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TREATING MODEL UNCERTAINTIES IN STRUCTURAL DYNAMICS

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SUMMARY

This paper is an initial report on a study to develop a methodology for treating uncertainties in the dynamic models of a structural system. The uncertainties are quantified using probability interpreted as a measure of plausibility of a hypothesis given specified information. Three approaches for calculating the statistical moments of the uncertain response induced by the uncertainties in the model are examined for a simple case of a linear oscillator. It is concluded that a numerical integration approach has the best potential of the three methods.

INTRODUCTION

Motivation Modeling the dynamic behavior of structures requires estimating the model parameters $\underset{\sim}{\theta}$, such as the structural periods and dampings in a modal approach, or the stiffness and mass distributions. Different methods are employed in estimating $\underset{\sim}{\theta}$, from empirical code-type formulas to very detailed, finite element methods. Even the most elaborate and detailed methods, though, lead to parameter values which have associated uncertainties because of the numerous assumptions made when modeling the geometry, material properties, constitutive laws, and boundary conditions of the structural members. In addition, there is an uncertain model error in the calculated response since any mathematical model gives only an approximate description of the real dynamics. This study has three goals:

Goal 1: To provide the engineer doing seismic design with a tool to go beyond checking the nominal dynamic response to specified excitations for a preliminary design; the engineer will be able to examine the associated uncertainty in the response due to the fact that the completed structure will not have precisely the model parameter values that were assumed, and also due to the fact that no model gives an exact description of a structure's dynamics.

Goal 2: To provide a rational method for utilizing test data from a structure to improve response predictions. This activity falls within the realm of system identification (Ref. 1).

Goal 3: To provide a framework to enable model uncertainties, in addition to excitation uncertainties, to be incorporated into a seismic risk methodology. Note that classical random vibration theory focuses on the uncertainty in the response resulting from a prescribed uncertainty in the excitation, but the structural dynamics are assumed to be described exactly.

<u>Quantification of Uncertainties Using Probability</u> We employ probability to quantify the uncertainties involved but use a "Bayesian" interpretation, that is, we treat probability as a multi-valued logic for plausible reasoning. Specifically, the probability of a given b, $P(a|b)$, denotes a measure of the plausibility of the proposition a given the information stated in proposition b. From this standpoint, all probabilities are conditional, since the plausibility of a proposition depends on the relevant information available. We remark that for most of the applications we are interested in, the common interpretation of probability as a relative frequency of occurrences in the long run does not make sense. Also, the calculus of probability logic is defined by the axioms of mathematical logic together with three additional axioms (Ref. 2), but this leads to essentially the same calculus as the Kolmogorov axioms of "mathematical" probability defined as a measure on a σ-algebra of sets. Refs. 3 and 4 give further background relating to probability logic.

Let $p(\underset{\sim}{\theta}|i)$ be the joint pdf (probability density function) describing the uncertainty in the model parameters $\underset{\sim}{\theta}$. Symbol i is used here to denote the information used either to assign or to compute this probability function. Let $p(y_k(t)|i)$ be the pdf describing the uncertainty in the structural response at time t at degree of freedom k. The specific task addressed in this paper is: given $p(\underset{\sim}{\theta}|i)$ and the mean and variance of the model error, compute the mean and variance of $y_k(t)$. It is assumed that the excitation is explicitly prescribed in the information i, although the approach can be generalized to include uncertainties in the excitation as well. We plan to report at a later date on this generalization and on a method for computing the full response distribution, $p(y_k(t)|i)$ rather than just its first two moments.

<u>Choices for Parameter pdf</u>

1) Choose $p(\underset{\sim}{\theta}|i)$ subjectively based on past experience dealing with similar structures: a convenient mathematical form is chosen which is roughly consistent with the engineer's judgement regarding the relative plausibilities of different values of $\underset{\sim}{\theta}$. Often knowing one parameter θ_j would not influence judgement of plausibilities of the other parameters, so the parameters are mutually irrelevant to one another. The joint pdf can then be taken as the product of the separate pdf's $p(\theta_j|i)$.
2) Choose $p(\underset{\sim}{\theta}|i)$ by the maximum entropy principle (Ref. 5) which produces the greatest uncertainty in the parameters consistent with the specified constraints. For example, if the set of plausible parameter vectors Θ is a bounded region, then it produces the uniform distribution if there are no other constraints, and it produces a truncated uncorrelated multi-dimensional Gaussian distribution if only the means and variances are specified for each θ_j.
3) Choose the "posterior" distribution derived via Bayes Theorem from a subjective "prior" distribution and test data, if available. This requires using probabilistic system identification to process the data.

MOMENTS OF THE UNCERTAIN RESPONSE

<u>Theory</u> The task is to determine the expected value and variance of the structural response based on a structural model and a description of the uncertainties associated with it. First,

we express the structural response at degree of freedom k at time t by:

$$y_k(t) = x_k(t|\underset{\sim}{\theta}) + \epsilon_k(t|\underset{\sim}{\theta}) \tag{1}$$

where $x_k(t|\underset{\sim}{\theta})$ is the corresponding (linear or nonlinear) model response for the parameter values $\underset{\sim}{\theta}$, and $\epsilon_k(t|\underset{\sim}{\theta})$ is the model error for the model with parameter values $\underset{\sim}{\theta}$. Second, the uncertainty in the model parameters is described by the pdf $p(\underset{\sim}{\theta}|i)$ and the uncertainty in the model error is described by just the first two moments:

$$E\big[\epsilon_k(t|\underset{\sim}{\theta})\big] = 0 \quad \text{and} \quad E\big[\epsilon_k^2(t|\underset{\sim}{\theta})\big] = \sigma_k^2(t|\underset{\sim}{\theta}) \ , \tag{2}$$

where σ_k^2 needs to be prescribed. Taking the mean error for a given model described by $\underset{\sim}{\theta}$ to be zero is reasonable if it is judged that any particular positive error is equally plausible as the corresponding negative error with the same magnitude. It follows that:

$$E\big[y_k(t)|\underset{\sim}{\theta}\big] = x_k(t|\underset{\sim}{\theta}) \quad \text{and} \quad E\big[y_k^2(t)|\underset{\sim}{\theta}\big] = x_k^2(t|\underset{\sim}{\theta}) + \sigma_k^2(t|\underset{\sim}{\theta}) \tag{3}$$

The desired expected value and variance of the structural response are therefore:

$$\begin{aligned} E\big[y_k(t)\big] &= \int_\Theta E\big[y_k(t)|\underset{\sim}{\theta}\big] p(\underset{\sim}{\theta}|i) d\underset{\sim}{\theta} \\ &= \int_\Theta x_k(t|\underset{\sim}{\theta})\, p(\underset{\sim}{\theta}|i) d\underset{\sim}{\theta} = E\big[x_k(t)\big] \end{aligned} \tag{4}$$

i.e., expected structural response is the mean model response, and:

$$\begin{aligned} V_{ar}\big[y_k(t)\big] &= E\big[y_k^2(t)\big] - E^2\big[y_k(t)\big] \\ &= \int_\Theta E\big[y_k^2(t)|\underset{\sim}{\theta}\big]\, p(\underset{\sim}{\theta}|i) d\underset{\sim}{\theta} - E^2\big[x_k(t)\big] \\ &= \int_\Theta \big\{x_k(t|\underset{\sim}{\theta}) - E\big[x_k(t)\big]\big\}^2 p(\underset{\sim}{\theta}|i) d\underset{\sim}{\theta} + \int_\Theta \sigma_k^2(t|\underset{\sim}{\theta})\, p(\underset{\sim}{\theta}|i) d\underset{\sim}{\theta} \\ &= V_{ar}\big[x_k(t)\big] + E\big[\sigma_k^2(t)\big] \end{aligned} \tag{5}$$

i.e., variance of structural response is variance in model response due to uncertain parameters plus mean variance in the model error. If the variance of the model error is taken to be constant, then $E\big[\sigma_k^2(t)\big] = \sigma_k^2$, which could be chosen based on the experience gained from applying system identification methods to structures. It remains to evaluate the integrals for the mean and variance of the model response, which cannot usually be done analytically. Therefore, the problem reduces to finding an accurate and efficient approximate method.

<u>Methods of Computation</u>

(i) Monte Carlo methods based on sampling using simulations. High accuracy requires large sample sizes and therefore these methods can become very expensive computationally.

(ii) Second-moment approach (SMA) which involves expanding to second order the model response $x(t|\underset{\sim}{\theta})$ in a Taylor series about the expected value $\bar{\underset{\sim}{\theta}}$ of $\underset{\sim}{\theta}$ (Ref. 6). We show that for dynamic problems this does not work well.

(iii) Fourier-series approach (FSA) which involves expanding the model response $x(t|\underset{\sim}{\theta})$ in a Fourier series w.r.t. $\underset{\sim}{\theta}$. This new approach works well if the number of parameters is small, but becomes cumbersome if the parameter space is of high dimension.

(iv) Numerical integration is straight-forward but computationally costly for high-dimensional parameter spaces. It turns out that it is advantageous to uniformly discretize the cumulative probability rather than uniformly discretize the range of each parameter. The method can be extended to high dimensions using cumulative conditional probability distributions, but we illustrate the idea in this paper with a one-dimensional example.

APPLICATION TO UNCERTAIN LINEAR OSCILLATOR

Introduction Let

$$\ddot{x} + 2\varsigma\omega\dot{x} + \omega^2 x = f(t) \; ; \qquad x = x(t|\omega) \; , \quad x(0|\omega) = 0 \; , \quad \dot{x}(0|\omega) = 0 \tag{6}$$

be the model of an oscillator starting from rest. For illustrative purposes, assume that the damping ς and excitation $f(t)$ are deterministic (known precisely), but that the natural frequency ω is uncertain. Use $p(\omega|i)$ to describe the plausibility of possible values of ω over a finite range $\Omega = [\omega_1, \omega_2]$. The solution of (6) for the model response is

$$x(t|\omega) = \int_0^t f(t-\tau)\, h(\tau|\omega) d\tau \; ; \qquad h(\tau|\omega) = \frac{1}{\omega_d} e^{-\varsigma\omega\tau} \sin \omega_d \tau \tag{7}$$

where $\omega_d = \omega\sqrt{1-\varsigma^2}$. The problem is to determine $E[x(t)]$ and $V_{ar}[x(t)]$ for a particular choice of $p(\omega|i)$. In what follows, we suppress the information i in the notation.

Second-Moment Approach SMA starts with

$$x(t|\omega) \simeq x(t|\bar{\omega}) + \left.\frac{\partial x(t|\omega)}{\partial \omega}\right|_{\omega=\bar{\omega}} \Delta\omega + \frac{1}{2} \left.\frac{\partial^2 x(t|\omega)}{\partial \omega^2}\right|_{\omega=\bar{\omega}} (\Delta\omega)^2 \tag{8}$$

where $\Delta\omega = (\omega - \bar{\omega})$ and $\bar{\omega}$ is the mean value of ω. This leads to

$$E[x(t)] \simeq x(t|\bar{\omega}) + \frac{1}{2} \left.\frac{\partial^2 x(t|\omega)}{\partial \omega^2}\right|_{\omega=\bar{\omega}} \sigma_\omega^2 \; , \quad V_{ar}[x(t)] \simeq \left(\left.\frac{\partial x(t|\omega)}{\partial \omega}\right|_{\omega=\bar{\omega}} \right)^2 \sigma_\omega^2 \tag{9}$$

Note that these approximations are independent of $p(\omega)$ except for $\bar{\omega}$ and $\sigma_\omega^2 = V_{ar}[\omega]$. The problem is that the quadratic approximation in (8) can be very poor over Ω, since $x(t|\omega)$ has an oscillatory type of behavior w.r.t. ω. For example, for free vibrations of an undamped linear oscillator starting with unit displacement, $x(t|\omega) = \cos \omega t$. Thus, (9) gives:

$$E[x(t)] = \cos \bar{\omega} t \left[1 - \frac{\sigma_\omega^2}{2} t^2 \right] \; , \quad V_{ar}[x(t)] = \sigma_\omega^2 t^2 \sin^2 \bar{\omega} t \tag{10}$$

This is obviously misleading, since each $x(t|\omega)$ has a bounded amplitude of unity and so $E[x(t)]$ and $V_{ar}[x(t)]$ cannot become unbounded with time. In fact, if ω is uniformly distributed over the interval Ω, then:

$$E[x(t)] = \int_\Omega x(t|\omega)\, p(\omega) d\omega = \int_{\omega_1}^{\omega_2} \cos \omega t \frac{1}{(\omega_2 - \omega_1)} d\omega = \frac{\cos(\bar{\omega} t) \sin(\sqrt{3}\,\sigma_\omega t)}{\sqrt{3}\,\sigma_\omega t} \tag{11}$$

where $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$ and $\sigma_\omega^2 = \frac{1}{12}(\omega_1 - \omega_2)^2$. We conclude that although SMA allows efficient computation of the mean and variance of the response, its accuracy is too questionable.

Fourier Series Approach   As an alternative, consider a truncated Fourier series expansion w.r.t. ω over the interval $\Omega = [\omega_1, \omega_2]$, rather than a truncated Taylor series:

$$x(t|\omega) \simeq x_L(t|\omega) + a_0(t) + \sum_{n=1}^{N} [a_{sn}(t)\sin(nu\omega) + a_{cn}(t)\cos(nu\omega)] \tag{12}$$

where $u = 2\pi/(\omega_2 - \omega_1)$ and $x_L(t|\omega) = b_0(t) + b_1(t)\omega$ is a linear function of ω between $x_1(t) = x(t|\omega_1)$ and $x_2(t) = x(t|\omega_2)$. x_L is introduced to reduce the effects of Gibbs' phenomenon at the endpoints in the truncated Fourier series. Thus, $a_0(t)$ and the $a_{sn}(t)$, $a_{cn}(t)$ are actually the Fourier coefficients of $[x(t|\omega) - x_L(t|\omega)]$ over Ω. By evaluating only the first few terms of the Fourier series, a good approximation for $x(t|\omega)$ over Ω is achieved. It is possible to derive analytical expressions for the derivatives of $a_0(t)$, $a_{sn}(t)$ and $a_{cn}(t)$ which involve convolutions of the excitation $f(t)$ with known functions. The time histories of these Fourier coefficients can then be evaluated numerically by using an FFT algorithm, allowing the moments to be computed. For example, if $p(\omega)$ is chosen to be a uniform distribution over $\Omega = [\omega_1, \omega_2]$, then:

$$E[x(t)] = \int_{\Omega} x(t|\omega)\, p(\omega) d\omega = x_L(t|\bar{\omega}) + a_0(t) \tag{13}$$

where $\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2)$. A closed-form expression for $V_{ar}[x(t)]$ can also be derived for this case.

Numerical Integration   One approach to approximate the integrals $\int_{\Omega} x^k(t|\omega)\, p(\omega) d\omega$ $(k = 1,2)$ required for $E[x(t)]$ and $V_{ar}[x(t)]$ is to use $\sum_{i=1}^{N} x^k(t|\omega_i)\, p(\omega_i)\Delta\omega$ where $\Delta\omega = (\omega_2 - \omega_1)/N$ and $\omega_i = \omega_1 + (i - 1/2)\Delta\omega$, $i = 1, 2, \ldots, N$, that is, the interval $\Omega = [\omega_1, \omega_2]$ is subdivided into N equal intervals. It is more efficient, however, to choose the ω's by subdividing the range of $P(\omega)$ into N equal subintervals where $P(\omega) = \int_{-\infty}^{\omega} p(\omega) d\omega$, that is, choose the ω_i's so that the probability of ω lying in any subinterval is $1/N$. This gives as an approximation:

$$\int_{\Omega} x^k(t|\omega)\, p(\omega) d\omega = \frac{1}{N} \sum_{i=1}^{N} x^k(t|\omega_i) \tag{14}$$

where $P(\omega_i) = (i - 1/2)/N$, $i = 1, 2, \ldots, N$. When many time steps are involved, numerical integration is less efficient than FSA for a single parameter, but it is more readily adapted to higher-dimensional parameter spaces than FSA. We note also that this numerical integration approach applies equally well to linear and nonlinear models, since it simply requires a set of model responses $x(t|\underset{\sim}{\theta}_i)$ to be computed.

Figure 1 shows the mean and standard deviation time histories computed by the three methods for a uniformly distributed frequency in the range $3 \leq \omega \leq 4$ rad/s. [For FSA, $N = 3$ in (12)] The oscillator was subjected to a base acceleration given by the 1940 El Centro NS record. The FSA method is nearly indistinguishable in these plots from the "exact" moments computed by numerical integration using a fine discretization. The SMA, on the other hand, gives a poor approximation. We also used numerical integration to compute the moment time histories for a truncated Gaussian distribution for ω over [3,4], with $\bar{\omega} = 3.5$ rad/s and $\sigma = 0.25$ rad/s. These plots differed from the uniform distribution case by less than about 10%.

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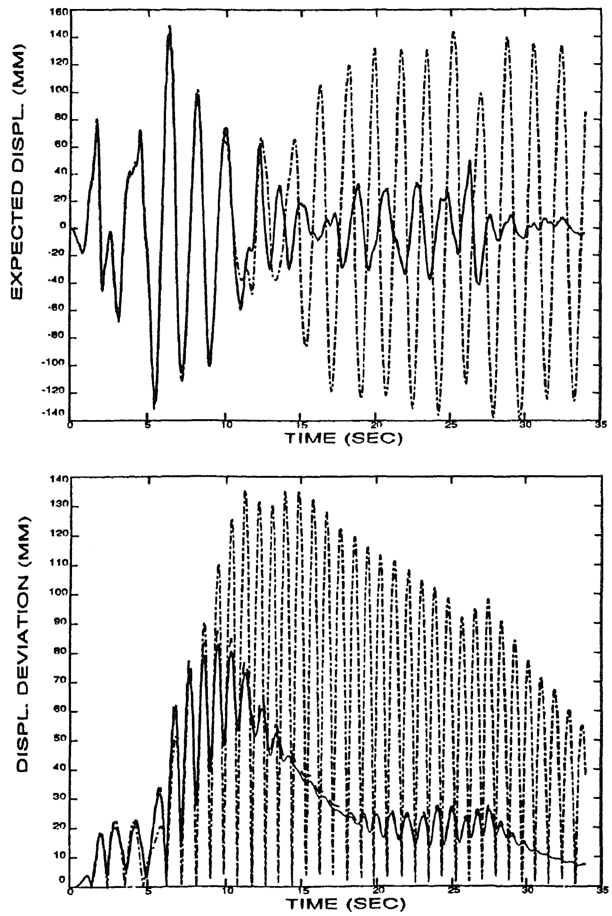

Figure 1 Expected displacement and displacement deviation for 5%-damped uncertain linear oscillator with assigned $p(\omega)$ uniform over [3,4]. (— numerical integration; -·-·- SMA; - - - - FSA).