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ON THE FIRST PASSAGE PROBLEM AND ITS APPLICATION TO EARTHQUAKE ENGINEERING

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SUMMARY

Methods and formulas related to the first passge probability and its corresponding reliability are reviewed in this paper. It is found that some formulas can be obtained from others by careful derivations. It is also found from the numerical examples that the independent peaks assumption results in very good approximation for the first passage probability for wide-band processes. As for the narrow-band processes, it is found that the Poisson envelope crossings assumption results in the best approximation for double-sided crossing problems. Some newly developed techniques more relevant to seismic problems are also briefly reviewed in this paper. A numerical example is provided to show the applicability of these techniques.

INTRODUCTION

The first passage problem is highly related to the reliability of the structure when the earthquake excitation and structural response are modeled as random processes. It is also known as the first excursion problem or first crossing problem. The direct topic of the first passage problem is the first passage time which is defined as the time that the random structural response or its envelope, passes a given single- or two-sided threshold for the first time. It can represent the probability of catastrophic failure of a structure if a proper threshold is specified, although this usually requires a full-fledged nonlinear response consideration. Due to the random nature of the excitation and response, the first passage time is also a random variable distributed with respect to time. The mean value and mean square value of the first passage time can usually give us valuable information about when such a catastrophic failure may occur.

The term "first passage failure," however, does not necessarily indicate that a structure fails immediately following the first time the response passes a given threshold level. Instead, the first passage probability is frequently used to measure the probability that the structural response exceeds certain design limit conditions such as onset of yielding at least once within a specific time interval. In these cases, the first passage failure generally involves only linear response consideration, and the first passage problem is meaningful as a measure of likehood for the structure to be in a design limit state, an important indicator for structural safety.

EXPECTED CROSSING RATE

The expected crossing rate is a very important parameter in some approximations related to the first passage problem. According to Rice (Ref. 1), the expected crossing rate for a random process $X(t)$ to cross a level b with positive slope (up-crossing rate) is

$$\nu_b(t) = \int_0^\infty \dot{x} f(b, \dot{x}, t) d\dot{x} \tag{1}$$

where $f(x, \dot{x}, t)$ is the joint probability density function of X and $\dot{X}$. In studying a linear time-invariant system's response to a Gaussian zero-mean nonstationary random excitation having an evolutionary power spectral density function, Yang (Ref. 2) derived the following expected up-crossing rate for the nonstationary response process $X(t)$:

$$\begin{aligned}\nu_b(t) = \frac{\sigma_{\dot{x}}\sqrt{1-\rho_1^2}}{2\pi\sigma_x} \exp\left(-\frac{b^2 K_{11}}{2}\right) &\left\{1 - bK_{12}\sqrt{\frac{2\pi}{K_{22}}} \exp\left(\frac{b^2 K_{12}^2}{2K_{22}}\right)\right. \\ &\left.\times\left[1 - \Phi\left(\frac{bK_{12}}{\sqrt{K_{22}}}\right)\right]\right\}\end{aligned} \tag{2}$$

where σ_x and $\sigma_{\dot{x}}$ are the standard deviations of $X(t)$ and $\dot{X}(t)$, respectively; $\Phi(\cdot)$ is the standard Gaussian distribution function; $\rho_1\sigma_x\sigma_{\dot{x}} = \rho_1\sigma_x(t)\sigma_{\dot{x}}(t) = E[X(t)\dot{X}(t)]$; $K_{11} = \sigma_{\dot{x}}^2/B$; $K_{12} = -\rho_1\sigma_x\sigma_{\dot{x}}/B$; $K_{22} = \sigma_x^2/B$; and $B = B(t) = \sigma_x^2\sigma_{\dot{x}}^2(1-\rho_1^2)$. The same result but in a different expression has also been derived by Howell and Lin (Ref. 3).

Extending Cramer and Leadbetter's stationary envelope definition (Ref. 4) Yang also derived the following expected up-crossing rate for the envelope of a nonstationary response process $X(t)$ (Ref. 2):

$$\begin{aligned}\nu_b^e(t) = \frac{b\sqrt{\Delta}}{\sigma_x^2\sqrt{2\pi}} \exp&\left[-\frac{b^2\left(\Delta + \frac{C^2}{4\sigma_x^2}\right)}{2\Delta\sigma_x^2}\right] \\ &\times\left\{1 + \frac{Cb}{2\sigma_x^2}\sqrt{\frac{\pi}{2\Delta}} \exp\left(\frac{C^2 b^2}{8\Delta\sigma_x^4}\right) \operatorname{erf}\left(-\frac{Cb}{2\sigma_x^2\sqrt{2\Delta}}\right)\right\}\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\Delta &= \lambda_2 - \frac{\lambda_1^2}{\sigma_x^2} - \frac{C^2}{4\sigma_x^2} - Q \\
\lambda_j &= \int_0^\infty \left(\omega + \dot{\phi}\right)^j |\psi(t,\omega)|^2 [2S(\omega)]\, d\omega; \quad j = 1, 2 \\
\dot{\phi} &= \frac{\partial\phi(t,\omega)}{\partial t} \\
\phi &= \tan^{-1}\left\{\frac{\mathrm{Im}[\psi(t,\omega)]}{\mathrm{Re}[\psi(t,\omega)]}\right\} \\
C &= 2\int_0^\infty |\psi(t,\omega)| \left[\frac{\partial|\psi(t,\omega)|}{\partial t}\right] [2S(\omega)]\, d\omega \\
&= 2E\left[X(t)\dot{X}(t)\right] \\
Q &= \int_0^\infty \left[\frac{\partial|\psi(t,\omega)|}{\partial t}\right]^2 [2S(\omega)]\, d\omega
\end{aligned} \tag{4}$$

$S(\omega)$ is the double-sided power spectral density function of a stationary random process and $\psi(t,\omega)$ is a deterministic function. If the random process considered is real, then $\psi(t,\omega)$ will also be a real function and the expressions in Eqs. (3) and (4) can be simplified. In particular, if ψ is a function of time only, the excitation becomes a uniformly modulated random process and expressions in Eqs. (3) and (4) will be even more simplified. For this special case, according to Langley (Ref. 5), Eqs. (2) and (3) can be written in the following expressions:

$$\nu_b(t) = \sqrt{\frac{1-\rho_1^2}{2\pi}} \left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) \exp\left(-\frac{b^2}{2\sigma_x^2}\right) [\phi(q) + q\Phi(q)] \tag{5}$$

$$\nu_b^e(t) = \sqrt{1-\rho_1^2-\rho_2^2} \left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) \left(\frac{b}{\sigma_x}\right) \exp\left(-\frac{b^2}{2\sigma_x^2}\right) [\phi(r) + r\Phi(r)] \tag{6}$$

where $\rho_2 \sigma_x \sigma_{\dot{\hat{x}}} = \rho_2 \sigma_x(t) \sigma_{\dot{\hat{x}}}(t) = E[X(t)\dot{\hat{X}}(t)]$; $\phi(\cdot)$ is the standard Gaussian probability density function; $\hat{X}(t)$ is the Hilbert tranformation of $X(t)$; and q and r are defined as

$$q = \frac{\rho_1}{\sqrt{1-\rho_1^2}} \left(\frac{b}{\sigma_x}\right) \qquad r = \frac{\rho_1 b}{\sigma_x \sqrt{1-\rho_1^2-\rho_2^2}} \tag{7}$$

It can be shown that Eq. (5) is exactly the same as Eq. (2).

In studying the transient as well as stationary structural response to a step-function-of-time modulated random excitation, Krenk et al. (Ref. 6) also derived an expression for the expected up-crossing rate which can be shown to reduce to Eqs. (2) ans (3).

For the special case that $X(t)$ is a stationary random process Eqs. (2) and (5) and Eqs. (3) and (6) reduce respectively to

$$\nu_b = \frac{1}{2\pi} \frac{\sigma \dot{x}}{\sigma_x} \exp\left(-\frac{b^2}{2\sigma_x^2}\right) \qquad \nu_b^e = \sqrt{\frac{1-\rho_2^2}{2\pi}} \left(\frac{b\sigma_{\dot{x}}}{\sigma_x^2}\right) \exp\left(-\frac{b^2}{2\sigma_x^2}\right) \tag{8}$$

where $\rho_2 = \sqrt{\lambda_1^2/(\lambda_0\lambda_2)}$, and λ_0 is defined similarly to λ_1 and λ_2 but with a different subscript (see Eq. (4)).

POISSON AND OTHER ASSUMPTIONS

Although Poisson approximation is an easy way to calculthe the first passage probability, it is asymptotically exact only when the process is Gaussian stationary and threshold level increases to infinity (Ref. 4). For a threshold level of practical interest, the result from Poisson assumption is either too conservative or nonconservative depending on different situations. In order to improve this, Vanmarcke derived another approximate formula for the first passage probability (Ref. 7). The derivation is based on the clump size consideration and several other assumptions such as the exponential distribution of the envelope excursion length, the independence of the clumps and the independence of the recurrence times. The final result is

$$P(t,b) = 1 - \exp\left\{-\nu_0 t \left[\frac{1-\exp\left(-\frac{\delta b}{\sigma_x}\sqrt{\frac{\pi}{2}}\right)}{\exp\left(\frac{b^2}{2\sigma_x^2}\right)-1}\right]\right\} \tag{9}$$

where ν_0 is the zero crossing rate which can be calculated from Eq. (1) or (8); $\delta = (1-\lambda_1^2/\lambda_0\lambda_2)^{1/2}$ is a band-width measure which has a value between zero and one and is defined based on the following spectral moments of the single-sided power spectral density function:

$$\lambda_n = \int_0^\infty \omega^n \, [2S(\omega)] \, d\omega, \qquad n = 0, 1, 2 \tag{10}$$

It is noted that Eq. (10) is the stationary case of that defined in Eq. (4). Vanmarcke derived

Eq. (9) for a stationary narrow-band Gaussian process, but claimed that it is also applicable to wide-band processes and nonstationary processes by considering the parameters S, λ_n, δ, σ_x and ν_0 time dependent instead of being constant (Ref. 7). It has been criticized that the theoretical argument for doing so has never been justified (Ref. 2). Particularly, it was pointed out that the transient nonstationary structural response is not a uniformly modulated random process. Although many nice properties of the stationary random process or envelope process can be extended to the uniformly modulated random process, it has never been proved that this is true for other nonstationary processes such as the transient structural response considered by Corotis et al. (Ref. 8). It is also doubtful that the clump phenomenon which is particularly prominent for stationary narrow-band processes is still a reasonable assumption for highly wide-band and/or nonstationary processes.

In spite of the questions mentioned above, because of its simplicity Vanmarcke's formula has been widely used by many researchers (Ref. 8). Many empirical or semi-empirical formulas have also been derived or proposed based on Vanmarcke's original formula. In his paper, Yang (Ref. 2) has also shown that Vanmarcke's formula can be written as follows after both crossing rates of the process and its envelope are obtained,

$$P(t,b) = 1 - \exp\left\{ - \int_0^t 2\nu_b(\tau) \left(\frac{1 - \exp\left[-\frac{\nu_b^e(\tau)}{2\nu_b(\tau)} \right]}{1 - \left[\frac{\nu_b(\tau)}{\nu_0(\tau)} \right]} \right) d\tau \right\} \tag{11}$$

It is noted that Eq. (11) includes both stationary and nonstationary cases.

RECENT DEVELOPMENTS

Most of the recent developements on the first passage problem focus on the application of those basic techniques discussed previously to vector processes or to load combination problems. In both cases, the expected crossing rate has been studied by several researchers (Refs. 9,10,11). After this crossing rate is obtained, the reliability of the structure can be estimated.

In the load combination problem, it is usually assumed that the component processes are statistically independent and the combination of these processes follows a linear rule. In real situations, both independence and linear combination may not be true. If the statistical dependence and nonlinear combination of these loads are to be considered, one may have to study the same problem from a vector process point of view (Refs. 12,13). The reliability of the structure is then considered to be probability that the vector process stays within the linear or nonlinear limit state surface describing the failure criterion in a given time interval. Similarly to other dynamic reliability analyses, the expected crossing rate (or more specifically, the expected out-crossing rate) plays an important role in the analysis.

In order to define the expected out-crossing rate, consider a random vector-process (or multi-variate process) $\mathbf{X}(t)$ and two complementary domains D and D^* such that

$$\mathbf{X}(t) = \{X_1(t) \quad X_2(t) \quad \ldots \quad X_m(t)\}^T \tag{12}$$
$$D = \{\mathbf{X} : \quad g(\mathbf{X}) > 0\} \tag{13}$$
$$D^* = \{\mathbf{X} : \quad g(\mathbf{X}) \leq 0\} \tag{14}$$

where T represents the transpose of a vector. Eq. (13) (or (14)) indicates that D (or D^*) is a domain in the m-dimensional space in which the components of $\mathbf{X}$ satisfy the condition $g(\mathbf{X}) > 0$ (or $g(\mathbf{X}) \leq 0$). Let F_D denote the limit state surface (interface between the domains D and D^*). Then, the expected out-crossing rate ν_D can be obtained by (Ref. 14)

$$\nu_D = \int_{F_D} dS_m \int_0^\infty \dot{y} f_{\mathbf{X}\dot{Y}}\,(\mathbf{x}, \dot{y})\, d\dot{y} \tag{15}$$

where dS_m is the surface element on the interface F_D and $f_{\mathbf{X}\dot{Y}}(\mathbf{x}, \dot{y})$ is the joint probability density function of $\mathbf{X}(t)$ and $\dot{Y}(t)$. The out-crossing rate ν_D given above is equal to the expected rate at which $\mathbf{X}(t)$ will cross the interface F_D from the domain D into the domain D^* at time t. The parameter $\dot{Y}(t)$ in Eq. (15) represents the normal component of the velocity of vector process $\mathbf{X}(t)$ crossing the limit space. It can be calculated from

$$Y(t) = \mathbf{n}^T \mathbf{X}(t) \qquad \text{and} \qquad \dot{Y}(t) = \mathbf{n}^T \dot{\mathbf{X}}(t) \tag{16}$$

where $\mathbf{n}$ is the unit outward vector defined on, and normal to, the interface F_D. The unit vector $\mathbf{n}$ is outward if its sense is directed from D to D^*. For a one-dimensional case, Eq. (16) reduces to Eq. (1).

Depending on the limit state surface, *i.e.* $g(\mathbf{X})$, expressions for the expected out-crossing rate can be derived from Eq. (15). For example, Shinozuka et al. (Ref. 12) have derived analytical expressions and upper bound of ν_D for those limit states defined by hyper-planes and by hyper-spheres, respectively. It was shown that their expression of the expected out-crossing rate for a hyper-sphere limit state reduces to a formula derived earlier by Veneziano et al. (Ref. 13) provided that the components of $\mathbf{X}(t)$ and $\dot{\mathbf{X}}(t)$ are uncorrelated. It was also pointed out that a hyper-polyhedral limit state surface can always be used with relative ease to approximate a limit state surface of any shape either by inscribing or describing the surface.

NUMERICAL EXAMPLES AND CONCLUSION

Numerical examples are studied in order to examine the accuracy of many formulas proposed to find the expected crossing rate and the first passage probability. It is found that for a stationary wide-band Gaussian process, the assumption that peaks occur independently or crossings occur following a Poisson process gives us reasonably accurate result for the first passge probability. As for the narrow-band Gaussian process, the present study shows that one of the best approximations for the two-sided crossing probability is to use the envelope crossing rate and assume crossings occur according to a Poisson process. Two of the results are shown in Fig. 1 in which the dashed curves from Eq. (11) and simulation results based on a sample of five hundred each are also plotted for comparison. More detail results can be found in Shinozuka and Wu (Ref. 15).

More recent example in which the first passage analysis was performed in relation to seismic structural safety assessment include the papers by Shinozuka et al. (Ref. 12) and Tzavelis and Shinozuka (Ref. 16).

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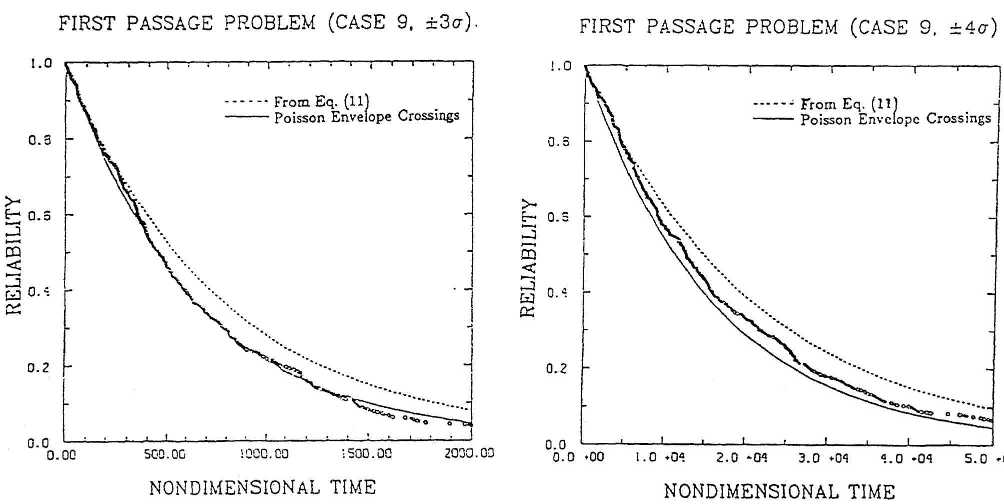

Fig. 1 Reliability of a stationary narrow-band random process