On Two Dimensional Liquefaction Analysis

N. Yoshida
Tohoku Gakuin University, Japan

Y. Ohya
Port and Airport Research Institute, Japan

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SUMMARY:
A new formulation, named a simple two-dimensional formulation, is presented for the two-dimensional liquefaction analysis. Three-dimensional constitutive model is required even in two-dimensional analysis in the plane strain condition, but two-dimensional constitutive model can be used in the new formulation, which makes analysis very simple. Difference between them appears in the definition of mean stress; it is average of two normal stresses. The formulation is made by keeping the S wave and P wave velocities are same with that in the three-dimensional analysis. Then the bulk modulus becomes different from the ordinary definition. In addition, several differences appear between elastic constants. A case study is made at the Akita port damaged during the 1983 Nihonkai-chubu earthquake and showed significant difference of excess porewater pressure generation compared with the plane strain analysis.

Keywords: Liquefaction, two-dimensional analysis, Poisson's ratio, Bulk modulus

1. INTRODUCTION

Since the space is originally three-dimension, an analysis of the ground is to be ideally made in a three-dimensional space. There are several difficulties, however, in making the three-dimensional analysis. For example, computer power is not sufficient in the practical use, and the soil data for the three-dimensional analysis is difficult to obtain because it is costly. Therefore, a one-dimensional analysis is the most popular method and a two-dimensional analysis is carried out to a special case such as an important structure.

The plane strain condition and the plane stress condition are well known assumption to make the two-dimensional analysis. These methods work to reduce the number of unknown variables, but it requires a three-dimensional constitutive model because the normal stress perpendicular to the plane of the analysis exists in the plane strain condition, and the normal strain perpendicular to the plane of the analysis exists in the plane stress condition. The plane strain condition is relevant in the two-dimensional analysis of the ground among these two two-dimensional formulations; therefore it is focused on in this paper.

The normal stress perpendicular to the plane appears in two places.

The one is in the yield condition, but it can be escaped. The Mohr-Coulomb yield criterion, for example, is frequently used in the analysis of soil, which uses three normal stresses. In the practice, however, the stress perpendicular to the plane of analysis is frequently neglected as it usually becomes middle principal stress. Therefore, a three-dimensional constitutive model is not necessary if this normal stress is not considered in the constitutive model.

Another place is evaluation of the mean normal stress, which is necessary to evaluate the strength and elastic moduli based on the principle of effective stress. Therefore, three-dimensional constitutive model is necessary in the liquefaction analysis in which evaluation of the mean normal stress is necessary.

Using a three-dimensional constitutive model is, however, troublesome work in the engineering practice and use of two-dimensional constitutive model is more preferable. It can become possible to evaluate the mean stress as sum of two normal stresses in the plane of consideration. In other word, if we consider the pure two-dimensional plane, we can use a two-dimensional constitutive model. This condition is named a simple two-dimensional condition in this paper, and the formulation in this condition is presented with a case history of the liquefaction analysis of the damaged quay wall during the 1983 Nihonkai-chubu earthquake.

2. THREE DIMENSIONAL FORMULATION

The stress-strain relationships are usually defined as follows by using the Young's modulus E and the Poisson's ratio ν.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1-2\nu)(1+\nu)}[D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{2.1}$$

where coefficient matrix $[D]$ is expressed as

$$[D] = \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{2.2}$$

Here, the shear strain is not a tensor strain but an engineering strain. This equation is used as the fundamental equation in this paper. In the analysis of the ground, this equation is frequently expressed by using the bulk modulus K and the shear modulus G, because both the volumetric change and the shear deformation are primary interest in the geometrical analysis. This equation is written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} K+\frac{4G}{3} & K-\frac{2G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K+\frac{4G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K-\frac{2G}{3} & K+\frac{4G}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{2.3}$$

In the liquefaction analysis, quantities related to the volumetric change (confining stress or mean stress σ_m and volumetric strain ε_v), the one-dimensional rebound modulus B, the Poisson's ratio ν, and the coefficient of earth pressure at rest K_0 are important. Here, as all the stresses in this paper is effective stresses, prime which is usually used to distinguish the effective stress from the total stress is not used. In addition, K_0 is defined under the one-dimensional compression loading of the elastic material. They are defines as follows:

$$\sigma_m = (\sigma_x + \sigma_y + \sigma_z)/3 \tag{2.4}$$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{2.5}$$

$$\sigma_m = (\sigma_x + \sigma_y + \sigma_z)/3 = K\varepsilon_v = K(\varepsilon_x + \varepsilon_y + \varepsilon_z) \tag{2.6}$$

$$B = K + 4G/3 \tag{2.7}$$

$$\nu = \frac{3K - 2G}{2(3K + G)} \tag{2.8}$$

$$K_0 = \frac{K - 2G/3}{K + 4G/3} = \frac{\nu}{1-\nu} \tag{2.9}$$

Here, one-dimensional rebound modulus is important because it is directly related to the P wave velocity V_p as

$$V_p = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{K + 4G/3}{\rho}} \tag{2.10}$$

where ρ denotes density.

3. TWO-DIMENSIONAL FORMULATION

When considering a constitutive model in the two- dimensional space, the volumetric change characteristics is to be expressed as

$$\sigma_m = (\sigma_x + \sigma_y)/2 = \tilde{K}\varepsilon_v = \tilde{K}(\varepsilon_x + \varepsilon_y) \tag{3.1}$$

This equation is derived by setting σ_z=0 and ε_z=0, and replace 3 into 2 in Eq. (2.6), but it is impossible because both stress and strain cannot be specified at the same time. Under the plane strain condition, for example, σ_z is not zero as ε_z is zero. This indicates that the framework of the constitutive model is different from the three-dimensional space when Eq. (3.1) is used. In order to emphasize it, tilde,"~", is put above the variable.

Then the problem occurs how to obtain the elastic moduli from the conventional elastic moduli. It is clear that the bulk modulus K defined in the three-dimensional space cannot be used instead of $\tilde{K}$. When deriving the conversion equation, it is important to consider what quantities are same both in the three-dimensional space and in the two-dimensional space. There are three elastic moduli that are frequently used in the engineering practice, which are E, K, and G. Among them, we assume that E and G are the same in the three- and two-dimensional analyses, in order to keep the S wave and the P wave velocities are same in two formulations.

Three strains are set zero in the plane strain condition because displacement in the direction perpendicular to the plane of interest does not occur, i.e., ε_z=γ_{yz}=γ_{zx}=0. Relationships that τ_{yz}=τ_{zx}=0 are automatically derived. Then the stress-strain relationships yields,

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} K+4G/3 & K-2G/3 & 0 \\ K-2G/3 & K+4G/3 & 0 \\ K-2G/3 & K-2G/3 & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{3.2}
$$

Since σ_z is not zero but changes in the plane strain condition, the constitutive model is essentially to be defined in the three-dimensional space.

The stress σ_z is included in the confining stress and the yield condition. Under the assumption that the middle principal stress does not affect the yielding, however, σ_z does not appear in the yield condition. This assumption is used in many constitutive models, and is accepted widely. Therefore, only the difference between Eqs. (2.6) and (3.1) remains.

The normal stresses σ_x and σ_y are expressed from Eq. (2.1) as

$$
\begin{aligned}
\sigma_x &= \left\{ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \right\} \varepsilon_x + \frac{E\nu}{(1+\nu)(1-2\nu)} \varepsilon_y \\
\sigma_y &= \left\{ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \right\} \varepsilon_y + \frac{E\nu}{(1+\nu)(1-2\nu)} \varepsilon_x
\end{aligned} \tag{3.3}
$$

Then we obtain sum of the normal stresses as

$$
\sigma_x + \sigma_y = \left\{ \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} + \frac{E\nu}{(1+\nu)(1-2\nu)} \right\} (\varepsilon_x + \varepsilon_y) = \frac{E}{(1+\nu)(1-2\nu)} (\varepsilon_x + \varepsilon_y) \tag{3.4}
$$

The relation among ν, E, and G is known to be expressed as

$$
G = \frac{E}{2(1+\nu)} \tag{3.5}
$$

in the three-dimensional space. These terms including the Poisson's ratio yields

$$
\begin{aligned}
&\nu = \frac{E-2G}{2G}, \\
&1+\nu = 1 + \frac{E-2G}{2G} = \frac{E}{2G}, \\
&1-\nu = 1 - \frac{E-2G}{2G} = \frac{4G-E}{2G}, \\
&1-2\nu = 1 - \frac{E-2G}{G} = \frac{3G-E}{G}
\end{aligned} \tag{3.6}
$$

Substitution of this equation into Eq. (3.3) results in

$$\sigma_x = \left\{ \frac{E\dfrac{4G-E}{2G}}{\dfrac{E}{2G}\dfrac{3G-E}{G}} \right\} \varepsilon_x + \frac{E\dfrac{E-2G}{2G}}{\dfrac{E}{2G}\dfrac{3G-E}{G}} \varepsilon_y = \frac{G(4G-E)}{3G-E}\varepsilon_x + \frac{G(E-2G)}{3G-E}\varepsilon_y$$
$$\sigma_y = \frac{G(4G-E)}{3G-E}\varepsilon_y + \frac{G(E-2G)}{3G-E}\varepsilon_x \tag{3.7}$$

Therefore we obtain the sum of the normal stresses as

$$\sigma_x + \sigma_y = \left\{ \frac{G(4G-E)}{3G-E} + \frac{G(E-2G)}{3G-E} \right\} (\varepsilon_x + \varepsilon_y) = \frac{2G^2}{3G-E}(\varepsilon_x + \varepsilon_y) \tag{3.8}$$

This is the fundamental equation in the simple two-dimensional formulation.
The bulk modulus $\tilde{K}$ in the simple two-dimensional formulation is obtained by comparing Eqs. (3.1) and (3.8) as

$$\tilde{K} = \frac{G^2}{3G-E} = \frac{G^2}{3G - \dfrac{9KG}{3K+G}} = K + \frac{1}{3}G \tag{3.9}$$

and the Young's modulus is obtained by means of G and $\tilde{K}$ as

$$E = \frac{G(3\tilde{K}-G)}{\tilde{K}} \tag{3.10}$$

Finally we obtain the coefficient of the normal stress-strain relationships in Eq. (3.8) as

$$\frac{G(4G-E)}{3G-E} = \frac{G(4G - \dfrac{G(3\tilde{K}-G)}{\tilde{K}})}{3G - \dfrac{G(3\tilde{K}-G)}{\tilde{K}}} = \frac{4\tilde{K} - (3\tilde{K}-G)}{1} = \tilde{K} + G$$
$$\frac{G(E-2G)}{3G-E} = \frac{G(\dfrac{G(3\tilde{K}-G)}{\tilde{K}} - 2G)}{3G - \dfrac{G(3\tilde{K}-G)}{\tilde{K}}} = \frac{G(3\tilde{K}-G) - 2G\tilde{K}}{3\tilde{K} - 3\tilde{K} + G} = \tilde{K} - G \tag{3.11}$$

Then the stress-strain relationships are obtained as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \tilde{K}+G & \tilde{K}-G & 0 \\ \tilde{K}-G & \tilde{K}+G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{3.12}$$

The one-dimensional rebound modulus B is obtained as

$$\tilde{B} = \tilde{K} + G = K + \frac{4}{3}G = B \tag{3.13}$$

This definition is same with the three-dimensional analysis. It means that the P wave velocity is same with that in three-dimensional analysis, which is the reason why we used the same Young's modulus E for both three-dimensional and two-dimensional analyses.

The normal stress is written as

$$\sigma_x = (\tilde{K} + G)\varepsilon_x + (\tilde{K} - G)\varepsilon_y$$
$$\sigma_y = (\tilde{K} - G)\varepsilon_x + (\tilde{K} + G)\varepsilon_y \tag{3.14}$$

Here, the following equation is obtained by putting σ_x=0

$$\varepsilon_x = -\frac{\tilde{K} - G}{4\tilde{K}G}\sigma_y$$
$$\varepsilon_y = -\frac{\tilde{K} + G}{\tilde{K} - G}\varepsilon_x = \frac{\tilde{K} + G}{4\tilde{K}G}\sigma_y \tag{3.15}$$

Therefore, the Poisson's ratio for the simple two-dimensional formulation yields

$$\tilde{\nu} = -\frac{\varepsilon_x}{\varepsilon_y} = \frac{\tilde{K} - G}{\tilde{K} + G} = \frac{K - \frac{2}{3}G}{K + \frac{4}{3}G} = \frac{\nu}{1 - \nu} \tag{3.16}$$

On the other hand, the following equation is obtained by putting ε_x=0,

$$\sigma_x = (\tilde{K} - G)\varepsilon_y,\ \sigma_y = (\tilde{K} + G)\varepsilon_y \tag{3.17}$$

Therefore, the coefficient of earth pressure at rest K_0 yields

$$K_0 = \tilde{\nu} \tag{3.18}$$

4. AN EXHIBITING EXAMPLE

The Akita port located at the O-hama district, Akita city, which was damaged during the 1983 Nihonkai-chubu earthquake, is analysed to show the difference between the plane strain condition and the simple two-dimensional condition. **Figure 1** shows cross-section of the port (Iai and Kameoka, 1993).

The constitutive model is composed with shear deformation, volumetric change characteristics and stress-dilatancy model. The multi-spring model (Yamazaki et al, 1985) is used for shear deformation, and incremental elastic model is used for volumetric change characteristics. Finally, a generalized stress-dilatancy model is used to consider dilatancy effect (Ohya et al., 2009).

Figure 2 shows modeling of the structure in **Figure 1**. This model is once analyzed by Ohya (Ohya, 2009). The bulk modulus K is evaluated under the plane strain condition as

$$K = \frac{2G(1+\nu)}{3(1-2\nu)} \tag{11}$$

When the theory based on the simple two-dimension, this equation yields

$$K = \frac{G}{1-2\nu} \tag{12}$$

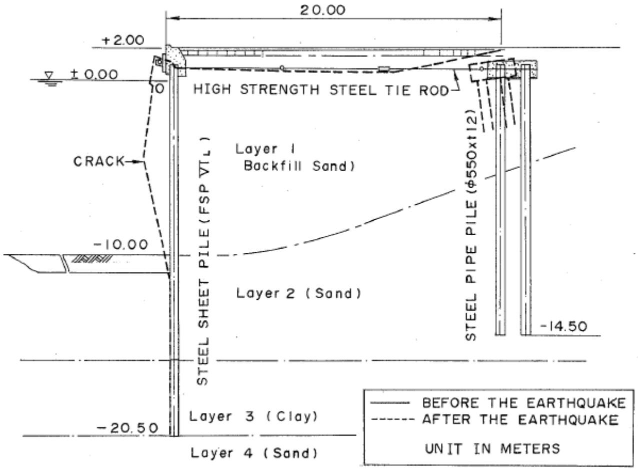


Figure 1. Cross-section of Akita port and damage caused by earthquake

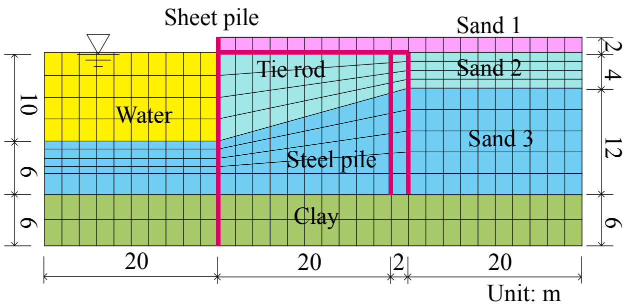


Figure 2. FEM mesh

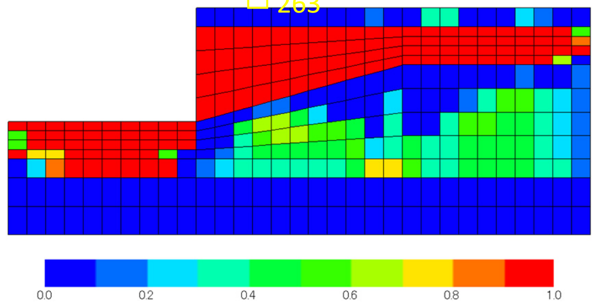
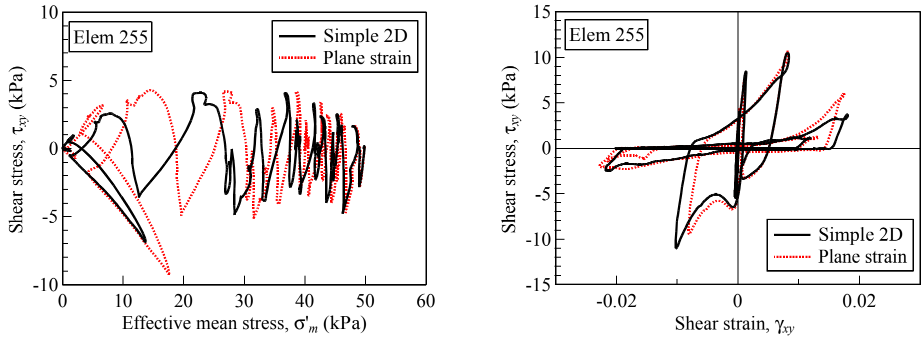


Figure 3. Excess porewater pressure ratio at the end of analysis

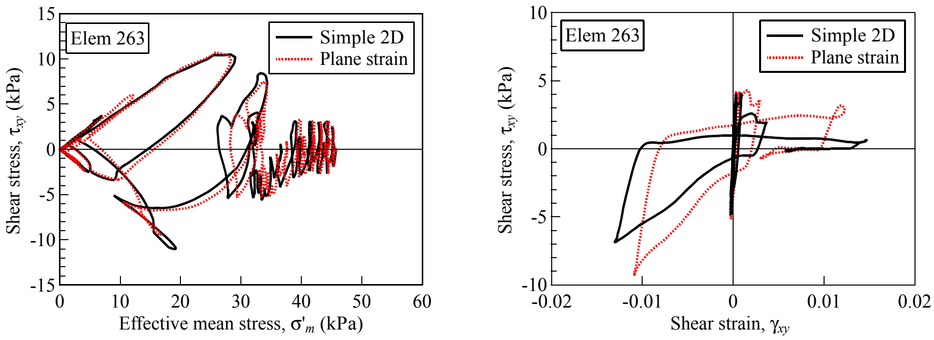
The sheet pile quay wall, steel pile and tie rod are modelled into beam element. The intersection between the pile and soil element is modelled so that horizontal displacements are same, but vertical displacement is free to each other. Excess reduced integration scheme (one-point Gauss integration) is used for water so that it resists only volume change (Ohya et al., 2010). On the other hand, improved integration scheme is used for soil element (Ohya and Yoshida, 2008) in order to escape from volume locking.

Mechanical properties used in the analysis are summarized in **Table 1**. The bulk modulus under simple two-dimensional condition is about 1.15 times larger than that under the three dimensional or plane strain condition.

Figure 3 shows excess porewater pressure ratio at the end of the analysis. All fill below the water table liquefied. In addition, Original sand ground just in front of the quay wall also liquefied, which makes quay wall to move toward the sea easier.



(a) Element 255 (near steel pile)



(b) Element 263 (back of quay wall)

Figure 4. Typical behavior of elements

Table 1. Mechanical properties

	Sand 1	Sand 2	Sand 3	Clay
ρ (t/m^3)	1.8	2.0	2.0	1.7
n	0.8	0.8	0.8	0.8
G_0 (kN/m^2)	33800	33800	72200	74970
$\tilde{K}$ (kN/m^2)	84500	84500	180500	187430
K (kN/m^2)	73230	73230	156430	156430
σ_{m0} (kN/m^2)	50	50	110	140
$\sin\phi_f$	0.602	0.602	0.656	0.629
h_{max}	0.3	0.3	0.3	0.3
ν	0.3	0.3	0.3	0.5

Note) Sand 1: Fill above the water table; Sand 2: Fill below the water table; Sand 3: Sand; Clay: Holocene clay; ρ: density; n: porosity; G_0: elastic shear modulus; $\tilde{K}$: bulk modulus under simple shear condition; K: bulk modulus; σ_{m0}: reference confining stress; ϕ_f: internal friction angle; h_{max}: maximum damping ratio; ν: Poisson's ratio

Figure 4 shows the behavior of two elements, which locates near the steep pile (element 255) and backfill ground (element 263). Both elements completely liquefy. The behaviors are quite similar between the simple two-dimensional analysis and a plane strain analysis. However, it is also noted that the simple two-dimensional analysis liquefies earlier than the plane strain analysis.

5. DISCUSSION

The stress-strain relation derived in this paper does not include the third stress σ_z and the third strain ε_z, because formulation is made in the pure two-dimensional space. The final form of the stress-strain relation is obtained as Eq. (3.12). This formulation is called a simple two-dimensional formulation in order to distinguish from other formulations. As G is kept constant between the three- and two-dimensional analyses, Lame's constant λ changes keeping μ (=G) constant if the stress-strain relationships are written by means of the Lame's constants. Among the elastic moduli frequently used in the engineering practice, only the bulk modulus and the Poisson's ratio are different from the conventional ones, which can be evaluated from the conventional elastic moduli as shown in **Table 2**. The S wave and the P wave velocities are hold in the new formulation. However, the bulk modulus includes shear modulus; it changes according to the nonlinear shear deformation of soil in exact sense. In the practical situation, however, this change may be neglected in order to make the analysis simple when the change of the shear modulus is not significant.

In the engineering practice, shear modulus is usually obtained with easy test such as, for example, the PS logging. On the other hand, the bulk modulus is difficult to evaluate. Therefore, it is frequently obtained by assuming a relevant Poisson's ratio and the bulk modulus is evaluated from the Poisson's ratio and the shear modulus. Here, the Poisson's ratio 1/3 is frequently assumed because the coefficient of earth pressure at rest becomes 0.5. However, in the simple two-dimensional formulation, the Poisson's ratio is to be 0.5 in order to make the coefficient of earth pressure at rest to be 0.5. The Poisson's ratio is close to 0.5 under nearly no volumetric change condition in the three-dimensional analysis whereas it is 1.0 under the simple two-dimensional analysis.

In the computer program in making the two-dimensional analysis, the normal stress perpendicular to the plane of analysis may not be output even when the plane strain condition is used. It means that the user cannot recognize whether the analysis is made by the plane strain condition or by the simple two-dimensional condition. The method to evaluate the elastic moduli is different between two formulations. Therefore, the engineer needs to check the formulation that he is going to use.
Since the bulk modulus in the simple two-dimensional analysis is larger than that in the three dimensional analysis, excess porewater generates earlier in the simple two-dimensional analysis. This can be probed through the case history.

Table 2. Conversion of elastic modulus for simplified two-dimensional formulation

	E, G	E, ν	E, K	G, ν	G, K	ν, K
$\tilde{K}$	$\dfrac{G^2}{3G-E}$	$\dfrac{E}{2(1+\nu)(1-2\nu)}$	$\dfrac{9K^2}{9K-E}$	$\dfrac{G}{1-2\nu}$	$K+\dfrac{1}{3}G$	$\dfrac{3K}{2(1+\nu)}$
$\tilde{\nu}$	$\dfrac{E-2G}{4G-E}$	$\dfrac{\nu}{1-\nu}$	$\dfrac{3K-E}{3K+E}$	$\dfrac{\nu}{1-\nu}$	$\dfrac{3K-2G}{3K+4G}$	$\dfrac{\nu}{1-\nu}$

6. CONCLUDING REMARKS

A new formulation named a simple two-dimensional formulation is presented. This formulation is necessary when the effective confining stress is evaluated from the two normal stresses working in the plane of analysis. It is essential for soil to consider the effective stress dependency of the elastic moduli as well as shear strength because of the effective stress principle. Therefore, three-dimensional formulation becomes necessary in the constitutive models that use three normal stresses in defining the effective confining stress. On the other hand, the simple two-dimensional analysis presented here makes the development of the constitutive models very simple as it uses only two normal stresses working in the plane of the analysis.

The S wave velocity and the P wave velocity are same with that in the three-dimensional analysis in this formulation, but the bulk modulus includes shear modulus only in this formulation. There are some differences between two formulations. Therefore, some notes are necessary to use this formulation.

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