THE CULLMAN ELLIPSE AS A STRATEGIC TOOL FOR SEISMIC GRADING OF STRUCTURAL CONFIGURATIONS

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ABSTRACT

Using Structural Analysis Programs and modeling a building the same way as to analyze it, one can generate Cullman Ellipses (CE) floor by floor (rigid diaphragms), for horizontal forces. (CE) are only one of many quadratic and quartic forms associated with linear equation systems pertaining to structures. By applying torsional moments and rotating loads to each floor in succession, one can generate a chain of affine ellipses ending in a Cullman Ellipse. Such Ellipses can indicate, using different graphical ways, the torsional amplification factors corresponding to each frame. Torsional kernels can also be derived from them. After processing several building configurations one can conclude that it is not the shape of the floor plans what makes a building "irregular", but the framing configuration what matters. A quartic, a "Booth Curve" (the inverse of an ellipse), changes shape when irregularities appear. It is a quick, visual tool for grading.

Keywords: Cullman Ellipse, Central kernel, Quadratic forms, Torsion in Buildings, Torsional amplification factors.

1. INTRODUCTION

Cullman's Ellipse, (Belluzzi, 1969) the old friend of long gone engineers, is only one of the many quadratic forms associated with linear structural analysis. Structural analysis, in itself, is no more than a large system of linear equations (Lanczos, 1988). Because of this, there are many other associated quadratic forms which apparently never have been of any use in structural engineering. They can all be derived from a single quadratic form, the deflections ellipse, resulting from an affine transform (Aleksandrov, 1999), a constant rotating force, its locus being a circle of forces, its response an ellipse of displacements, or either its geometrical inverse, the Booth's Ovals or "Beans", so called because of its form. With another series of five successive affine transforms, we can get a series of six Ellipses. The ellipses of: 1) deflections; 2) flexibility; 3) rigidity; 4) radius of gyration squared; 5) radius of gyration; and 6) Cullman's Ellipse (The transpose of the Ellipse of radius of gyration). (Paz, Peña, 2011)

2. EXPERIMENTAL PROCEDURES

All six ellipses contain and use two invariants of any structural configuration: the two principal and mutually perpendicular axes (having constant orientations and variable magnitudes). The axes of these ellipses become the only pieces of data necessary to draw the six successive ellipses (all with variable magnitudes, some oblate, some other prolate, keeping the principal axes directions constant). It is also possible to plot them knowing only two conjugate diameters which appear when instead of a "rotating load" we take only two perpendicular axes of loading, with arbitrary orientation, as it is done for most projects, when one just follows the codes, which usually require only two arbitrary perpendicular directions of loading to perform the structural analyses. The main author of this paper began working with the torsional behavior of buildings some years ago, (Paparoni 1991, Paparoni & Hummelgens 2000) and found a behavioral analogy between the structural response of framed spatial structures under the simultaneous actions of translation and torsion and the well known behavior of beam sections under axial loads and flexural moments (Or eccentrically applied normal forces). There are many formulations used in this last case, we chose the well known Rankine formulation $s=(P/A)*(1 \pm ec/r^2)$. The second term, in brackets, represents a non-dimensional parameter, which can be interpreted as an "stress amplification factor", but we can say that it is also a polarity relation, geometrically speaking (Projective geometry). Analytically it is a relation; the product of two numbers equals the square of a third number, accompanied by a certain conic, an ellipse for linear and stable structures. The Rankine formula locates the rotation axes (neutral axes) in the same plane of the section. When $e*c=r^2$ the amplification factor value is two. The Cullman ellipse is a quadratic form which, especially in the past, has been associated with a given structure, member or section. It is determined by a long series of manual calculations and normally, in structural textbooks of the era, to solve planar structures, where the three parameters, r, c and e (a radius of gyration, a distance from the center of the ellipse to an edge or corner, or point, and the eccentricity of a force, respectively) are all contained in the same plane of the ellipse.. This ellipse establishes, for a certain structural problem, a relationship between a force vector, the ellipse itself and their corresponding instantaneous centre of rotation, as a sort of kinematic algorithm, (Belluzzi, 1969) This arrangement changes if one wants to apply the same relation of parameters to a building floor, considered as a plane diaphragm. The force is now coplanar with the diaphragm; the axis of rotation is now perpendicular to the diaphragm; e and c are coplanar with the diaphragm. This demands the transposition (or 90 degrees of rotation) for the radius of gyration ellipse to become the classical Cullman Ellipse. (e, must be perpendicular to the force).

There is another difference, in a beam section we have flexural moments and axial forces, in a floor diaphragm we have shearing forces (coplanar with the diaphragm) and torsional moments. In the Rankine formula the section is described with three geometrical parameters only (A, Ixx, Ixy), in the diaphragm we have two mechanical parameters, a Linear Rigidity and a Torsional rigidity, and two geometrical parameters, e and c. The Cullman Ellipse has the same kinematic properties in both cases (beam section or diaphragm), specifically, the instantaneous center of rotation in the plane of the diaphragm is the equivalent of the rotational axes (neutral axes) in the beam section. The polarity relation is different if we take the traces of frames or walls as the equivalents of the beam section boundaries. All this can be simplified and expressed with relatively short formulas, but two pieces of information are rather difficult to have at hand immediately, the rigidity centers position , and the principal axes of the ellipse of displacements. Some structural programs give the center, but not the axes, floor by floor.

We need these two pieces of data to start with the process. We decided to use the perturbation method as a starting point, using single arbitrary "round value" loads, one floor at a time, applying each load every 10 degrees of azimuth, and getting the respective deflections ∂x and ∂y, which is the customary form in most structural programs. Going this way we could get the relations we were looking for without having to develop the necessary new algorithms, by using the single mentioned loadings. We had a confirmation, ∂x and ∂y plotted together, in cartesian coordinates, floor by floor have, as geometrical loci, perfect ellipses.

But when we tried to plot it in Polar Coordinates, taking as angle, each azimuth of every rotating load, and as radius the resultant (real) deflection $\sqrt{(\partial x2 + \partial y2)}$ we got a curve not usually seen on structural analysis books, a Booth's Oval or "Bean" or "Dog bone", as a the deflections locus. This curve is very interesting. It envelopes an ellipse, having the same principal axes and four points in common, such points determine the magnitudes and directions of the principal axes of deformation, floor by floor. These axes determine the ellipse of deflections mentioned before. It also makes clear that a building, under multidirectional loadings (as it happens with seismic or wind forces) does not respond in a simple way, with the displacements being collinear with the applied forces; instead, most real deflections tend to concentrate towards the most flexible direction of the building, not at all as we tend to imagine it according to the customary and old structural convention of using only two directions of loading. Another interesting fact was to find that the Booth's ovals or "beans" are the geometrical inverses of the inscribed ellipses contained in them (relatively rotated 90°). Once we get the necessary information about the positions of the rigidity centers and the principal axes magnitudes and azimuths, the successive ellipses come naturally in succession, being relatively simple successive affine transforms. Later in the effort, we found that one can go straight to simple formulas from the initial deflections ellipse to Cullman's Ellipses (floor by floor). We should add that in order to make these calculations, we also need the torsional floor rigidities at the beginning of the process. These are obtained by applying constant moments and calculating their respective torsional rotations. This very same process can identify the rigidity centers and this particular verification must be performed at the beginning of the described process to be able to apply the loads right at the rigidity centers. As one can see, the initial processes developed in this research to find the Cullman ellipse corresponding to each floor, are rather long and complicated looking. When the calculations were completed, we compared our results using the old faithful Cullman ellipse with those calculated using a structural analysis program. Using several different structural configurations, we found very small discrepancies when comparing the results of the ellipses with the results of the new fashionable structural analysis programs. So, old and new tools were successfully combined.

We are now in the process of simplifying the procedures, applying the results to existing buildings (rather than model buildings), and trying to adapt the process as an optimization tool or a grading tool. Up to now, we are convinced about the validity this methodology. We can get applicable results, we have seen that the normal, symmetric, regular, with perpendicular framing directions respond in rather simple ways, and the code prescriptions can be applied as they are. We have also found that the "extreme configurations" where form precedes over function, can have tremendously complex behaviors, but also, if apply the methods proposed here one can come to acceptable solutions, using the knowledge gained by the old faithful Cullman Ellipse, in another guise. The big advantage is that one does not work with mountains of numbers, but only with geometrical figures. The Cullman Ellipses can be superimposed over the Technical Drawings of each building, at the same scale, or can be transformed into "level curves" showing the interaction between Lateral loadings and Axial Twisting of Buildings, taking into account also their Torsional Properties.

3. PRACTICAL USE OF THESE TOOLS:

The long sequence of 6 ellipses was followed in order to comply with the initial hypothesis about the invariance of principal directions in all of them, and to get clear ideas about the changing shapes and orientations of the successive ellipses. The circular rotating loading generates in succession, first two similarly oriented ellipses of deflections and flexibility, then, another two, the ellipse of rigidities (or linear stiffness's), the ellipse of rho^2 (radiuses of gyration squared) and followed by the ellipse of rhos, both mutually angled with a 90° difference from the preceding ones. Then comes the Cullman ellipse, with orientation and shapes akin to the ellipse of rigidities and 90° off of the initial ellipse of displacements. These ellipses, expressed as matrices, must contain a tremendous amount of information, in the form of functions or shapes. Each set of ellipses is equivalent, for the whole building, to two geometrically orthogonal matrices with the information pertinent to these directions only. The information about the other intermediate directions is contained in the infinitude of points of the geometrical loci of each ellipse. If we want to use easier formulas, for the Cullman ellipse only, we propose the following expression, which gives the magnitudes of the principal axes (the directions can be inferred from the ellipse of deflections, which is always needed as a reference.

See chapter six

4. GRADING STRUCTURES:

How do we use the Cullman ellipse to grade Structural Configurations: The ellipse can be used in two guises, as a single ellipse, superimposed on the building drawings, at their same scales. Then it can be interpreted as "level curves" (a set of ellipses, centrally homothetic with the initial ellipse indicating several values of the "amplification factor") beginning with 1 (one) right at the rigidity center, 2 at the boundary of the original Cullman ellipse, then less than 2 inside the original ellipse, more than 2, when it is outside the ellipse contour. Then we can say the following: All frames (their traces in the diaphragm), inside each one of the ellipses will have amplification factors less than the indicated in each “level curve” when an horizontal force, coplanar with the diaphragm is secant to that curve.

5. ILLUSTRATIONS

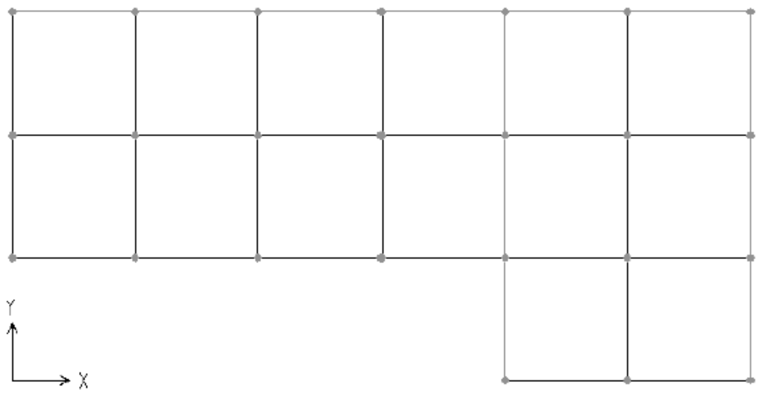

Figure 1. Asymmetrical frame

The following frame is asymmetrical; the dimensions of its beams are 0.30 x 0.50 m, while its columns are 0.3x0.3 m. The building has 9 floors and each floor height is 3.00 m

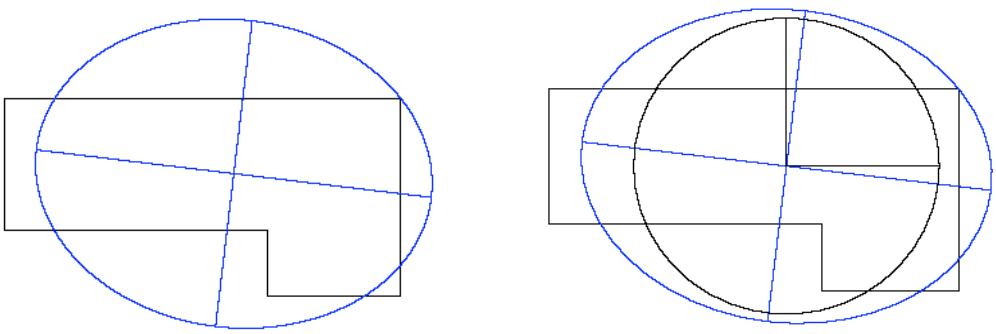

Figure 2. Comparison between the Cullman Ellipses in the first and last floor.

The outer ellipse corresponds to the last floor, while the inner ellipse corresponds to the first floor. This graphic illustrates that there is a change in the Cullman ellipse’s dimensions and axis if the building height is increased, whether the frame is symmetrical or not.

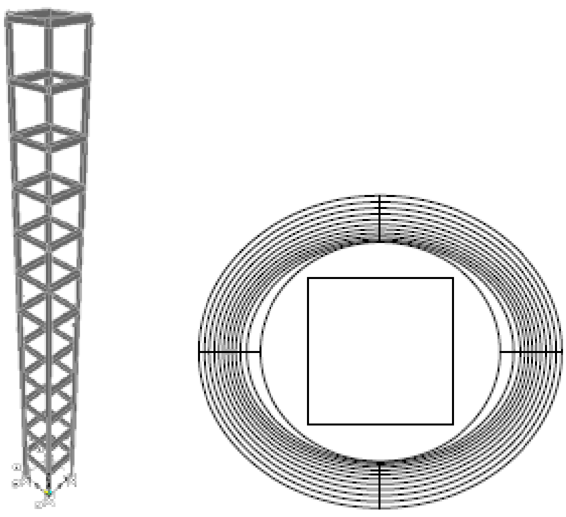

Figure 3. Twelve story high framed tower and Cullman Ellipse for each floor.

In this example, there is a twelve story high frame, each floor has a 3 mt high separation. As for the Cullman ellipse, there is a difference between each floor ellipse.

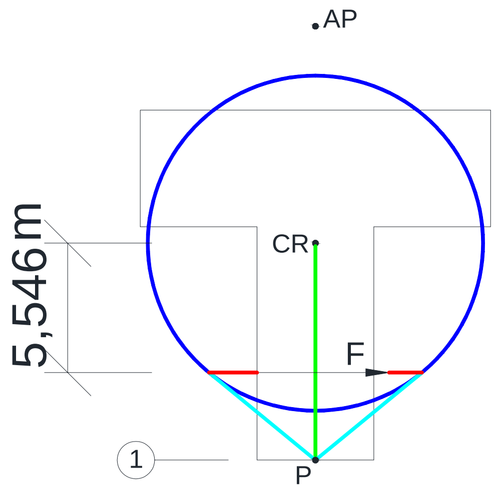

Figure 4. Force position (F) inducing an amplification factor of 2 on the frame (P).

Polarity relationship between the pole P (frame) and the polar (F), AP indicates the antipole position (instantaneous rotation center).

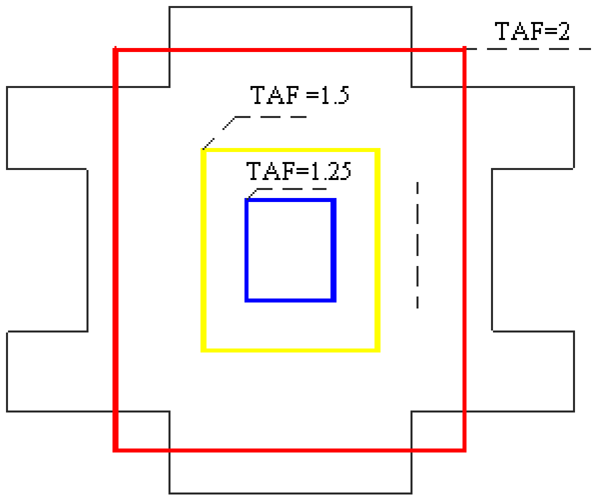

Figure 5: Central torsional kernels referred to the Torsional Amplification Factor (TAF) =2, TAF = 1.5 y TAF =1.25)

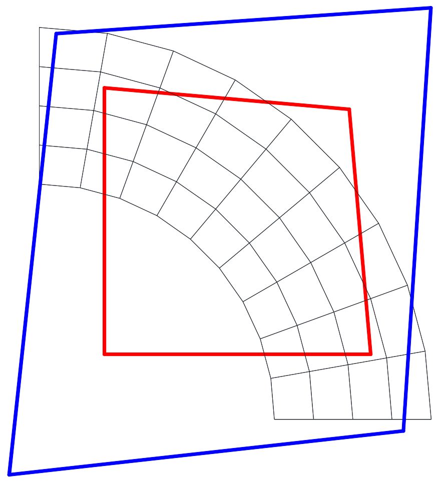

Figure 6. Central torsional kernel for the original configuration (inner kernel) and the optimized one (outer kernel).

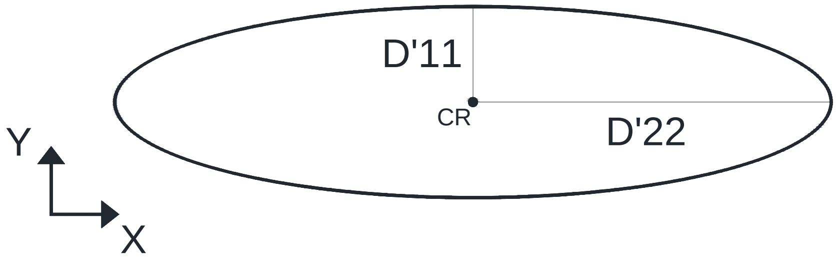

Figure 7. Orientation of a deflections ellipse with principal axes directions coincident with the reference system selected.

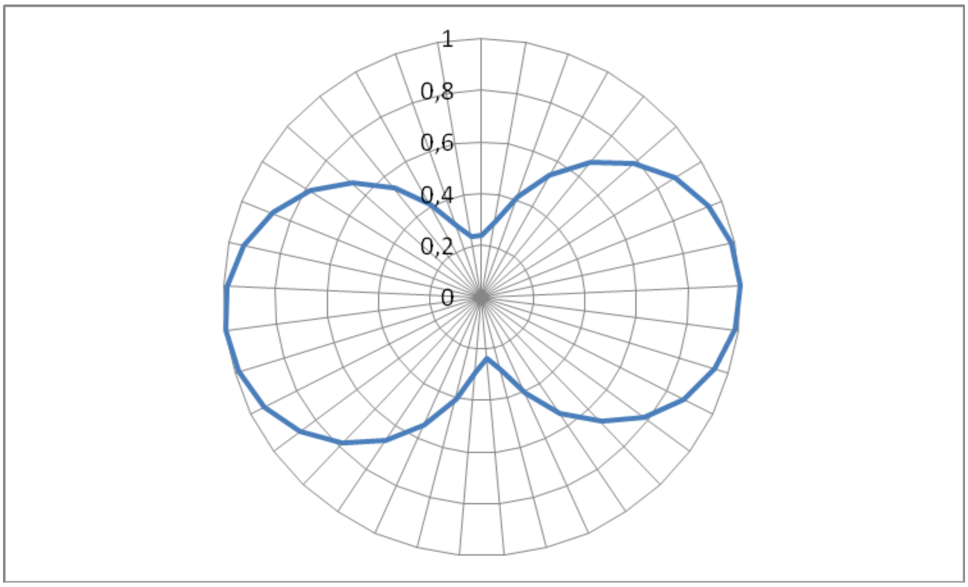

Figure 8. Booth curve, bean-shaped, for the deflections ellipse of the figure 6.

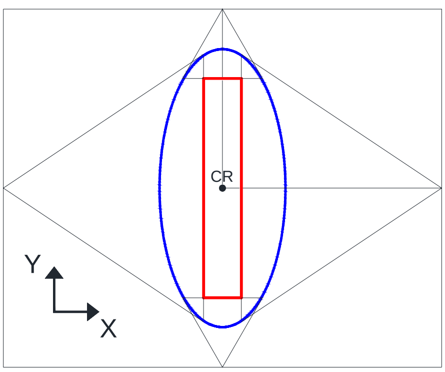

Figure 9. Central torsional kernel for the Cullman ellipse of figure 7

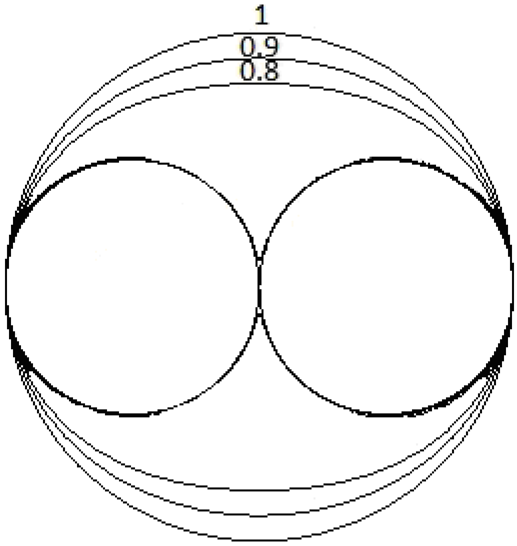

Figure 10: Regularly framed structures

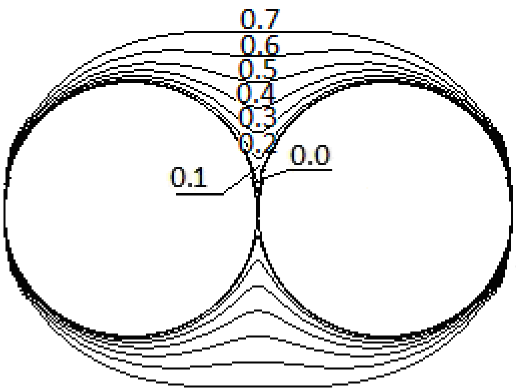

Figure 11: Irregularly framed structures

6. QUICK FORMULATIONS TO DETERMINE THE CULLMAN ELLIPSE

The following expressions resume the results which can be obtained once the Torsional and the two Principal longitudinal rigidities (directions and magnitudes) are known, ρ represents the radiuses of the Cullman Ellipse, α is the twisting angle induced by the Torsional Moment M. Once ρi) and ρj) are found, the Cullman ellipse can be drawn using these principal axes and their corresponding magnitudes and directions [Notice that the Cullman Ellipse has the same orientation and relative magnitudes as the Rigidity Ellipse] [Notice also that by crossing indexes i and j in these formulas, the Cullman Ellipse is already rotated the necessary 90º. (Transposed)

$$\rho_i \equiv \left(\sqrt{\frac{M_z}{\alpha_z}}\right)\times\left(\sqrt{\frac{\delta_j}{F_j}}\right);\Leftrightarrow \rho_j \equiv \left(\sqrt{\frac{M_z}{\alpha_z}}\right)\times\left(\sqrt{\frac{\delta_i}{F_i}}\right);$$

In words:

Radius of Gyration ρ (i) = √ Torsional Rigidity (z) / √Longitudinal Rigidity (j)

Radius of Gyration ρ (j) = √ Torsional Rigidity (z) / √Longitudinal Rigidity (i)

Being i and j the principal directions of the deflections ellipse; δ their magnitudes and M and F, Respectively, the applied Moment and the applied force (Perturbations induced on the system). So, the successive ellipses have only a mathematical interest.

6.1 Ellipses and booth curves expressed in polar coordinates:

Ellipses: $$\rho = \frac{ab}{\sqrt{\left(a^2\cos^2\theta + b^2\sin^2\theta\right)}}$$

Where θ is the azimuthal angle of the applied Force.

Booth Curve: $$\rho = \sqrt{\left(a^2\cos^2\theta + b^2\sin^2\theta\right)}$$

Where θ is the same as above

These two curves appear orientated perpendicularly to each other. They share 4 points in common.

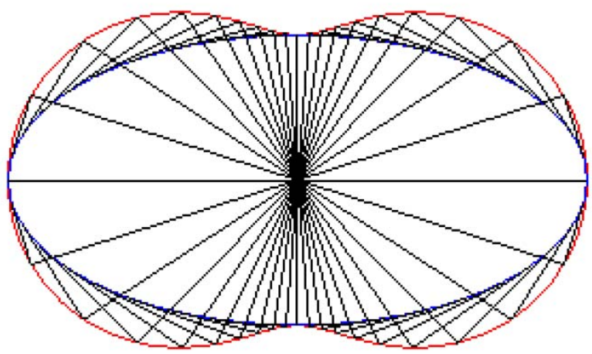

Figure 12. Relationship between a Booth curve and its inscribed Ellipse, and its 4 common points, at the extremities of each semi axis.

This figure shows correlated points on booth curves, the radii are common, the angles, the same on both. Only at the principal axes copuntuality between the two curves exist. All others have “phase shifts”. The radii shown are perpendicular to ellipse tangents.

Operational Sequence to obtain the Culmann Ellipse (Ž= displacement; θ = load azimuth)

$$\{\partial i;\partial j\}_1 \mapsto \left\{\frac{\partial}{F}\right\}_2^{-1} \mapsto \left\{\frac{F}{\partial}\right\}_3 * \frac{\theta}{M} \mapsto \left\{\rho^2\right\}_4^{\frac{1}{2}} \mapsto \{\rho\}_5 \lrcorner^{90^\circ} = \left[\rho_{\mathrm{Culmann}}\right]_6 = \left\langle \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\rangle_{\mathrm{canonic}} \text{ or } (\rho;\theta)_{polar}$$

All operations are performed on a;b (principal axes), then transpose axes (a,b becomes b,a) : (5)

Dimensional Checks : 1: [L]; 2 : [L/F]; 3 : [F/L]; 4 : [L^2]; 5 : [L]; 6 : [L]

REFERENCES

Paparoni, M. and Chacón D. (2004). *Quadratic Forms as Functional Representations of Loading Cases for Seismic design.* **13th World Conference on Earthquake Engineering.** Vancouver, B.C., Canada

Paparoni, M. and Hummelgens P. (2000). *Un Tratamiento Matemático de la Rigidez Torsional de una Planta de Edificio con Pórticos en Direcciones Arbitrarias*. **Revista Tekhné Universidad Católica Andrés Bello**, **N°4.** Caracas, Venezuela. Pages 79-85.

Belluzzi, O. (1942) Scienza delle Costruzioni. Volume Secondo. La teoria dell'elisse di elasticitá., Zanichelli, Bologna. Chapter XVII. Pages 1-73.

Lanczos ,C. (1988) Applied Analysis. Dover Publications Inc. New York. Pages 81-130

Paz, O.; Peña, O. (2011). *Configuraciones estructurales extremas, una búsqueda de variables sistémicas definitorias*, Universidad Católica Andrés Bello, Caracas

Paparoni, M. (1991), Dimensionamiento de Estructuras Altas de Concreto Armado. Sidetur, Caracas.Chapter 9. Pages 149-175.

Google. Mathematical Encyclopedia. (in French)., Courbes de Booth. http://www.mathcurve.com/courbes2d/booth/booth.stme.