ,  $[M_G]$ ,  $[M_Q]$  and  $[M_E]$  are complex transfer matrices to be determined. They are functions of the matrices  $[H_{11}]$ ,  $[H_{21}]$ ,  $[H_{12}]$  and  $[H_{22}]$  formed by the material constants. Let's

notice that if  $\{q_a\}$  and  $\{p_b\}$  have been specified, the solution of  $\{q\}$  and  $\{p\}$  in the interval  $[z_a, z_b]$  is well defined. Zhong W.X. *et al.* (2004) solve the Eq. (3.6) using the PIM, and higher precision is achieved. The basic concept and the equations for the development in this paper are summarized.

In order to obtain the transfer matrices  $[M_F]$ ,  $[M_G]$ ,  $[M_Q]$  and  $[M_E]$  as exactly as possible, in the PIM, the thickness of every layer  $h_r = z_r - z_{r-1}$  (r = 1, 2, ..., l) is firstly divided into  $2^{N_1}$  sublayers of equal thickness. Then each sublayer is further divided into  $2^{N_2}$  mini-layers of equal thickness  $\tau$  (in this paper, N<sub>1</sub>=6 and N<sub>2</sub>=20 will be selected). Since  $\tau$  is extremely small, the transfer matrices  $[M_F(\tau)], [M_Q(\tau)]$  and  $[M_E(\tau)]$  can be found in terms of Taylor series expansion. With increasing terms of Taylor expansion, any desired accuracy of the results can be reached. In this paper, four terms of Taylor's series is considered sufficient.

From (3.6), combination of two adjacent intervals leads to the new transfer matrices as shown in formula (3.7).

$$\begin{bmatrix} M_F^c \end{bmatrix} = \begin{bmatrix} M_F^2 \end{bmatrix} (\begin{bmatrix} I \end{bmatrix} + \begin{bmatrix} M_G^1 \end{bmatrix} \begin{bmatrix} M_Q^2 \end{bmatrix})^{-1} \begin{bmatrix} M_F^1 \end{bmatrix}$$

$$\begin{bmatrix} M_G^c \end{bmatrix} = \begin{bmatrix} M_G^2 \end{bmatrix} + \begin{bmatrix} M_F^2 \end{bmatrix} (\begin{bmatrix} M_G^1 \end{bmatrix}^{-1} + \begin{bmatrix} M_Q^2 \end{bmatrix})^{-1} \begin{bmatrix} M_E^2 \end{bmatrix}$$

$$\begin{bmatrix} M_Q^c \end{bmatrix} = \begin{bmatrix} M_Q^1 \end{bmatrix} + \begin{bmatrix} M_E^1 \end{bmatrix} (\begin{bmatrix} M_Q^2 \end{bmatrix}^{-1} + \begin{bmatrix} M_G^1 \end{bmatrix})^{-1} \begin{bmatrix} M_F^1 \end{bmatrix}$$

$$\begin{bmatrix} M_E^c \end{bmatrix} = \begin{bmatrix} M_E^1 \end{bmatrix} (\begin{bmatrix} I \end{bmatrix} + \begin{bmatrix} M_Q^2 \end{bmatrix} \begin{bmatrix} M_G^1 \end{bmatrix})^{-1} \begin{bmatrix} M_E^2 \end{bmatrix}$$
(3.7)

where the superscript 1 and 2 denote the matrices associated with the original two intervals and the superscript c denotes the newly combined matrices. It is therefore important that as the combination is processed in a mini-layer,  $[M_F(\tau)]$  and  $[M_E(\tau)]$  are very small because  $\tau$  is very small. They are computed and stored independently to avoid losing effective digits. Hence it is necessary to replace  $[M_F]$  and  $[M_E]$  in Eq. (3.7) by  $[I]+[M_F]$  and  $[I]+[M_E]$  respectively.

As all intervals having equal thickness and identical material constants proceed above, combination of such intervals are performed easily. For each pass of combination, transfer matrices  $[M_F]$ ,  $[M_G]$ ,  $[M_Q]$  and  $[M_E]$  are merged together to form a new one. And each pass of combination reduces the number of total intervals by a half. Proceeding in this way, any desired accuracy can be achieved in the sense that its precision is limited only by the precision the computer acquired.

The combination of the interval matrices of the *l* layers into global interval matrices is similar to the Eq. (3.7). Finally we can obtain the relationship between displacement and force vectors  $\{q_0\}$ ,  $\{q_l\}$  and  $\{p_0\}$ ,  $\{p_l\}$  at the two ends  $z_0$  and  $z_l$  respectively in the form of Eq. (3.6).

In the first kind which is mentioned in the section 2, the radiative conditions in the semi-infinite space are firstly addressed (Zhong W.X. et al. 2004).

$$\{p_l\} = [R_{\infty}]\{q_l\} \tag{3.8}$$

To satisfy the continuity conditions at the interface between the layered soil and half-space, the relationship between the nodal displacement vector  $\{q_0\}$  and dual vector  $\{p_0\}$  at the surface can be formulated using Eqs. (3.6) and (3.8) as:

$$\{p_0\} = \left( \left[ M_Q \right] + \left[ M_E \right] \left[ R_\infty \right] \left( \left[ I \right] + \left[ M_G \right] \left[ R_\infty \right] \right)^{-1} \left[ M_F \right] \right) \{q_0\} = \left[ S(k) \right] \{q_0\}$$
(3.9)

For the second kind, i.e. the stratified soil rests on rigid bedrock, the displacement vector at  $z_i$  must be zero. Therefore we can obtain a different relationship between  $\{q_0\}$  and  $\{p_0\}$ :

$$\{p_0\} = \left( \left[ M_{\mathcal{Q}} \right] + \left[ M_{\mathcal{E}} \right] \left[ M_{\mathcal{G}} \right]^{-1} \left[ M_{\mathcal{F}} \right] \right) \{q_0\} = \left[ S(k) \right] \{q_0\}$$

$$(3.10)$$

## 4. DYNAMIC STIFFNESS MATRICES FOR RIGID STRIP FOOTING



Figure 4.1. Nodal model of the rigid footing

Now discretize the interface between the footing and the stratified soil into N-1 elements with N nodal points. Between adjacent nodal points, the interaction stresses imposed on the stratified soil by the rigid footing are uniformly distributed (see Fig. 4.1). Transform Eqs. (3.9) and (3.10) back to frequency-space domain by inverse Fourier transform, then we can derive

$$\begin{cases} \tilde{u}(x,0,\omega) \\ \tilde{w}(x,0,\omega) \end{cases} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \begin{bmatrix} \bar{F}_{xx}(k) & \bar{F}_{xz}(k) \\ \bar{F}_{zx}(k) & \bar{F}_{zz}(k) \end{bmatrix} \begin{cases} -\bar{\tau}_{xz}(\kappa,0,\omega) \\ -\bar{\sigma}_{z}(\kappa,0,\omega) \end{cases} e^{i\kappa x} d\kappa$$
(4.1)



Figure 4.2. Vertical flexibility coefficients in the wave-number domain

Five points Gauss integral is used for the integration of the Eq. (4.1). For the case of the purely elastic solid,  $\xi = 0$ , the procedure for evaluating the integrals of these equations is more involved because the singularities of the integrand are on the real axis. To illustrate this, the vertical flexibility coefficient  $\overline{F}_{zz}(k)$  of a layered soil consisting of a layer of depth *d* resting on a visoelastic half-space is presented. The shear wave velocity of the half-space  $c_{s2}$  is twice that of the layer  $c_{s1}$ . Poisson's ratio and damping ratio are equal to 1/3 and 0.001 respectively for the whole layered soil. The dimensionless frequency  $\omega d / c_{s1}$  is selected as 6. The vertical flexibility coefficient  $\overline{F}_{zz}(k)$  is plotted as a function of the dimensionless wave number kd in the Fig. 4.2. As the figure show, there are four sharp peaks of the real and imaginary parts for four distinct wave numbers. And the values of the peaks are finite. It's worth noting that if the damping ratio is selected as zero, the peaks will be

infinite except the first one. However, this is no limitation for practical applications, because for real materials, a viscoelastic model with small value of  $\xi$  is more appropriate than a purely elastic model. In this paper, the damping ratio  $\xi$  is assumed to be 0.001 representing the purely elastic case. Numerical evaluation of the integrals of Eq. (4.1) for a viscoelastic soil is straightforward, because no singularities or branch points of the integrand appear on the real axis.

Then we can obtain the complex amplitude of the displacement at nodal point m(m=1,2,...N) is

$$\begin{cases} \tilde{u}^m_{xx} & \tilde{u}^m_{xz} \\ \tilde{w}^m \end{cases} = \begin{bmatrix} \tilde{u}^m_{xx} & \tilde{u}^m_{xz} \\ \tilde{u}^m_{zx} & \tilde{u}^m_{zz} \end{bmatrix} \begin{cases} \tilde{f}_x \\ \tilde{f}_z \end{cases} \quad \text{or} \quad \{\tilde{u}\} = \begin{bmatrix} \tilde{F}^m \end{bmatrix} \{\tilde{f}\}$$
(4.2)

where  $\left\lceil \tilde{F}^{m} \right\rceil$  and  $\left\{ \tilde{f} \right\}$  denote the dynamic flexibility influence coefficients and the complex amplitude of the interaction force between the two nodal points of the m element respectively. The complex-valued dynamic influence coefficient  $\tilde{u}_{zz}^m(x,0,\omega)$  is expressed as follows:

$$\tilde{u}_{zz}^{m}(x,0,a_{0}) = \frac{1}{\mu} \Big[ f_{zz}(a_{0}) + ig_{zz}(a_{0}) \Big]$$
(4.3)

The location x (superscript) of the nodal point m is assumed to be b/20. The real and imaginary parts  $f_{zz}(\omega)$  and  $g_{zz}(\omega)$  are real-valued functions of the dimensionless frequency  $a_0$ . The dynamic flexibility influence coefficients  $f_{zz}(\omega)$  and  $g_{zz}(\omega)$  are evaluated over a range of the dimensionless frequency  $a_0$  for different upper limit of integral of 5, 10, 20, 50 and 100. The results are presented in Fig. 4.3.



Figure 4.3. Vertical flexibility influence function

The figures clearly show the effect of the upper limit of the integral. As the upper limit of the integral increases to 100, the real and imaginary parts converge to a steady value. So we can choose a limited value instead of the infinite as the upper limit of the integral.

Employ the principle of virtual work to state that the work done by the interaction forces is the same whether it is expressed in the nodal point or nodal interval. Then we can obtain the relationship between the nodal point displacements  $\{\tilde{u}\}\$  and the nodal point forces  $\{\tilde{t}\}\$  as

$$\{\tilde{t}\} = [D]^T ([D] [\tilde{F}])^{-1} [D] \{\tilde{u}\} = [S(\omega)] \{\tilde{u}\}$$

$$(4.4)$$

where [D] is a simple displacement transformation matrix between the nodal point displacement and

the interval displacement.  $\left[\tilde{F}\right]$  is the dynamic flexibility influence coefficients for the whole system.  $\left[S(\omega)\right]$  is the dynamic stiffness matrix for the stratified soil.

The relationship between the displacements and forces of the rigid footing is derived.

$$\{P\} = \left[\tilde{H}\right]^{T} \left[S(\omega)\right] \left[H\right] \{\Delta\} = \left[S_{f}(\omega)\right] \{\Delta\}$$

$$(4.5)$$

where  $[S_f(\omega)]$  represents the dynamic stiffness of the rigid footing.  $\{P\}$  contains H, P and M which represent the amplitudes per unit length of the horizontal force, the vertical force, and the moment applied to the rigid footing respectively.  $\{\Delta\}$  contains  $\Delta_1$ ,  $\Delta_2$  and  $\varphi$  which represent the amplitude of the horizontal displacement, vertical displacement and rotation of the centroid of the rigid footing respectively.

#### **5. NUMERICAL EXAMPLES**



Figure 5.1. Abridged general view of the numerical examples





**Figure 5.2.** Dynamic stiffness of rigid strip footing, layer on half space,  $\xi = 0.05$  ((a) horizontal (b) vertical (c) rocking (c) coupling)

A single layer of depth *d* and of Possion's ratio  $v_L = 0.33$  resting on a half space also with Possion's ratio  $v_R = 0.33$ . The ratios of the shear wave velocities  $c_s^R / c_s^L$  and of the mass densities  $\rho_R / \rho_L$  equal 2 and 1 respectively.  $\xi = 0.05$  is assumed for both layer and the half space. A rigid strip footing with half-width b = d rests on the surface of the stratified soil (see Fig. 5.1(a)). The dynamic stiffness matrix is defined by the matrix Eq. (5.1).

$$S_{f}(a_{0}) = \pi \mu (k(a_{0}) + ia_{0}c(a_{0})), \quad a_{0} = \omega b / c_{s}$$
(5.1)

where  $k(a_0)$ ,  $c(a_0)$ , and  $a_0$  denote the frequency-dependent spring and damping coefficients and the dimensionless frequency respectively.  $c_s$  represents the shear wave velocity  $c_s = \sqrt{G/\rho}$  at the surface of the stratified soil.

The dynamic stiffness coefficients obtained for horizontal, vertical, rocking and coupling between horizontal and rocking motions are shown in Fig.5.2 (a)-(d) respectively. The results are compared to a reference solution obtained by Wolf using the boundary-integral method (Wolf 1983). In general, the agreement of the results is satisfactory.



#### 5.2 A Rigid Strip Footing Rests on Layered Soil with Rigid Bedrock

**Figure 5.3.** Dynamic stiffness of rigid strip footing, two layers built-in at its base,  $\xi = 0.001$ ((a) horizontal (b) vertical (c) rocking (d)coupling)

Two layers resting on the rigid rock is studied in this example. The ratios of the shear wave velocities  $c_s^2/c_s^1$  and of the mass densities  $\rho_2/\rho_1$  equal 2 and 1 respectively. The Poisson's ratio is 0.33 for both layers. A rigid strip footing with half-width *b* rests on the surface of the stratified soil (see Fig. 5.1(b)). The dynamic stiffness matrix is defined by the matrix Eq. (5.1). And the corresponding dynamic stiffness coefficients obtained for horizontal, vertical and rocking motion with  $\xi = 0.001$  are

shown in Fig.5.3. The results are compared with the results obtained through the modified SBFEM. In general, the dynamic stiffness coefficients of layered soil resting over rigid base are strongly frequency-dependent. As the figures show, the agreement between the results obtained by the proposed method and modified SBFEM is excellent.

#### 5.3 A Rigid Strip Footing Rests on a Multi-layered Half-space

Layer	$\lambda_r$	$\mu_r$	V <sub>r</sub>	$h_r$	ξ <sub>r</sub>
1	λ	μ	1/4	b	
2	3.0 <i>λ</i>	1.5 <i>µ</i>	1/3	0.5 <i>b</i>	0.05
4	3.75 λ	2.5 μ	0.3	Semi-infinite	

 Table 5.1. The Material Properties Of The Layers And Half-space

The case with two layers resting on a half space is considered. The material properties of the layers and half space are presented in the Table 1. A rigid strip footing with half-width b rests on the surface of the stratified soil (Fig. 5.1(d)). The dynamic stiffness matrix is defined by the matrix Eq. (5.1). The dynamic stiffness coefficients obtained for horizontal, vertical, rocking and coupling between horizontal and rocking motion are shown in Fig.5.4. Until now, because there is no available reference solution about this example to compare with, only our results are presented.



Figure 5.4. Dynamic stiffness of rigid strip footing, three layers on half-space, with damping ((a)spring coefficients (b)damping coefficients)

## 6. CONCLUSIONS

A new approach based on domain-transformation, dual vector representation of wave motion equation and precise integration method is proposed for a rigid strip footing perfectly bonded to the surface of elastic or visoelastic half space. The efficiency and accuracy are verified by four numerical examples provided and comparisons made with the results available in the literature. The approach can handle problem containing multi-layered stratum with less computational effort, which achieving higher accuracy. The difficulties such as overflow inherent in search procedures using transmission matrices can be avoided and no limitation is imposed on the thickness of the layer.

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