# Multi-Component Nonlinear Stochastic Dynamic Analysis Using Tail-Equivalent Linearization Method

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#### SUMMARY:

This paper extends the Tail-Equivalent Linearization Method, TELM, to the case of a nonlinear structure subjected to multiple stochastic excitations. Following the original formulation, Fujumura Der Kiureghian (2007-09), the method employs a discrete representation of the stochastic inputs and the first-order reliability method, FORM. Each component of the Gaussian excitation is expressed as a linear function of standard normal random variables. For a specified response threshold of the nonlinear system, the tail equivalent linear system, TELS, is defined in the standard normal space by matching the "design point" of the equivalent linear and nonlinear responses. This leads to the identification of the TELS in terms of a unit-impulse response function or, equivalently, a frequency response function for each component of the input excitation. The method is demonstrated through its application to an asymmetric one story building with non-degrading hysteretic behavior. The results obtained by TELM are in close agreement with Monte Carlo simulation results.

Keywords: random vibration, tail probability, multi-components excitation, linearization methods.

# **1. INTRODUCTION**

Safety analysis of structures subjected to stochastic excitation, such as earthquake, wind or wave loading, usually requires consideration of nonlinear behaviour. Furthermore, for highly reliable structures, the interest is in the tail region of the response distribution. The conventional equivalent linearization method, ELM, which is often used for such problems, is based on the assumption of Gaussian response, which may not be a good approximation for the tail region. The TELM is a recent alternative based on the First-Order Reliability Method, FORM, which aims to solve this class of problems with good accuracy in the tail region.

In TELM, the input process is discretized and represented by a set of standard normal random variables. Each response threshold defines a limit state surface with the "design point" being the point on the surface that is nearest to the origin. Linearization of the limit-state surface at this point uniquely and non-parametrically defines a linear system, denoted as Tail-Equivalent Linear System, TELS. The tail probability of the TELS response for the specified threshold is equal to the first-order approximation of the tail probability of the nonlinear system response for the same threshold.

Once the TELS is defined for a specific response threshold, methods of linear random vibration analysis are used to compute various response statistics, such as the mean crossing rate and tail probabilities of local and extreme peaks. The method has been developed for application in both time, Fujimura and Der Kiureghian (2007-09), and frequency domain, Garrè and Der Kiureghian (2010), and it has been applied for inelastic structures as well as structures experiencing geometric nonlinearities.

This paper describes an extension of TELM for multi-component stochastic excitations. It is shown that the concept of a single TELS for a given threshold also applies in the case of multiple excitations. The specific case of an inelastic structure subjected to two horizontal ground motion components is

developed. For this purpose, use is made of an existing inelastic constitutive model, for which an efficient method is developed for computing the response gradient that is needed for finding the design point. An example application demonstrates the accuracy of TELM by comparison with "exact" results obtained by Monte Carlo simulation.

# 2. STOCHASTIC REPRESENTATION OF INPUT EXCITATIONS

The stochastic excitation in TELM analysis is represented by a linear combination of a set of basis functions  $\mathbf{s}(t) = [s_1(t), \dots s_n(t)]$  with independent standard normal random coefficients  $\mathbf{u}(t) = [u_1(t), \dots u_n(t)]^T$ :

$$F(t) = \mathbf{s}(t)\mathbf{u} \tag{2.1}$$

In the case of multiple components of excitation, assuming statistical independence between the components, each component is modelled as Eqn. 2.1 and collected in a vector form:

$$F(t) = \begin{bmatrix} F^{(1)}(t) \\ \vdots \\ F^{(m)}(t) \end{bmatrix} = \begin{bmatrix} s^{(1)}(t)u \\ \vdots \\ s^{(m)}(t)u \end{bmatrix}$$
(2.2)

Originally, TELM was developed in the time domain, using filtered white-noise representation of the input excitation, Fujimura and Der Kiureghian (2007). Following this formulation, the *j*th component of the excitation may be defined as

$$F^{(j)}(t) = \left[\eta^{(j)}(t) * W^{(j)}(t)\right] U(t) = \int_0^t \eta^{(j)}(t-\tau) W^{(j)}(\tau) d\tau$$
(2.3)

where \* denotes convolution, U(t) is the unit step function,  $W^{(j)}(\tau)$  is the white-noise process and  $\eta^{(j)}(t-\tau)$  is the impulse response function, IRF, of a stable linear filter. TELM implementation of Eqn.2.3 requires discretization of the time axis. For a selected time step  $\Delta t$  and initial time  $t_0 = 0$ , we approximate the white noise  $W^{(j)}(\tau)$  with the following rectangular wave process

$$\widehat{W}^{(j)}(t) = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} W^{(j)}(\tau) d\tau, \quad t_{i-1} < t < t_i \quad i = 1, 2, \dots n$$
(2.4)

The resulting process is band-limited at frequency  $\overline{\omega} = \pi/\Delta t$  [rad/s] and, for a given white noise spectral density  $S^{(j)}$ , it has variance  $\sigma^{(j)2} = 2\pi S^{(j)}/\Delta t$ . Defining the standard normal random variables as  $\widehat{w}_i^{(j)}(t_i)/\sigma$ , Eqn. 2.4 can be written in the form

$$\widehat{W}^{(j)}(t) = \mathbf{s}^{(j)}(t)\mathbf{u}$$

$$s_i^{(j)}(t) = \sigma^{(j)}, \ t_{i-1} < t < t_i \ i = 1, 2, \dots n$$
(2.5)

$$= 0$$
, otherwise (2.6)

The discrete version of Eqn. 2.3,  $\hat{F}^{(j)}$ , is obtained by replacing  $W^{(j)}(t)$  with  $\widehat{W}^{(j)}(t)$ 

$$\widehat{F}^{(j)}(t) = \left[\eta^{(j)}(t) * \widehat{W}^{(j)}(t)\right] U(t) = s^{(j)}(t) \boldsymbol{u}$$
(2.7)

$$s_{i}^{(j)}(t) = \sigma^{(j)} \int_{t_{i-1}}^{t_{i}} \eta^{(j)}(t-\tau) d\tau, \ t_{i-1} < t < t_{i} \quad i = 1, 2, \dots n$$
(2.8)

$$= 0, t < t_{i-1}$$
(2.9)

An alternative to the above formulation is to perform the discretization in the frequency domain. Following Shinozuka (1975-91) and Deodatis (1991), the basis functions  $s_i^{(j)}(t)$  are selected as the sine and cosine functions. The representation is the canonical Fourier series with random coefficients.

For a selected frequency discretization step  $\Delta \omega$ , and given  $[\omega_0 \dots \omega_m]$  with  $\omega_i = \omega_{i-1} + \Delta \omega$  Eqn. 2.7 is written as:

$$\hat{F}^{(j)}(t) = \sum_{k=1}^{K} \sigma_k^{(j)} \left[ u^{(j)}{}_k \sin(\omega_k t) + \bar{u}^{(j)}{}_k \cos(\omega_k t) \right] = s^{(j)}(t) u^{(j)}$$
(2.10)

$$s_{k}^{(j)}(t) = [s_{k}^{(j)}(t), \dots, s_{K}^{(j)}(t); \bar{s}_{k}^{(j)}(t), \dots, \bar{s}_{K}^{(j)}(t)]$$
(2.11)

$$s_k^{(j)}(t) = \sigma^{(j)} \sin(\omega_k t)$$
 (2.12)

$$\bar{s}_k^{(j)}(t) = \sigma^{(j)} \cos(\omega_k t) \tag{2.13}$$

In contrast to the time domain discretization that leads to a band limited process, the frequency domain discretization leads to a periodic process. For a given  $\Delta\omega$ , Eqn.2.10 produces a process having the period  $T = 2\pi/\Delta\omega$ . The above discretization was first employed in TELM analysis by Garrè and Der Kiureghian (2009) for a marine application involving nonlinear loading and elastic material. In this paper we use this formulation while including inelasticity in the material behaviour.

#### **3. TELM REVIEW**

### 3.1. Time domain TELM

The governing equation of a stable system subject to stochastic input can be written as

$$L[X(t)] = F(t) \tag{3.1}$$

where L[X(t)] is a differential operator. If the system is linear, the response can be obtained by convolving its IRF with the input excitation:

$$X(t) = \left[h(t) * \hat{F}(t)\right] U(t) = \sum_{i=1}^{n} \int_{0}^{t} h(t-\tau) s_i(\tau) d\tau u_i = \boldsymbol{a}(t) \boldsymbol{u}$$
(3.2)

where the h(t) is the IRF of the linear system and  $a_i(t) = \int_0^t h(t-\tau)s_i(\tau)d\tau$ . If the system is nonlinear, a numerical solution can be used to compute the response X(t). Given the representation in Eqn. 2.1, the response X(t) is either an implicit or explicit function of standard normal random variables, i.e.  $X(t, \boldsymbol{u})$ . Given a response threshold of interest x, at a specific time  $t_x$ , the tail probability is defined as  $Pr[x < X(t_x, \boldsymbol{u})]$ . Reliability theory is then used to compute the tail probability by defining a limit state function  $g(x, t_x, \boldsymbol{u}) = x - X(t_x, \boldsymbol{u})$  and rewriting the probability statement as  $Pr[g(x, t_x, \boldsymbol{u}) < 0]$ . In particular TELM employs the first order reliability method, FORM, in which a first order approximation of the probability is computed by defining in the standard normal space the so-called design point  $\boldsymbol{u}^*$  which belongs to the limit state surface  $g(x, t_x, \boldsymbol{u}) = 0$  and has minimum distance from the origin. This distance is known as the reliability index. The significance of this point is described by Koo, Der Kiureghian, and Fujimura (2005). If the system is linear the limit state surface is an hyperplane with gradient  $\boldsymbol{a}(t)$  and the design point and the reliability index are given in closed form as:

$$\boldsymbol{u}^{*}(\mathbf{x}, \mathbf{t}_{\mathbf{x}}) = \frac{x}{\|\boldsymbol{a}(t_{x})\|} \frac{\boldsymbol{a}(t_{x})^{T}}{\|\boldsymbol{a}(t_{x})\|}$$
(3.3)

$$\beta(\mathbf{x}, \mathbf{t}_{\mathbf{x}}) = \frac{x}{\|\boldsymbol{a}(t_{x})\|}$$
(3.4)

moreover the gradient  $a(t_x)$  can be written explicitly in terms of a given design point:

$$a(t_x) = \frac{x}{\|\mathbf{u}^*\|} \frac{\mathbf{u}^{*T}}{\|\mathbf{u}^*\|}$$
(3.5)

and the tail probability has the simple solution

$$Pr[x < X(t_x, \boldsymbol{u})] = \Phi[-\beta] \tag{3.6}$$

where  $\Phi[]$  is the standard normal cumulative probability function. In the more general nonlinear case, first the design point  $u^*$  is computed with a constrained optimization algorithm. The limit-state function is then expanded in Taylor series at the design point:

$$g(x, t_x, u^*) = x - [X(t_x, u^*) + \nabla_{u^*} X(t_x, u^*) \cdot (u - u^*) + \text{h.o.t}]$$
(3.7)

The first order approximation of  $Pr[g(x, t_x, u) < 0]$  is then obtained by keeping the linear terms. This corresponds to approximating the limit-state surface by its tangent hyperplane at the design point. Let a(t) denoting the gradient vector of this hyperplane. Then, for known  $s_i(t)$ , the set of equations

$$\sum_{i=1}^{n} h(t_x - t_j) s_i(t_j) \Delta t = a_i(t_x)$$
(3.8)

can be solved for the IRF h(t) of the TELS represented by the tangent hyperplane. Once the IRF of the TELS is obtained, methods of linear random vibration are used to compute the statistics of the response for the specific threshold, x.

#### 3.2. Frequency domain TELM

The governing equation of a stable linear system in frequency-domain can be written as

$$\mathcal{F}\{L[X(t)]\}(\omega) = \mathcal{F}\{F(t)\}(\omega)$$

$$\mathcal{L}\{\bar{X}(\omega)\} = \bar{F}(\omega)$$
(3.9)
(3.10)

where  $\mathcal{F}$  is the classical Fourier transform operator and  $\overline{X}(\omega)$  and  $\overline{F}(\omega)$  are the Fourier transforms of the response and the input excitation respectively. The steady-state response is obtained as

$$\bar{X}(\omega) = \mathcal{F}\{h(t) * F(t)\}(\omega) = H(\omega) \bar{F}(\omega)$$
(3.11)

where  $H(\omega) = \mathcal{F}{h(t)}(\omega)$  is frequency response function, FRF, of the system. Given the stochastic representation Eqn. 2.10, using Eqn. 3.11, the steady state response of a linear system is obtained in the time domain as:

$$X(t) = \mathcal{F}^{-1}\{\bar{X}(\omega)\}(t) = \sum_{k=1}^{K} \sigma_k |H(\omega_k)| [\sin(\theta_k t)u_k + \cos(\theta_k t)\bar{u}_k] = \boldsymbol{a}(t)\boldsymbol{u}$$
(3.12)  
$$\boldsymbol{a}(t) = [\boldsymbol{a}_k(t) - \boldsymbol{a}_k(t)] = \bar{\boldsymbol{a}}(t) - \bar{\boldsymbol{a}}_k(t) - \bar{\boldsymbol{a}}_k(t) = [\boldsymbol{a}_k(t) - \boldsymbol{a}_k(t)]$$
(3.13)

$$\boldsymbol{a}(t) = [a_k(t), \dots, a_K(t); \bar{a}_k(t), \dots, \bar{a}_K^{(j)}(t)]$$
(3.13)

$$a_k(t) = \sigma_k |H(\omega_k)| \sin(\theta_k t)$$
(3.14)

$$a_k(t) = \sigma_k |H(\omega_k)| \cos(\theta_k t)$$
(3.15)

in which  $\theta_k = \omega_k t + \varphi_k$  and  $|H(\omega_k)|$  and  $\varphi_k$  respectively represent the modulus and the phase of the FRF of the linear system. It easy to show, Garrè and Der Kiureghian (2009), that the following relationships exist between the elements of the gradient vector  $\mathbf{a}(t_x)$  and the FRF:

$$|H(\omega_k)| = \sqrt{a_k (t_x)^2 + \bar{a}_k (t_x)^2} / \sigma_k$$
(3.16)
$$\tan(\theta_k) = -\sigma_k (t_x) / \bar{\sigma}_k (t_x)^2 / \sigma_k$$
(3.17)

$$\tan(\theta_k) = -a_k(t_x)/\bar{a}_k(t_x) \tag{3.17}$$

Given a general nonlinear system and a stochastic input described by (2.10), the design point  $u^*$ , is first determined and the gradient vector of the tangent plane,  $a(t_x)$ , is computed from Eqn. 3.5. The latter in conjunction with Eqn. 3.16 and Eqn. 3.17 uniquely defines the FRF of the TELS. Once the FRF is determined, methods of linear random vibration are used to compute the statistic of the nonlinear response for the specified threshold x.

### 4. MULTI-COMPONENT TELM ANALYSIS

In the case of a multi-component excitation, a specific response quantity of a stable linear system in the time-domain is given by superposition over the input components

$$X(t) = \sum_{j=1}^{m} \left[ h^{(j)}(t) * F^{(j)}(t) \right] U(t)$$
(4.1)

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \int_{0}^{t} h^{(j)}(t-\tau) s_{i}^{(j)}(\tau) d\tau u_{i}^{(j)} = \boldsymbol{a}(t) \boldsymbol{u}$$
(4.2)

where  $\boldsymbol{a}(t) = [\boldsymbol{a}^{(1)}(t), \dots \boldsymbol{a}^{(m)}(t)]$  and  $\boldsymbol{u}(t) = [\boldsymbol{u}^{(1)T}(t), \dots \boldsymbol{u}^{(m)T}(t)]^T$ . Similarly, in the frequency domain the response is given by:

$$\bar{X}(\omega) = \sum_{j=1}^{m} \left[ H^{(j)}(\omega) * \bar{F}^{(j)}(\omega) \right]$$
(4.3)

Given the inputs described in Eqn. 2.10, the steady-state response of the linear system is given by:

$$X(t) = \sum_{j=1}^{m} \sum_{k=1}^{K} \sigma_{k}^{(j)} \left| H^{(j)}(\omega_{k}) \right| \left[ \sin\left(\theta_{k}^{(j)}t\right) u_{k}^{(j)} + \cos\left(\theta_{k}^{(j)}t\right) \bar{u}_{k}^{(j)} \right] = \boldsymbol{a}(t)\boldsymbol{u}$$
(4.4)

where  $\boldsymbol{a}(t) = [\boldsymbol{a}^{(1)}(t), \dots \boldsymbol{a}^{(m)}(t); \boldsymbol{a}^{(1)}(t), \dots \boldsymbol{a}^{(m)}(t)]$  and  $\boldsymbol{u}(t) = [\boldsymbol{u}^{(1)T}(t), \dots \boldsymbol{u}^{(m)T}(t); \boldsymbol{a}^{(1)T}(t), \dots \boldsymbol{u}^{(m)T}(t); \boldsymbol{a}^{(1)T}(t), \dots \boldsymbol{u}^{(m)T}(t); \boldsymbol{a}^{(1)T}(t), \dots \boldsymbol{u}^{(m)T}(t); \boldsymbol{a}^{(1)T}(t), \dots \boldsymbol{u}^{(m)T}(t)]^T$ . In the multi-component excitation case, both the vectors of basis functions  $\boldsymbol{s}(t)$  and  $\boldsymbol{a}(t)$ , as well as the standard normal vector  $\boldsymbol{u}$ , are partitioned in m sub-vectors. For a general nonlinear system, the usual constrained optimization algorithm is employed to compute the complete design point. Then partition (j) represents the design point for the *j*th input component. Given the complete design point  $\boldsymbol{u}^*$ , the complete  $\boldsymbol{a}(t)$  can be computed from Eqn. 3.6 in the standard normal space of dimension  $n \times m$ , where *n* stands for the number of random variables for each discretized excitation (assuming this number to be the same). In the time domain context, the IRF for the *j*th input component is computed by solving:

$$\sum_{l=1}^{n} h^{(j)}(t_x - t_l) \, s_i^{(j)}(t_l) \Delta t = a_i^{(j)}(t_x), \text{ for each } (j)$$
(4.5)

In the frequency domain, the FRF associated with the *j*th input component is:

$$|H(\omega_k)|^{(j)} = \sqrt{(a_k^{(j)})^2(t_x) + (\bar{a}_k^{(j)})^2(t_x)} / \sigma^{(j)}_k$$
(4.6)

$$\tan(\theta_k) = -a_k^{(j)}(t_x)/\bar{a}_k^{(j)}(t_x)$$
(4.7)

With the IRF or FRF for the response quantity of interest with respect to each input excitation component determined, methods of linear random vibration are used to compute response statistics of interest for the specified threshold, x.

# **5. NUMERICAL EXAMPLE**

An asymmetric three degree of freedom, one story bay frame system with non-degrading hysteretic structural members is considered. The in-plane inelastic behaviour of each frames is described through a non-degrading Bouc-Wen model, which is governed by the following set of differential equations:

$$m\ddot{X}^{L}(t) + c\dot{X}^{L}(t) + k[\alpha X^{L}(t) + (1 - \alpha)Z(t)] = F^{L}(t)$$
(5.1)

$$\dot{Z}(t) = -\gamma |\dot{X}(t)| |Z(t)|^{(\bar{n}-1)} Z(t) - \eta |Z(t)|^{\bar{n}} \dot{X}^{L}(t) + A \dot{X}^{L}(t)$$
(5.2)

where  $X^L$  is the local displacement of the frame, the parameter  $\alpha$  controls the degree of hysteresis,

and Z(t) follows the Bouc-Wen hysteresis law (4.2). Frames are assumed to have negligible out ofplane stiffness. The geometry and material properties of the structural system are listed in Table 5.1. The excitation is a bi-directional base motion described by  $F^{(1)}(t) = -m\ddot{U}_g^{(1)}(t)$  and  $F^{(2)}(t) =$  $-m\ddot{U}_{g}^{(2)}(t)$ , where  $\ddot{U}_{g}^{(1)}(t)$  and  $\ddot{U}_{g}^{(2)}(t)$  are statistically independent components of white noise in 1 and 2 directions having spectral density  $S^{(1)} = S^{(2)} = S$ . Given a response quantity of interest X(t), e.g. the horizontal displacement of frame 1,  $X_1^L(t)$ , first the limit state function for a specific threshold x is computed with Newmark integration scheme applied to the governing equation

$$\boldsymbol{M}\ddot{\boldsymbol{X}}(t) + \boldsymbol{C}\dot{\boldsymbol{X}}(t) + \boldsymbol{R}_{in}[\boldsymbol{X}(t)] = \boldsymbol{P}\boldsymbol{F}(t)$$
(5.1)

The mass matrix, M, the damping matrix, C, and the restoring force,  $R_{in}$ , are constructed from the local frame properties while X(t) and F(t) are the vectors storing the global response and the global excitation. The gradient is computed by the DDM algorithm proposed by Zhang and Der Kiureghian (1993). Once the complete design point is obtained, the TELS is identified in the manner described earlier. In the following analysis, the response quantity of interest is the displacement  $X_1^L(t)$  of frame 1. It is of interest to examine the so called design-point excitation and design-point response, which respectively represent the most likely excitation and response for a specific input, response threshold and time. Fig. 5.2 shows the design point bi-directional excitation and the global response of the system for thresholds  $x = 3\sigma_0$  and  $t_x = 6s$  where  $\sigma_0^2 = \pi Sk/c\omega_0^2$  is the root-mean square response of the linear system. Interesting is to notice that the structure achieves the threshold with zero slope in the global direction 2 and the excitation has zero value at time  $t_x$ .

Finally, given a particular discretization scheme, one can directly compute the IRF or the FRF of the TELS for each input of interest. Fig. 5.3 compares the IRFs and FRFs of three TELSs corresponding to the thresholds  $x = \sigma_0$ ,  $x = 2\sigma_0$  and  $x = 5\sigma_0$ , where  $\omega_0/2\pi = 1.88$ [Hz] is the frequency of the translational mode.

| Table 5.1. Structural and Excitation Properties |                    |                    |                |                              |                                  |                  |       |
|---|--------------------|--------------------|----------------|------------------------------|----------------------------------|------------------|-------|
| Structural Properties                           | <i>a</i> [m]       | <i>b</i> [m]       | e[m]           | $\rho$ [kg/m <sup>2</sup> ]  |                                  |                  |       |
| Ĩ   | 20                 | 30                 | 3              | 500                          |                                  |                  |       |
|   |                    |                    | -              |                              |                                  |                  |       |
| <b>Bouc-Wen Properties</b>                      | α                  | c[kNs/m]           | <i>k</i> [kNm] | $\gamma[1/m^{\overline{n}}]$ | $\eta[1/m^{\overline{n}}]$       | ñ                | Α     |
| Frame 1, 2 and 3                                | 0.1                | 1.51E3             | 2.51E3         | $1/2\sigma_0^{\overline{n}}$ | $1/2\sigma_0^{\overline{n}}$     | 3                | 1     |
|   | (1) 2 3            | (2) 2 3            |                |                              |                                  |                  |       |
| Excitation Properties                           | $S^{(1)}[m^2/s^3]$ | $S^{(2)}[m^2/s^3]$ | $t_x[s]$       | $\Delta t[s]$                | $\Delta \omega_{[\Pi_{\sigma}]}$ | ω<br>[μ_7]       | n     |
|   |                    |                    |                |                              | $\frac{1}{2\pi}$                 | $\frac{1}{2\pi}$ |       |
| time domain                                     | 1                  | 1                  | 6              | 0.01                         |                                  | 50               | 2x600 |
| frequency domain                                | 1                  | 1                  | 6              |                              | 0.1                              | 10               | 2x200 |



Figure 5.1 Mathematical models of the structure



Figure 5.2 Design point excitation, and global response to the design point excitation for a threshold  $x = 3\sigma_0$ and a time  $t_x = 6[s]$ 



Figure 5.3 FRFs and IRFs for a specific TELS

# 6. RANDOM VIBRATION ANALYSIS

By repeated TELM analysis, a sequence of design points for an ordered set of thresholds  $x_1 < x_2 < \cdots < x_p$  at a specific time  $t_x$  is obtained. From this sequence, it is possible to directly compute the

first-order approximation of the CDF as:

$$F_{X(t_x)}(x,t_x) \approx 1 - \Phi[-\beta(x,t_x)] \tag{6.1}$$

and the first order approximation of the PDF as:

$$f_{X(t_x)}(x, t_x) \approx \phi[-\beta(x, t_x)]/||a(t_x)||$$
  
(6.2)

where  $\phi[]$  is the standard normal PDF. Fig. 6.1 shows the CDF and the PDF of the displacement response of frame 1 in direction 1. The figure reports TELM results with both time and frequency domain discretization approaches compared with result of crude Monte Carlo simulation with a sample size of 100,000. It is of interest to observe that the probability distribution is not Gaussian. In fact, in the log scale, the tail of the PDF tends to be linear instead of parabolic like for a Gaussian distribution. Moreover the IRF and/or FRF of each TELS can be used for time or frequency-domain analysis to compute other statistics of interest such as mean up crossing rate and first passage probability. For stationary excitations a convenient approach is to compute these statistics is the frequency-domain. For each threshold of interest the *q*th spectral moment of the response is obtained as:

$$\lambda_{q}(x) = 2 \int_{0}^{\infty} \left[ \sum_{i=1}^{m} \left| H^{(j)} \right| S^{(j)}(\omega) \right] \omega^{q} d\omega$$
(6.3)

where  $S^{(j)}(\omega)$  is the power spectral density of the *j*th excitation component. Once the spectral moments are known, classical solutions can be used. For example, for the mean rate of up-crossing rate, we can use the formula  $\nu^+(x) = (2\pi)^{-1} \sqrt{\lambda_2/\lambda_0} \exp(-0.5x^2/\lambda_0)$ , while for the first passage probability the solution proposed by Vanmarcke (1975) can be employed.



Figure 6.1 CDF and PDF of the tail probability for response of frame 1

#### 7. CONCLUSIONS

The extension of the Tail Equivalent Linearization Method for multiple stochastic excitations is developed. Following the original work, the method is based on a discrete representation of each excitation component in terms of standard normal random variables. Two discretization methods are used, the original time-domain version and the later frequency-domain version. In both formulations, the multicomponent excitation and response belong to the  $\mathbb{R}^{mXn}$  standard normal space. In this space the equivalent linear system is defined by matching its design point with that of the nonlinear system for a specific threshold. For each TELS, the IRF or FRF for each component is determined. Once the

IRFs or the FRFs are determined, linear random vibration analysis is employed to determine the statistics of interest.

The multicomponent TELM analysis is a straightforward extension of the original TELM formulation, which offers a series of advantages to the conventional ELM or simulation methods. First, TELM is able to capture the non-Gaussian distribution of the nonlinear response. Second, TELM is not a parametric method and does not requires the selection of a linear model or a set of model parameters as in ELM. The advantage over the classical simulation methods lies in its efficiency. In fact TELM is able to accurately predict small tail probability values which are infeasible with classical simulation methods. The efficiency of TELM lies in efficient computation of the gradient in the improved HLRF algorithm. The number of random variables employed thus is crucial. Both versions, the time-domain and the frequency-domain, give similar and accurate results, even though, the two discretizations have different frequency contents. In particular, for time step  $\Delta t = 0.01$ [s], the time domain discretization has a cut off frequency of 50[Hz], while the frequency-domain has a cut off frequency of 10[Hz]. The similarity of the results suggests that the influence of high-frequency content of the excitation is negligible. In the time-domain formulation, the frequency content is dictated by the numerical analysis discretization step and most of the random varibles are used to describe non-important high frequencies. In this context, the frequency-domain formulation offers a much more efficient way to employ TELM. On the other hand, the frequency rate  $\Delta \omega$  dictates the periodicity of the excitation and response, e.g. for  $\Delta \omega/2\pi = 0.1$ [Hz], the periodicity T = 10[s] which necessitates a selection of  $t_x \leq T[s]$ . Higher values of  $t_x$  requires smaller  $\Delta \omega$  and larger number of random variables.

The drawbacks of multicomponent TELM analysis are the same as those for the single component TELM analysis. In particular, TELM requires considerably more analysis than ELM if one is interested only in the first and second moments. For second-moment analysis, ELM is the appropriate method while TELM is effective for accurate estimation of tail probabilities. Moreover because TELM is based on FORM, there is no measure of the error due to the approximation and thus the accuracy of TELM cannot be estimated in advance. The numerical investigation shows the importance of considering both excitation components in computing the statistics of the response for coupled systems.

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