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RANDOM RESPONSE OF HYSTERETIC SINGLE-DEGREE-OF-FREEDOM SYSTEMS SUBJECTED TO EARTHQUAKE-LIKE EXCITATIONS

Yutaka MATSUSHIMA[1]

[1]Institute of Engineering Mechanics, University of Tsukuba, Tsukuba, Japan

SUMMARY

The nonlinear random response of the single-degree-of-freedom system with the slip-type hysteretic restoring force characteristic is dealt with. The non-white excitation which has the Tajimi's spectrum is assumed as the model of earthquake-like disturbance. The attention is focussed on the equivalent natural frequency, the hysteretically dissipated energy, the maximum displacement and on the probability of safety. The approximate solutions are derived on the basis of the theoretical investigation. They are compared with the digital estimates obtained from the Monte Carlo simulation. The agreements between the both are satisfactory over the wide ranges of related parameters.

INTRODUCTION

The objective of this paper is to clarify the fundamental characters of nonlinear behavior of hysteretic structures excited by random seismic motions. The structure having the high frequency does not usually receive the great amount of energy from the seismic motion in the elastic stage. The reduction of equivalent frequency caused by the damage, however, makes the input energy greater, which results in the increase of damage and therefore in the further decrease of frequency in a progressive way. This study is motivated partly by the necessity of understanding such phenomena.

INPUT-OUTPUT SYSTEM

It is assumed that the undamped single-degree-of-freedom system rested on the ground is suddenly subjected to the random noise which is taken as the ground acceleration. The equation of motion is given by

$$\ddot{x} + f(x,\dot{x}) = -U(t)N(t) \ , \tag{1}$$

where x is the displacement of the system and $\cdot$ means the derivative with respect to time t. U(t) is the unit step function. N(t) denotes the stationary random process having the following power spectral density proposed by Tajimi (Ref.1):

$$S(\omega) = S_0 \cdot \frac{1 + 4h_g^2(\omega/\omega_g)^2}{\{1-(\omega/\omega_g)^2\}^2 + 4h_g^2(\omega/\omega_g)^2} \tag{2}$$

$S(\omega)$ is referred to as the both-sided spectral density with the angular frequency ω ranging from $-\infty$ to ∞. S_0 is the spectral density where $\omega=0$. $S(\omega)$

has a peak when $\omega \cong \omega_g$. h_g controls the sharpness of its peak. The value of h_g which makes this spectrum approximately fit the Housner's averaged response spectrum (Ref.2) is around 0.6 (Ref.1). Figure 1 illustrates the Tajimi's spectrum of this case.

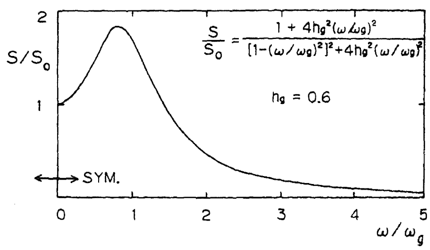

Fig.1 Tajimi's spectrum

$f(x,\dot{x})$ respresents a restoring force function which has the slip-type hysteretic characteristic as illustrated in Fig.2. This hysteresis is chosen for analysis, because this is the representative of the brittle behavior of structures which is accompanied by the degrading stiffness. The plastic stiffness after yielding is taken as zero. The mass rested on the origin excurses on the lines with arrows indicated for example. When the restoring force becomes zero, the displacement progresses towards the last slip displacement without the change of restoring force. This is the typical pattern of the character of structures with X-type steel bracings. The timber and reinforced concrete shear walls also have the similar property where the slippage takes place more or less. α, Δ and ω_0 given in the figure are the yield acceleration, yield displacement and natural angular frequency in the elastic region, respectively.

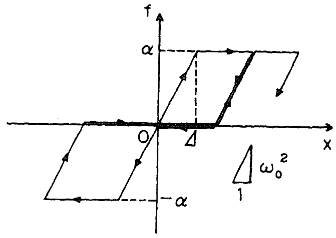

Fig.2 Slip-type hysteresis

EQUIVALENT NATURAL FREQUENCY

The initial conditions of Eq.(1) are $x=\dot{x}=0$ when $t=0$. In the nonstationary response, the system behaves elastically in the early stage. The expected time for the response to arrive at the elastic limit, which is denoted by t_0, is approximately estimated as follows:

The Tajimi's spectrum varies slowly with frequency. The elastic response goes through the narrow-banded process with the expected natural angular frequency ω_0. Therefore the expected energy given to the system per unit time will be equal to $\pi S(\omega_0)$. It is considered that the expected time for this energy to reach the maximum elastic potential energy $\alpha^2/(2\omega_0^2)$ is equal to t_0. Hence the following equation holds:

$$t_0 = \frac{\alpha^2}{2\omega_0^2} \cdot \frac{1}{\pi S(\omega_0)} \qquad (3)$$

Now it is convenient to define the following nondimensional quantities, in which T_0 is the natural period of the system in the elastic stage:

$$\tau_0 \equiv t_0/T_0,\ \xi \equiv \omega_0 S_0/\alpha^2 \text{ and } \nu \equiv \omega_0/\omega_g \qquad (4)$$

The substitution of Eq.(2) into the right hand side of Eq.(3) and the application of Eqs.(4) to Eq.(3) gives the following nondimensional time of elastic limit:

$$\tau_0 = \frac{1}{4\pi^2\xi} \cdot \frac{(1-\nu^2)^2 + 4h_g^2\nu^2}{1 + 4h_g^2\nu^2} \qquad (5)$$

The system goes into the plastic region when t is greater than t_0, and the apparent natural frequency changes. The amount of this change of frequency will be estimated as follows:

The expected cumulative plastic deformation per unit time is approximately equal to $\pi S(\omega_e)/(2\alpha)$. Here ω_e represents the expected equivalent natural angular frequency. The plastic deformation is defined by the deformation which excurses on the lines $f=\pm\alpha$. The cumulative plastic deformation x_p is the sum of the plastic deformation. The absolute value of velocity, when slipping on the x-axis in a free vibration with initial conditions $f=\pm\alpha$ and $\dot{x}=0$, equals α/ω_0. Since the slip deformation is considered equal to the plastic deformation, the time derivative of the expected equivalent natural frequency becomes

$$\frac{d\omega_e}{dt'} = -\frac{\omega_0\omega_e{}^2 S(\omega_e)}{\alpha^2}, \tag{6}$$

where $t'=t-t_0$, which is the time measured from when the system goes beyond the elastic limit. Applying Eq.(2) to the right hand side of Eq.(6) and integrating, finally one gets

$$\tau' = \frac{1}{2\pi\xi}\left[\frac{1-\beta}{\beta} + \frac{\nu^2(1-\beta)}{4h_g{}^2} + \frac{8h_g{}^2+1}{8h_g{}^3}(\tan^{-1}2h_g\nu\beta-\tan^{-1}2h_g\nu)\nu\right], \tag{7}$$

where the following nondimensional quantities are defined:

$$\tau' \equiv t'/T_0 \text{ and } \beta \equiv \omega_e/\omega_0 \tag{8}$$

This is the relation between the time and the expected equivalent natural frequency. β is the function of h_g, ν, ξ and τ'. β in the limit $h_g \to \infty$ is estimated as

$$\beta = \frac{1}{1+2\pi\xi\tau'}, \tag{9}$$

which is the value of β under white input. Equation (7) holds when $\tau' \geqq 0$, namely $\tau \geqq \tau_0$, where $\tau \equiv t/T_0$. β is taken as unity when $\tau<\tau_0$.

HYSTERETICALLY DISSIPATED ENERGY

The expectation $\bar{E}$ of the hysteretically dissipated energy E is zero, when $t \leqq t_0$. Since it is possible to consider that the energy given to the system is consumed exclusively by the hysteresis when $t \geqq t_0$, the following equation holds:

$$\frac{d\bar{E}}{dt'} = \pi S(\omega_e) \tag{10}$$

Dividing both sides of Eq.(10) by the corresponding sides of Eq.(6), one gets

$$\frac{d\bar{E}}{d\omega_e} = -\frac{\pi\alpha^2}{\omega_0\omega_e{}^2}. \tag{11}$$

Multiplying both sides of Eq.(11) by $d\omega_e$ and integrating, there is obtained

$$\bar{E} = \frac{\pi\alpha^2(\omega_0-\omega_e)}{\omega_0{}^2\omega_e}. \tag{12}$$

Here the following quantilty is defined in order to make E be dimensionless:

$$\bar{\lambda} \equiv \bar{E}/(\alpha\Delta) = \omega_0{}^2\bar{E}/\alpha^2 \tag{13}$$

$\bar{\lambda}$ stands for the expectation of cumulative ductility factor λ which is defined as the cumulative plastic deformation divided by the yield displacement. With the aid of Eq.(13), Eq.(12) is reduced to the following nondimensional form:

$$\bar{\lambda} = \pi(1-\beta)/\beta \tag{14}$$

This is the relation between $\bar{\lambda}$ and β. $\bar{\lambda}$-τ relation is obtained from Eqs.(7) and (14) through the aid of β, when $\tau \geqq \tau_0$. $\bar{\lambda}$ is taken as zero, when $\tau<\tau_0$. Substituting β given by Eq.(9) into the right hand side of Eq.(14), one has

$$\bar{\lambda} = 2\pi^2\xi\tau'. \tag{15}$$

This is identical with the expression already obtained as the expectation of cumulative ductility factor due to the white noise (Ref.3).

The variance V_E of the hysteretically dissipated energy is assumed zero, when $t \leqq t_0$. It is possible to write, like in the case of $\bar{E}$, as

$$\frac{dV_E}{dt'} = 2\pi S(\omega_e)\sigma_{\dot{x}}^2 \ , \tag{16}$$

when $t \geqq t_0$. Here $\sigma_{\dot{x}}$ represents the standard deviation of the response velocity $\dot{x}$. Since $\dot{x}$ can be assumed to vibrate sinusoidally with the amplitude $\omega_e(\bar{x}_p/2+\Delta)$ in an approximate sense, the variance of $\dot{x}$ will be written by

$$\sigma_{\dot{x}}^2 = \frac{\omega_e^2}{2}(\frac{\bar{x}_p}{2} + \Delta)^2 \ . \tag{17}$$

Substituting Eq.(17) into the right hand side of Eq.(16) and expressing in terms of β, the following nondimensional form is obtained:

$$\frac{dV_\lambda}{d\beta} = -\frac{\pi}{4}(\frac{\pi}{\beta} - \pi + 2)^2 \tag{18}$$

Multiplying both sides of Eq.(18) by $d\beta$ and integrating,

$$V_\lambda = \frac{\pi}{4}[(1-\beta)\{\frac{\pi^2}{\beta} + (\pi-2)^2\} + 2\pi(\pi-2)\ln\beta] \ . \tag{19}$$

This is the relation between V_λ and β, when $\tau \geqq \tau_0$. V_λ is taken as zero, when $\tau < \tau_0$.

MAXIMUM DISPLACEMENT

The ductility factor μ which is defined by the absolute value of maximum displacement divided by the yield displacement can be written by

$$\mu = \max[\lambda_+ , \lambda_-] + 1 \ , \tag{20}$$

when $\tau \geqq \tau_0$, where $\max[\cdot]$ means the maximum value of arguments. λ_+ and λ_- are the cumulative ductility factor in a positive zone and that in a negative zone, respectively. λ is the sum of these two. Supposed that λ_+ and λ_- which have the same probability distribution function $P_{\lambda_+}(\lambda_+)$ are statistically independent, the probability distribution function of μ becomes

$$P_\mu(\mu) = \mathrm{Prob}[\lambda_+ < \mu-1 \cap \lambda_- < \mu-1] = P_{\lambda_+}(\mu-1)^2 \ , \tag{21}$$

where $\mathrm{Prob}[\cdot]$ means the probability. The probability density function of μ, $p_\mu(\mu)$, is obtained by differentiating Eq.(21) with respect to μ as

$$p_\mu(\mu) = 2P_{\lambda_+}(\mu-1)p_{\lambda_+}(\mu-1) \ . \tag{22}$$

The probability distribution of λ_+ is now assumed to be the Gamma distribution (Ref.3). Since the expectation and variance of λ_+ are respectively

$$\bar{\lambda}_+ = \bar{\lambda}/2 \text{ and } V_{\lambda_+} = V_\lambda/2 \ , \tag{23}$$

finally one has

$$p_{\lambda_+}(\lambda_+) = \frac{\eta^\rho}{\Gamma(\rho)} \lambda_+^{\rho-1} e^{-\eta\lambda_+} \ , \tag{24}$$

where

$$\rho \equiv \bar{\lambda}_+^2/V_{\lambda_+} = \bar{\lambda}^2/(2V_\lambda) \text{ and } \eta \equiv \bar{\lambda}_+/V_{\lambda_+} = \bar{\lambda}/V_\lambda \ . \tag{25}$$

$\Gamma(\cdot)$ represents the Gamma function. In addition,

$$P_{\lambda_+}(\lambda_+) = \int_0^{\lambda_+} p_{\lambda_+}(\lambda_+)d\lambda_+ \ . \tag{26}$$

$p_\mu(\mu)$ is obtained by applying Eqs.(24) through (26) to Eq.(22) with $\bar{\lambda}$ and V_λ given already. The expectation and variance of μ in the region $\tau \geqq \tau_0$ can be calculated respectively by

$$\bar{\mu} = \int_1^\infty \mu p_\mu(\mu)d\mu \qquad (27)$$

and

$$V_\mu = \int_1^\infty (\mu-\bar{\mu})^2 p_\mu(\mu)d\mu \ . \qquad (28)$$

If $\tau<\tau_0$, then $\mu<1$ and the system remains elastic on average. Therefore no detailed discussion about μ in this range is made here.

PROBABILITY OF SAFETY

The reliability functions $R_\lambda(\lambda_F)$ and $R_\mu(\mu_F)$ are defined by the probabilities that λ and μ do not exceed the prescribed λ_F and μ_F, respectively. Since both λ and μ monotonously increase with time, the reliability functions in the region $\tau \geqq \tau_0$ are obtained simply by integrating the corresponding probability density functions as

$$R_\lambda(\lambda_F) = \int_0^{\lambda_F} p_\lambda(\lambda)d\lambda \qquad (29)$$

and

$$R_\mu(\mu_F) = \int_1^{\mu_F} p_\mu(\mu)d\mu \ . \qquad (30)$$

In the above calulations $p_\lambda(\lambda)$ is regarded as the Gamma distribution with the expectation $\bar{\lambda}$ and the variance V_λ. $p_\mu(\mu)$ is as given in the previous section.

VERIFICATION BY MONTE CARLO SIMULATION

The digital simulation has been performed in order to verify analytical expressions obtained above. Three hundred sample functions having Tajimi's spectrum where h_g=0.6 have been generated on the basis of stochastic concept. The nonstationary nonlinear responses due to them have been numerically computed. The statistical treatment has been made on the results. The values of ν have been taken as 0,1,2 and 3. The case ν=0 corresponds to the white noise, and therefore the spectral density at the natural frequency is equal to S_0 from the outset. The initial values of spectral density for cases ν=1,2 and 3 are respectively 1.694, 0.458 and 0.181 of S_0. The values of ξ have been taken as 0.025, 0.05 and 0.1. On account of limited space, however, only the results of the case ξ=0.025 are explained below.

The β-τ relations with a parameter ν are displayed in Fig.3. The solid lines stand for approximate solutions and those

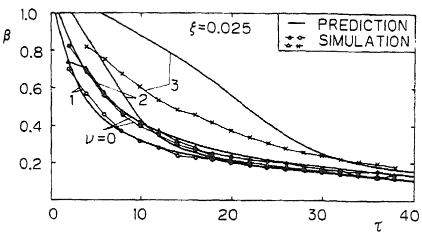

Fig.3 Time change of expectation of equivalent natural frequency

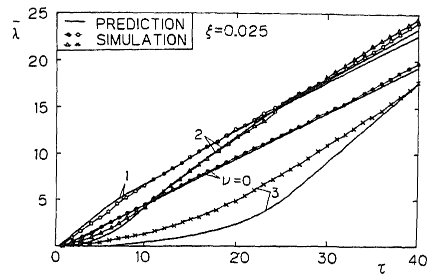

Fig.4 Time change of expectation of cumulative ductility factor

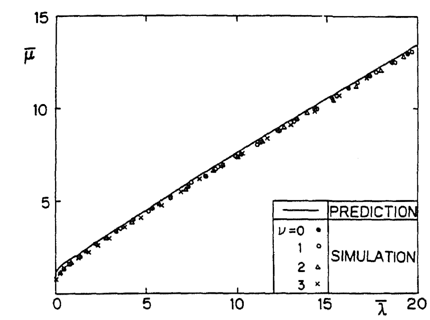

Fig.5 Relation between $\bar{\mu}$ and $\bar{\lambda}$

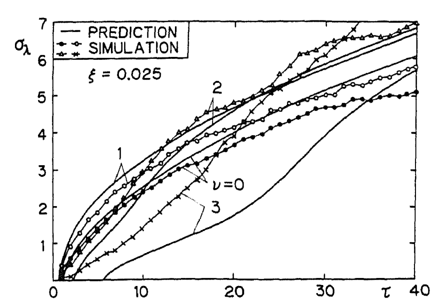

Fig.6 Time change of standard deviation of cumulative ductility factor

accompanied by symbols for simulation estimates. The degree of agreements between analytical and digital estimates is generally acceptable, although not so excellent when ν is great.

Figure 4 indicates $\bar{\lambda}$-τ relations in the same manner as in the previous figure. It is found that the theoretical prediction is accurate enough from the practical point of view. Figure 5 shows the relation between $\bar{\mu}$ and $\bar{\lambda}$. This is independent of h_g, ν and ξ, since $\bar{\mu}$ and $\bar{\lambda}$ are uniquely determined if only β is given. The symbols indicated are those where τ=2,4,6,........, 40. $\bar{\mu}$ and $\bar{\lambda}$ are approximately in a linear relation. The solutions are in good agreement with simulation estimates.

σ_λ-τ relations are shown in Fig.6. σ_λ is the standard deviation of λ which is equal to $\sqrt{V_\lambda}$. The degree of agreements between both estimates is generally acceptable, although not so good in the case ν=3. Figure 7 indicates the relation between σ_μ and σ_λ in the same manner as in Fig.5. This relation also does not depend on h_g, ν and ξ. The trend similar to as in Fig.5 can be pointed out.

Figures 8 and 9 display respectively $R_\lambda(\lambda_F)$-τ and $R_\mu(\mu_F)$-τ relations where λ_F=μ_F=10. It is natural that $R_\mu(\mu_F)$ is greater than $R_\lambda(\lambda_F)$, since the same value as λ_F is assigned on μ_F in these examples. It is recognized that the analytical solutions agree well with digital estimates.

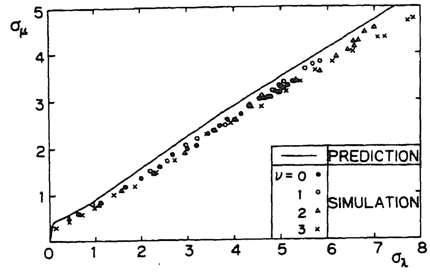

Fig.7 Relation between σ_μ and σ_λ

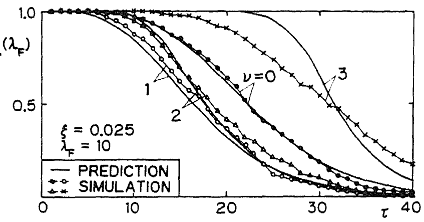

Fig.8 Time change of reliability indexed by cumulative ductility factor

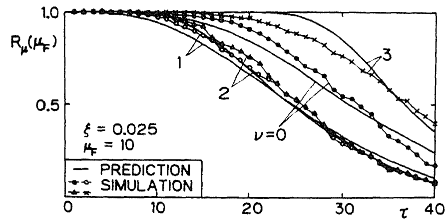

Fig.9 Time change of reliability indexed by ductility factor

CONCLUDING REMARKS

This paper deals with nonlinear random responses of single-degree-of-freedom systems with slip-type hysteretic restoring force characteristics, when subjected to earthquake-like excitations. The approximate solutions for the time change of equivalent frequency, cumulative ductility factor, ductility factor and reliability are derived on the basis of the theoretical investigation. They are compared and in good agreement with simulation estimates.

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