STOCHASTIC ESTIMATES OF HYSTERETIC STRUCTURAL SYSTEMS UNDER SEISMIC EXCITATIONS

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SUMMARY

A method of adaptive stochastic estimates of dynamic states of hysteretic structural systems including damage states as well as excitation and system parameters is presented. By making use of differential representations of hysteretic constitutive laws, measures of structural damages, and by considering the correlation between system and observation noises, the problem is formulated in the form of the Itô stochastic differential equations. The differential forms of the conditional probability density functions of state variables given observation data in a finite time interval are determined for filtering, smoothing and prediction problems, and their solution procedures are described.

INTRODUCTION

The objective of this paper is to find a method of adaptive stochastic estimates of state variables including measures of structural damages such as the cumulative plastic deformation ratio, low cycle fatigue damage factor as well as excitation and system parameters of hysteretic structural systems subjected to intense seismic excitations. The proposed method of adaptive stochastic estimates is based on the theory of continuous Markov vector processes, specifically relying on the Itô stochastic differential equations. By making use of the differential representations of hysteretic constitutive laws and of the measures of structural damages (Ref.1), parameter dynamics which describe their time-variation in differential forms as well as white noise differential equations governing seismic excitations, the system equations are expressed in the form of the Itô stochastic differential equations. On the other hand, the observation equations are supposed to be given in the similar equations to the system equations taking into account the correlation between system and observation noises.

FORMULATION BASED ON STOCHASTIC DIFFERENTIAL EQUATIONS

The basic equations for adaptive stochastic estimates of hysteretic structural systems are expressed as the following Itô stochastic differential equations consisting of system and observation equations (Ref.2):

\[ dZ_t = P_t(Z_t)dt + G_t(Z_t)dW_t, \quad Z_{t_0} = z_0 \]  
\[ dY_t = H_t(Z_t)dt + R_t(Z_t)dV_t, \quad Y_{t_0} = y_0 \]

In the above equations, \( Z_t \) (nx1) and \( Y_t \) (kx1) are vector-valued state and output
variables, respectively, $F_t(Z_t)$ (nx1), $G_t(Z_t)$ (nxm), $H_t(Z_t)$ (kx1) and $R_t(Z_t)$ (kx k) are vector- or matrix-valued nonlinear functions of time $t$ and state vector $Z_t$, and $W_t$ (mx1) and $V_t$ (kx1) are, respectively, normalized system and observation noises which are Wiener processes with unit intensity matrices and are in general considered to be correlated processes with a time-dependent correlation intensity matrix $J_t$ and are independent of vector-valued random initial conditions, $z_0$ (nx1) and $y_0$ (kx1). The system equations given by Eq. (1) are by themselves the Itô stochastic differential equations which can be constructed by making use of differential forms of hysteretic constitutive laws and measures of structural damages as well as the state space equations governing nonstationary seismic excitations and parameter dynamics.

The differential forms of hysteretic constitutive laws of anisotropic structural systems under multi-axial deformations are obtained by applying the conventional plasticity theory as follows:

\[ \dot{\varepsilon}_{ij} = \left[ \delta_{ij} - \frac{c_{ik}}{\frac{\partial\varepsilon}{\partial z_k} \frac{\partial\varepsilon}{\partial z_k}} \right] \frac{\partial\varepsilon}{\partial z_j} \]  \hspace{1cm} (3)

\[ \ddot{x} = \frac{1}{2} \frac{\partial}{\partial x^p} L_{x^p} \frac{\partial\varepsilon}{\partial z_j} \]  \hspace{1cm} (4)

\[ \frac{\partial z}{\partial x^p} = \frac{\varepsilon_c}{x - \varepsilon_c} \]  \hspace{1cm} (5)

\[ \phi_i = r \lambda_i x_j + (1-r) \lambda_i z_j \]  \hspace{1cm} (6)

In the above equations, summation convention is used, and $\phi_i$, $x_i$ and $z_i$ are restoring force, total deformation and elastic deformation of hysteretic element of the $i$th component, $x$, $x^p$, and $z$ are equivalent total deformation, plastic and elastic deformations, $\delta_{ij}$ and $c_{ij}$ are Kronecker's delta and dimensionless compliance matrix, respectively, and $r$ and $\lambda_i$ are rigidity ratio and rigidity matrix, respectively. The equivalent elastic deformation $z = z(z_1, z_2, \ldots, z_m)$ is determined from the yield condition with hardening, and governed by a one-dimensional constitutive law of piecewise or smooth hysteresis, while the multi-axial constitutive law is determined by assuming the associative flow rule and by making use of the concepts of equivalent force and deformation under multi-axial state.

The differential forms of measures of structural damages are, for instance, in the case of nonlinear low cycle fatigue, expressed in the following forms:

\[ \dot{\eta}_{ftn} = ab(\eta_{ftn})^{b-1} c_{F}^{-a} |x|^{a-1} |\dot{x}| = \varepsilon_{\eta_{ftn}} \]  \hspace{1cm} (7)

\[ \dot{\eta}_{fpn} = ab(\eta_{fpn})^{b-1} (1-r)^{a} c_{P}^{-a} |x-z|^{a-1} |\dot{x-z}| = \varepsilon_{\eta_{fpn}} \]  \hspace{1cm} (8)

where $\eta_{ftn}$ and $\eta_{fpm}$ are low cycle fatigue damage factors in terms of total and plastic deformations, respectively, $c_p$ and $c_p$ are one-sided ultimate total and plastic deformations. Specifically for the case where $b = 1$, Eqs. (7) and (8) reduce to the differential forms of the corresponding linear low cycle fatigue. Under the multi-axial state the equivalent total and plastic deformations, $x$ and $x - z$ may be used in Eqs. (7) and (8).

The differential forms of unknown excitation and system parameters are
given by either of the following equations:

\[ a_t = \frac{P}{p=0} a_t p^p, \quad a_t^{(p)} = a_t^{(p+1)}, \quad p = 0, 1, \ldots, P - 1, \quad a_t^{(P)} = 0 \]  \( (9) \)

\[ c_t = a_t^{\gamma} \exp(-\gamma \delta^2), \quad \tilde{c}_t = (\delta_t - \gamma \delta_t \delta - 1) c_t, \quad \tilde{\alpha} = \tilde{\beta} = \tilde{\gamma} = \tilde{\delta} = 0 \]  \( (10) \)

By making use of the above-mentioned differential forms, the system equations may be expressed in the form,

\[ dZ_t = (A_{t}Z_t + B_{t}(Z_t))dt + (C_{t} + D_{t}(Z_t))dW_t, \quad Z_{t_0} = z_0 \]  \( (11) \)

\[ Z_t = \left[ \begin{array}{llll} \mathbf{Z}_t^T & \mathbf{s}_t^T & \mathbf{n}_t^T & \mathbf{p}_t^T \end{array} \right]^T \]  \( (12) \)

where \( \mathbf{Z}_t, \mathbf{s}_t, \mathbf{n}_t, \mathbf{p}_t \) and \( \mathbf{z}_t \) are, respectively, the state sub-vectors concerning seismic excitations, hysteretic structural system, structural damages, and excitation and system unknown parameters. The state sub-vector \( \mathbf{z}_t \) is composed of \( x_i, \ z_i, \ i = 1, 2, \ldots, m \), \( x \) and the state variables which control the degradation of hysterectes.

On the other hand, the observation equations are also given in the similar form to the Itô stochastic differential equations. The quantity \( d\mathbf{y}/dt \) is observed during a finite time interval \([\mathbf{t}_0, \mathbf{t}]\) with the observation noise.

**BASIC EQUATIONS OF NONLINEAR STOCHASTIC ESTIMATES**

The solution processes \( Z_t \) and \( (Z_t, Y_t) \) are known to be Markovian, and characterized by Kolmogorov's forward and backward differential operators:

\[ L^{\ast}_{Z_t} Y_t = L^{\ast}_{Z_t} + L^{\ast}_{Y_t}, \quad L^{\ast}_{Z_t Y_t} = L^{\ast}_{Z_t} + L^{\ast}_{Y_t} + G^{\ast}_{Z_t Y_t} \]  \( (13) \)

and

\[ L^{\ast}_{Z_t} \psi = \frac{1}{2}(G_{t} \mathbf{R}_t \mathbf{R}_t^T) \psi_{\kappa} \psi_{\nu} + F_{t} \psi_{\kappa} \]

\[ L^{\ast}_{Y_t} \psi = \frac{1}{2}(R_t \mathbf{R}_t^T) \lambda \mu + H_t \psi_{\mu} \lambda \]

\[ G^{\ast}_{Z_t Y_t} \psi = \frac{1}{2}(G_{t} \mathbf{R}_t \mathbf{R}_t^T) \psi_{\kappa} \psi_{\nu} + \frac{1}{2}(G_{t} \mathbf{R}_t \mathbf{R}_t^T) \psi_{\kappa} \mu \]  \( (14) \)

where \( L^{\ast}_{Z_t}, \ L^{\ast}_{Y_t} \) and \( L^{\ast}_{Z_t Y_t} \) are forward operators concerning the processes \( (Z_t, Y_t) \), \( Z_t \) and \( Y_t \), \( G^{\ast}_{Z_t Y_t} \) is a forward operator associated with the correlation between system and observation noises, \( L^{\ast}_{Z_t Y_t}, \ L^{\ast}_{Z_t}, \ L^{\ast}_{Y_t} \) and \( G^{\ast}_{Z_t Y_t} \) are the corresponding backward operators and \( \psi \) is a scalar-valued differentiable function of \( t, Z_t \) and \( Y_t \). In Eq. (14), summation and differentiation conventions are used.

Under the minimum error variance criterion, the optimal estimator of a twice continuously differentiable function \( \phi(Z_t) \) and its time differential are expressed as follows (Ref.3):

\[ \langle \phi(Z_t) \rangle_t = \int_{\mathbb{R}^n} dz_t \phi(z_t) p_t(z_t | Y_t), \quad \tau \leq t \]  \( (15) \)

\[ d_{-\tau} \phi(Z_t) > t = \int_{\mathbb{R}^n} dz_t \phi(z_t) d_{-\tau} p_t(z_t | Y_t) \]  \( (16) \)

where \( p_t(z_t | Y_t) \) is conditional probability density function given the observation \( Y_t \) in \([t_0, t] \). By making use of Markovian properties of \( Z_t \) and \( (Z_t, Y_t) \), the fundamental equations governing the conditional probability density functions and the associated Itô-Dynkin type formulas governing the conditional expectations of \( \phi(Z_t) \) are obtained for filtering, fixed interval smoothing and prediction problems as follows (Ref.2):

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Filtering problem : $\tau = t$ (Ref. 4)

$$d_{\tau}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right) = L_{z_{\tau}}^{*}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right) d\tau + \left( H_{z_{\tau}}^{T}(z_{\tau})(R_{\tau}(z_{\tau})R_{\tau}^{T}(z_{\tau}))^{-1} \right)$$

$$- \left( H_{z_{\tau}}^{T}(Z_{\tau})(R_{\tau}(Z_{\tau})R_{\tau}^{T}(Z_{\tau}))^{-1} \right)_{\tau}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right) d\nu_{\tau}$$

$$- \left\{ \frac{\partial}{\partial z_{\tau}} \right\} \left( P_{\tau}(z_{\tau})G_{\tau}(z_{\tau})J_{\tau}R_{\tau}^{T}(z_{\tau})(R_{\tau}(z_{\tau})R_{\tau}^{T}(z_{\tau}))^{-1} \right)_{\tau} d\nu_{\tau}$$

(17)

and

$$d_{\tau} \Phi(\mathcal{Z}_{\tau}) = \left( H_{z_{\tau}}^{T}(\mathcal{Z}_{\tau})(R_{\tau}(\mathcal{Z}_{\tau})R_{\tau}^{T}(\mathcal{Z}_{\tau}))^{-1} \right)_{\tau} d\nu_{\tau}$$

$$+ \left( J_{\tau}R_{\tau}(\mathcal{Z}_{\tau})(R_{\tau}(\mathcal{Z}_{\tau})R_{\tau}^{T}(\mathcal{Z}_{\tau}))^{-1} \right)_{\tau} d\nu_{\tau}$$

(18)

with $\Phi(\mathcal{Z}_{\tau})_{\tau} = \Phi(\mathcal{Z}_{\tau})_{\tau}$. In the above equations,

$$d\nu_{\tau} = dY_{\tau} - H_{\tau}(\mathcal{Z}_{\tau})_{\tau} d\tau$$

(19)

which is called innovations process, and its statistical properties are the same as the observation noise.

Smoothing problem : $\tau < t$ and fixed interval $[\tau, t]$

$$d_{\tau}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right) = \frac{L_{z_{\tau}}^{*}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right)}{P_{\tau}(z_{\tau} \mid Y_{\tau})} d\tau - \frac{P_{\tau}(z_{\tau} \mid Y_{\tau})}{P_{\tau}(z_{\tau} \mid Y_{\tau})} \left[ \frac{L_{z_{\tau}}^{*}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right)}{P_{\tau}(z_{\tau} \mid Y_{\tau})} \right] d\tau$$

(20)

and

$$d_{\tau} \Phi(\mathcal{Z}_{\tau}) = \Phi(\mathcal{Z}_{\tau}) \frac{L_{z_{\tau}}^{*}P_{\tau} \left( Z_{\tau} \mid Y_{\tau} \right)}{P_{\tau}(Z_{\tau} \mid Y_{\tau})} d\tau - \frac{P_{\tau}(Z_{\tau} \mid Y_{\tau})}{P_{\tau}(Z_{\tau} \mid Y_{\tau})} \left[ \frac{L_{z_{\tau}}^{*}P_{\tau} \left( Z_{\tau} \mid Y_{\tau} \right)}{P_{\tau}(Z_{\tau} \mid Y_{\tau})} \right] d\tau$$

(21)

with $\Phi(\mathcal{Z}_{\tau})_{\tau} = \Phi(\mathcal{Z}_{\tau})_{\tau}$, $\tau \geq \tau \geq \tau_{0}$.

Prediction problem : $\tau > t$ and fixed interval $[\tau_{0}, \tau]$

$$d_{\tau}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right) = L_{z_{\tau}}^{*}P_{\tau} \left( z_{\tau} \mid Y_{\tau} \right) d\tau$$

(22)

and

$$d_{\tau} \Phi(\mathcal{Z}_{\tau}) = \frac{L_{z_{\tau}}^{*} \Phi(\mathcal{Z}_{\tau})}{\Phi(\mathcal{Z}_{\tau})} d\tau$$

(23)

with $\Phi(\mathcal{Z}_{\tau})_{\tau} = \Phi(\mathcal{Z}_{\tau})_{\tau}$, $\tau \geq t$.

The filtering is the most basic problem in the sense that both in the fundamental equation and the Itô-Dynkin type formula of the smoothing problem, the conditional probability density function of the associated filtering problem is included, and that the end condition of the Itô-Dynkin type formula of the smoothing problem and the initial condition of the prediction problem are given by the end condition of the associated filtering problem.

CONDITIONAL MOMENT EQUATIONS AND PROBABILITY DENSITY FUNCTIONS

The aim of stochastic estimates is to determine the conditional probability density functions (Ref. 3). Since it is difficult to solve directly the fundamental equation which is an extension of the Fokker-Planck equation, truncated conditional moment equations are numerically solved to obtain the time-dependent conditional probability density functions (Refs. 1-3). The outline of the solution procedure is shown in the case of filtering problem, since the filtering problem is the most fundamental and the same techniques are applicable to the other problems (Ref. 2).
By substituting $\phi(\tilde{Z}_\tau) = Z_\tau$ in Eq. (18), the first order conditional moment equations are obtained as follows:

$$
\frac{d\tau}{\tau} <Z_\tau>_\tau = <F_\tau(Z_\tau)>_\tau d\tau + K_\tau d\nu_\tau 
$$

(24)

where

$$
K_\tau = <S_\tau^1(Z_\tau) + S_\tau^2(Z_\tau)>_\tau 
$$

(25)

$$
S_\tau^1(Z_\tau) = H_\tau^1(Z_\tau)(R_\tau(Z_\tau)R_\tau^T(Z_\tau))^{-1} - \frac{\partial H_\tau(Z_\tau)(R_\tau(Z_\tau)R_\tau^T(Z_\tau))^{-1}}{\partial Z_\tau} + \frac{\partial}{\partial Z_\tau} \{G_\tau(Z_\tau)J_\tau R_\tau^T(Z_\tau)(R_\tau(Z_\tau)R_\tau^T(Z_\tau))^{-1}\}
$$

(26)

$$
S_\tau^2(Z_\tau) = G_\tau(Z_\tau)J_\tau R_\tau^T(Z_\tau)(R_\tau(Z_\tau)R_\tau^T(Z_\tau))^{-1}
$$

(27)

The Itô-Dynkin type formula for estimation error vector, $\tilde{Z}_\tau = Z_\tau - <Z_\tau>_\tau$ is determined as

$$
\frac{d\tau}{\tau} <\phi(\tilde{Z}_\tau)>_\tau = <L\tilde{Z}_\tau \phi(\tilde{Z}_\tau)>_\tau d\tau + \phi(\tilde{Z}_\tau)S_\tau^1(Z_\tau)
$$

$$
+ \left\{ \frac{\partial}{\partial Z_\tau} \phi(\tilde{Z}_\tau)S_\tau^2(Z_\tau) \right\} d\nu_\tau
$$

with $<\phi(\tilde{Z}_\tau)>_{t_0}, \tau \geq t_0$

(28)

where

$$
L\tilde{Z}_\tau \psi = \frac{1}{2}Q\tilde{Z}_\tau \psi + F\tilde{Z}_\tau \psi
$$

(29)

$$
Q_\tau(z_\tau) = G_\tau(z_\tau)G_\tau^T(z_\tau) + K_\tau R_\tau(z_\tau)J_\tau^T G_\tau^T(z_\tau) - G_\tau(z_\tau)J_\tau^T R_\tau^T(z_\tau)K_\tau
$$

(30)

$$
F_\tau'(z_\tau) = F_\tau(z_\tau) - K_\tau H_\tau(z_\tau)
$$

(31)

$$
\tilde{F}_\tau(z_\tau) = F_\tau(z_\tau) - <F_\tau(Z_\tau)>_\tau, \quad \tilde{H}_\tau(z_\tau) = H_\tau(z_\tau) - <H_\tau(Z_\tau)>_\tau
$$

(32)

Substitutions of the following equation into Eq. (28) may yield the central moment equations of higher than or equal to the second order.

$$
\phi(\tilde{Z}_\tau) = \sum_{s=1}^{n} \tilde{Z}_{t,s}^{k_s}, \quad \sum_{s=1}^{n} k_s \geq 2
$$

(33)

In order to evaluate the conditional expectations of nonlinear terms in the right-hand sides of the conditional moment equations and to truncate the conditional moment equations, an analytical form of the conditional probability density function expressed in terms of multi-dimensional Hermite polynomials and one-dimensional Laguerre polynomials is introduced. A fourth order expression is given in the form,

$$
P_\tau(\xi_1, \xi_2, \ldots, \xi_{n_1}, \eta_1, \eta_2, \ldots, \eta_{n_2}|Z_\tau)
$$

$$
= \omega(\xi_1, \xi_2, \ldots, \xi_{n_1}) \sum_{j=1}^{n_2} \sum_{i=1}^{n_2} \sum_{j=1}^{n_2} H_{l_1}^j(\xi_1, \xi_2, \ldots, \xi_{n_1}) L_{l_1}^{(B_j-1)}(\eta_1, \eta_2, \ldots, \eta_{n_2})
$$

$$
+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_2} B_{l_1j} H_{l_1}^j(\xi_1, \xi_2, \ldots, \xi_{n_1}) L_{l_1}^{(B_j-1)}(\eta_1, \eta_2, \ldots, \eta_{n_2})
$$

$$
+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_2} C_{l_1j} L_{l_1}^{(B_j-1)}(\eta_1, \eta_2, \ldots, \eta_{n_2})
$$

$$
+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_2} D_{l_1j} L_{l_1}^{(B_j-1)}(\eta_1, \eta_2, \ldots, \eta_{n_2})
$$

$$
+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_2} E_{l_1j} L_{l_1}^{(B_j-1)}(\eta_1, \eta_2, \ldots, \eta_{n_2})
$$

$$
+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_2} F_{l_1j} L_{l_1}^{(B_j-1)}(\eta_1, \eta_2, \ldots, \eta_{n_2})
$$

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\[
+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \sum_{l=k+1}^{n_2} E_{1jk} H_2^{i+1}(\xi_1, \xi_2, ..., \xi_{n_1}) L_2(\beta k^{-1}) \langle \nu_k \kappa_k \rangle \\
+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \sum_{l=k+1}^{n_2} E_{1jk} H_2^{i+1}(\xi_1, \xi_2, ..., \xi_{n_1}) L_1(\beta k^{-1}) \langle \nu_k \kappa_k \rangle L_1(\beta l^{-1}) \langle \nu_l \kappa_l \rangle 
\]

where
\[
\beta_j = \langle \eta_j > \langle \eta_j >^{-1} - 1, \quad \nu_j = \langle \eta_j > \langle \eta_j >^{-1} - 1, \quad \eta_j = \eta_j - \langle \eta_j > 
\]

In Eq. (34), \( w_1, h_1^1 \) and \( h_2^1 \) are an \( n_1 \)-dimensional normal density function, the first and second order \( n_1 \)-dimensional Hermite polynomials, and \( w_0, L_1(\beta^{-1}) \) and \( L_2(\beta^{-1}) \) are the gamma density function and the first and second order generalized Laguerre polynomials. The marginal density functions concerning Gaussian variables \( \xi_i \) and gamma variables \( \eta_j \) are, respectively, given by
\[
p_T(\xi_1, \xi_2, ..., \xi_{n_1} | Y_T) = w_1(\xi_1, \xi_2, ..., \xi_{n_1}) 
\]
and
\[
p_T(\eta_1, \eta_2, ..., \eta_{n_2} | Y_T) = \prod_{j=1}^{n_1} w_0(\eta_j) \prod_{i=1}^{n_1} \sum_{j=i+1}^{n_2} C_{i,j} L_1(\beta^{-1}) \langle \nu_i \eta_i \rangle L_1(\beta^{-1}) \langle \nu_j \eta_j \rangle 
\]

By making use of the definitions and orthogonality relationships of the polynomials, the coefficient functions appearing in Eqs. (34) to (37) can be expressed in terms of conditional first order moment functions and higher order central moment functions up to the fourth order. Substitution of Eq. (34) in Eqs. (24), (28) to (33), yields the truncated fourth order conditional moment equations which are solved numerically to determine the time-dependent conditional joint probability density functions as well as optimal estimators of statistics of state variables including measures of structural damages and unknown excitation and system parameters.

CONCLUDING REMARKS

The adaptive stochastic estimates of hysteretic structural systems are formulated in the form of the Itô stochastic differential equations. The fundamental equations and the associated Itô-Dynkin type formulas are given for filtering, fixed interval smoothing and prediction problems. The solution techniques of the time-dependent conditional probability density functions and the optimal statistics of state variables are shown by introducing a finite mixed-type series expansion of the non-Gaussian density function in terms of multidimensional Hermite polynomials and one-dimensional Laguerre polynomials.

REFERENCES