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STOCHASTIC RESPONSE OF A COULOMB SLIP SYSTEM SUBJECTED TO EARTHQUAKE EXCITATION USING EQUIVALENT LINEARIZATION

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SUMMARY

The applicability of statistical equivalent linearization for evaluating the nonstationary stochastic response of a strongly nonlinear hysteretic system subjected to typical earthquake excitation is investigated. The applicability is demonstrated for the Coulomb slip system which represents the most extreme hysteretic system encountered in earthquake engineering. It is shown that the response of a Coulomb slip system can be evaluated with high accuracy by non-Gaussian equivalent linearization where the non-Gaussian properties of the response are taken into account.

INTRODUCTION

The method of statistical equivalent linearization has remained an attractive tool over 30 years since its first formulation in order to estimate the first two moments of the stochastic response of a nonlinear system. Although the method gives sufficiently accurate results for many nonlinear problems of engineering interest, the conventional approach fails to predict the second moments of strongly nonlinear systems such as the Coulomb slip system. In [1], it is even claimed that no statistical linearization technique can predict accurately the second moments of the (displacements) response of a Coulomb slip system.From [1] follows that no amount of sophistication of the method will lead to correct second moments of a hysteretic system, especially when the excitation is strong, because all hysteretic systems exhibit to a certain degree physical features of a Coulomb slip system.

A closer look, however, reveals that the failure to predict accurately the displacement response (drift) of Coulomb slip system can be attributed to the conventional assumption of a jointly Gaussian distributed nonlinear response. The assumption of a Gaussian distributed response however, is not inherent to the technique of equivalent linearization. It will be shown at the beginning of this paper that a statistically "true" linear system does exist in case of a symmetrically distributed response and Gaussian white noise excitation with zero mean. Following [2], the adjective "true" is used when the first two moments of the nonlinear stochastic response are represented exactly by the linear response of the equivalent linear system.

Gaussian white noise excitation is a necessary condition for the existence of a statistically "true" linear system. Since white noise is not a suitable model for earthquake excitation, filtered shot noise [3] is used to model a typical earthquake excitation as an evolutionary process with asymptotic zero low frequency content.

TO THE EXISTENCE OF A STATISTICALLY "TRUE" EQUIVALENT LINEAR SYSTEM

Consider a n-th order nonlinear system which follows the differential equation

$$\frac{d\mathbf{y}}{dt} = \mathbf{g}(\mathbf{y}) + \mathbf{b}(t) \tag{1}$$

where $\mathbf{y}(t)$ is the state vector of the nonlinear response, $\mathbf{g}(\mathbf{y})$ a nonlinear asymmetric relation, i.e. $\mathbf{g}(\mathbf{y}) = -\mathbf{g}(-\mathbf{y})$, and $\mathbf{b}(t)$ is the excitation vector having only time modulated Gaussian white noise components. By using the method of equivalent linearization (EQL), the nonlinear eq.(1) is replaced by a linear differential equation

$$\frac{d\mathbf{x}}{dt} = [A(t)]\mathbf{x} + \mathbf{b}(t) \tag{2}$$

where $\mathbf{x}(t)$ is the linearized response vector and $[A(t)]$ the matrix to be determined by equivalent linearization. In the following, the required conditions on $[A(t)]$ are investigated which lead to identical first two moments

$$E\{\mathbf{x}\} = E\{\mathbf{y}\} \quad \text{and} \quad E\{\mathbf{x}\mathbf{x}^T\} = E\{\mathbf{y}\mathbf{y}^T\} \tag{3}$$

of the nonlinear response and the (stochastically "true" equivalent) linear response. Clearly, the first moments $E\{\mathbf{x}\}=E\{\mathbf{y}\}$ are identical for the above stated assumptions, since the nonlinear response $\mathbf{y}$ and the linear response $\mathbf{x}$ have both a symmetric distribution with zero mean, i.e. $E\{\mathbf{x}\}=E\{\mathbf{y}\}=\mathbf{0}$. Hence, only the condition for identical second moments needs to be investigated. In order to explore conditions for identical second moments, a procedure deriving the Lyapunov equation is applied for eq. (1):

$$E\{\frac{d\mathbf{y}}{dt}\mathbf{y}^T\} = E\{\mathbf{g}(\mathbf{y})\mathbf{y}^T\} + E\{\mathbf{b}\mathbf{y}^T\} = \mathbf{D_y} + \mathbf{F_y}$$
$$\text{where} \quad \mathbf{D_y} = E\{\,\mathbf{g}(\mathbf{y})\mathbf{y}^T\,\} \text{ and } \mathbf{F_y} = E\{\,\mathbf{b}\mathbf{y}^T\,\} \tag{4}$$

Since,

$$\frac{dE\{\mathbf{y}\mathbf{y}^T\}}{dt} = E\{\frac{d\mathbf{y}}{dt}\mathbf{y}^T\} + E\{\mathbf{y}\frac{d\mathbf{y}^T}{dt}\} = \mathbf{D_y} + \mathbf{D_y^T} + \mathbf{F_y} + \mathbf{F_y^T} \tag{5}$$

the second moments are uniquely determined by eq.(5) and the initial conditions.

Analogous considerations for eq.(2) lead to the following result:

$$\frac{dE\{\mathbf{x}\mathbf{x}^T\}}{dt} = E\{\frac{d\mathbf{x}}{dt}\mathbf{x}^T\} + E\{\mathbf{x}\frac{d\mathbf{x}^T}{dt}\} = \mathbf{D_x} + \mathbf{D_x}^T + \mathbf{F_x} + \mathbf{F_x}$$
$$\text{where} \quad \mathbf{D_x} = [A(t)E\{\mathbf{x}\mathbf{x}^T\}] \quad \text{and} \quad \mathbf{F_x} = E\{\mathbf{b}\mathbf{x}^T\} \tag{6}$$

Assuming for the time t=0 both systems at rest, i.e. $E\{\mathbf{x}(0)\}=E\{\mathbf{y}(0)\}=0$, eq.(3) will be satisfied if the right hand side of eq.(5) and eq.(6) are identical, i.e.:

$$\mathbf{D_x} + \mathbf{D_x^T} + \mathbf{F_x} + \mathbf{F_x^T} = \mathbf{D_y} + \mathbf{D_y^T} + \mathbf{F_y} + \mathbf{F_y^T} \tag{7}$$

The validity of $\mathbf{F_x} = \mathbf{F_y}$ has been shown for the assumed special case of a Gaussian white noise excitation with zero mean (See e.g. # 5.7 in [4]).

Hence, if

$$\mathbf{D_x} = \mathbf{D_y} \leftrightarrow [A(t)]E\{\mathbf{x}\mathbf{x}^T\} = E\{\mathbf{g}(\mathbf{y})\mathbf{y}^T\} \tag{8}$$

is satisfied, eq.(7) holds and consequently eq.(3) which states the definition of a statistically "true" equivalent linear system. Using eq.(3) and eq.(8) the "true" coefficient matrix $[A(t)\mathbf{b}]$ is found by

$$[A(t)] = E\{\mathbf{g}(\mathbf{y}(t))\mathbf{y}^T(t)\}\,E\{\mathbf{y}(t)\mathbf{y}^T(t)\}^{-1} \tag{9}$$

The above relation is also found by selecting $[A(t)]$ such that

$$E\{\|\mathbf{g}(\mathbf{y}(t)) - \mathbf{A}(t)\mathbf{y}(t)\|^2\} \rightarrow \text{minimum} \tag{10}$$

is minimized, which is the criterion applied right from its early beginning. Since the characteristics of the nonlinear response $\mathbf{y}(t)$ is not known, but can be found iteratively, eq.(9) is replaced in the conventional equivalent linearization technique by

$$[A'(t)] = E\{\mathbf{g}(\mathbf{x}(t))\mathbf{x}^T(t)\} \; E\{\mathbf{x}(t)\mathbf{x}^T(t)\}^{-1} \tag{11}$$

although it is well known that the nonlinear response is in general not Gaussian distributed.Since it is shown above that eq.(9) leads for the considered case to exact estimates for the variance of the nonlinear response quantities, biased estimates for strongly nonlinear systems are due to the unjustified assumption of jointly Gaussian response quantities only.

CONSIDERATION OF NON-GAUSSIAN MARGINAL DISTRIBUTIONS

For practical applications, eq.(9) is only useful for those cases for which it is possible to construct the joint distribution $f_{\mathbf{y}}(\mathbf{y})$ from known expectations $E\{\mathbf{x}\}=E\{\mathbf{y}\}$ and $E\{\mathbf{x}\mathbf{x}^T\}=E\{\mathbf{y}\mathbf{y}^T\}$ alone. In other words, if the shape of the joint distribution is known as function of $E\{\mathbf{y}\}$ and $E\{\mathbf{y}\mathbf{y}^T\}$, the "true" coefficient matrix can be established by the following procedure. Since eq.(3) holds for a "true" linear system, $f_{\mathbf{y}}(\mathbf{y})$ can be established as function of $E\{\mathbf{x}\}$ and $E\{\mathbf{x}\mathbf{x}^T\}$. Then, it is possible to evaluate a nonlinear relation

$$\mathbf{y}(\mathbf{x}) = F_{\mathbf{y}}^{-1}(F_{\mathbf{x}}(\mathbf{x})) \tag{12}$$

between the nonlinear response $\mathbf{y}(t)$ and the linearized (Gaussian) response $\mathbf{x}(t)$ which satisfies eq.(3). F denotes in eq.(12) the cumulative distribution and F^{-1} its inverse. By use of the nonlinear relation $\mathbf{y}(\mathbf{x})$, the expectation

$$E\{\mathbf{g}(\mathbf{y})\mathbf{y}^T\} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} [\mathbf{g}(\mathbf{y}(\mathbf{x}))\mathbf{y}^T(\mathbf{x})] f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \tag{13}$$

can be evaluated and subsequently eq.(9). Hence, the nonlinear relation $\mathbf{y}(\mathbf{x})$ allows one to take into account non-Gaussian properties of the response.

In practice, however, it is very difficult to formulate the shape of non-Gaussian joint distributions, although it is often possible to approximate the shape of probability distributions of components of the response vector $\mathbf{y}$, i.e. marginal distributions $f_{y_i}(y_i)$ as function of $E\{x_i\}$ and $E\{x_i^2\}$. An efficient way to determine realistic marginal distribution is the introduction of nonlinear transformations $y_i(x_i)$ established on the basis of physical considerations [5]. Any conceivable nonlinear transformation can be utilized, however under the condition that it meets eq.(3), i.e.: $E\{y_i(x_i)\} = E\{x_i\}$ and $E\{y_i(x_i)y_j(x_j)\} = E\{x_i x_j\}$. The nonlinear transformation $y_i(x_i)$ is then used to determine the marginal distributions $f_{y_i}(y_i)$. For constructing approximately the non-Gaussian joint distribution, the utilization of Nataf's [6,7] model is suggested.

GOVERNING EQUATIONS FOR THE GROUND MOTION AND FOR THE COULOMB SLIP SYSTEM

A stochastic model used extensively in the past to describe earthquake excitation in stochastic terms is filtered white noise [3,4]. For the purpose of a linear response analysis, uniformly modulated filtered white noise is shown to be adequate. When the structure reacts strongly nonlinear, however, the ground motion should be modeled as evolutionary process to account for the evolution of the frequency content with respect to time.

The nonlinear response of the Coulomb slip system and systems with similar physical properties (hysteretic systems) are strongly affected by the low frequency content of ground motion [1,3]. It is therefore important to represent realistically the low frequency content with zero low-frequency asymptote. For the results presented herein, the ground acceleration a(t) will be assumed to be filtered shot noise [3,4] described by the equation

$$a(t) = \sum_{k=1}^{M} 2\zeta_k \omega_k \dot{x}_k \quad ; \quad x_k(t) = \int_0^t h_{x_k}(t-\tau) \; s_k(\tau) \; d\tau \tag{14}$$

where a superseeded dot "·" indicates the derivative with respect to time and the filter motions $x_k(t)$ are defined as filtered shot noise where h_{xk} is the impulse response function of a simple linear oscillator and the shot noise $s_k(t)$ is modulated white noise,

$$s_k(t) = n_k(t)\frac{e^{-a_k t} - e^{-b_k t}}{C_k} \quad \text{with} \quad C_k = \left(\frac{a_k}{b_k}\right)^{\frac{a_k}{b_k - a_k}} - \left(\frac{a_k}{b_k}\right)^{\frac{b_k}{b_k - a_k}} \tag{15}$$

where the parameter a_k, b_k define the shape of the deterministic modulating function and $n_k(t)$ represents white noise with a constant intensity I_k. Using the representation in eq.(14), $M \geq 2$, and the above subsequent relations, allows one to model ground acceleration typical for earthquake excitation as evolutionary process with zero low-frequency asymptote.

The Coulomb slip system consists of a mass sliding on a flat horizontal surface with an constant friction coefficient, μ, where the surface has a random excitation $a(t)$. Let y_1 be the relative displacement of the mass and y_2 its velocity. Then, the slip motion is governed by the following set of the first order differential equations:

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = g(y_2; a(t), \mu g) = \begin{cases} -a(t) - \text{sgn}(y_2)\mu g & \text{for } y_2 \neq 0 \\ 0 & \text{for } y_2 = 0 \text{ and } |a(t)| \leq \mu g \\ -a(t) - \text{sgn}(a(t))\mu g & \text{for } y_2 = 0 \text{ and } |a(t)| > \mu g \end{cases} \tag{16}$$

where g is the acceleration of gravity. Note, that for a non-white colored excitation, no slipping ($y_2=0$) might occur temporarily.

NUMERICAL EXAMPLE

The parameters of the model of the ground motion described in the previous section have been found by adjusting them to a typical European (Friuli-1976) earthquake in the near field. These parameters are listed in Tab.1.

k	$S_k [m^2/sec^3]$	ω_k [rad/sec]	ξ_k [-]	$a_k [sec^{-1}]$	$b_k [sec^{-1}]$
1	0.15	1.0	0.6	0.2	0.22
2	0.30	3.0	0.4	0.2	0.50
3	0.80	6.0	0.1	0.3	1.00

Table 1: Parameters of the ground motion model

The variance of the ground acceleration may be seen from Fig.1. Since the modulating functions $f_k(t)$, defined in eq.(15) by the two parameters a_k, b_k, k=1,2,3, differ from each other, an evolutionary process can be generated. Its modulating function $a(t,\omega)$ shown in Fig.2. Note from Fig.2 that the evolutionary process has a slower decay of its low-frequency motion with a zero low-frequency asymptote. The constant friction coefficient μ of the Coulomb slip system (eq.(17)) has been selected such that sliding occurs after passing the threshold acceleration 0.1g, i.e. μ=0.1.

The statistically "true" equivalent linear system has been found by use of eq.(9) and performing statistics on a sample of simulated state vectors **y** (sample size 1000) of the non-linear response. A time step procedure using complex modal analysis [8] (time step 0.05 sec) is then applied to evaluate the second moments of the stochastic response of the time varying equivalent linear system. The results for the Coulomb slip system are presented in Fig.3 and Fig.4 where the standard deviation of the displacement and velocity responses are shown. The small deviation from the simulated results are clearly due to random fluctuations inherent to the finite sample size. Additionally, it is worth noting that much care was required to simulate the ground motion (filtered shot noise) with sufficient accuracy in the low-frequency range. To obtain the presented simulated results, a quite short time step (0.001 sec) in the numerical integration procedure was required (for the presented good agreement).

CONCLUSIONS

Based on the results presented herein the following conclusions can be drawn for the random response of strongly yielding structural systems which display Coulomb slip type behavior:

1. A statistically "true" equivalent linear system exists for nonstationary earthquake excitation which captures exactly the second moments of the random response of a strongly yielding system.
2. To improve the accuracy of equivalent linearization non-Gaussian response properties must be considered.
3. The non-Gaussian response properties can be taken into account within the framework of equivalent linearization by using nonlinear transformations y(x) between the nonlinear response y and the linearized response x.
4. The displacement response (drift) of a strongly yielding system depends strongly on the low-frequency content of the ground motion which should be modeled by filtered shot noise.

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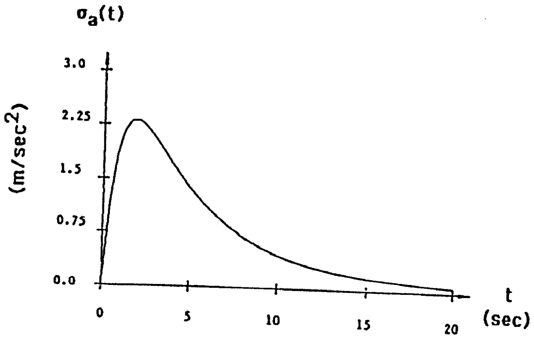

Fig. 1: Standard Deviation of Ground Acceleration $\sigma_a(t)$

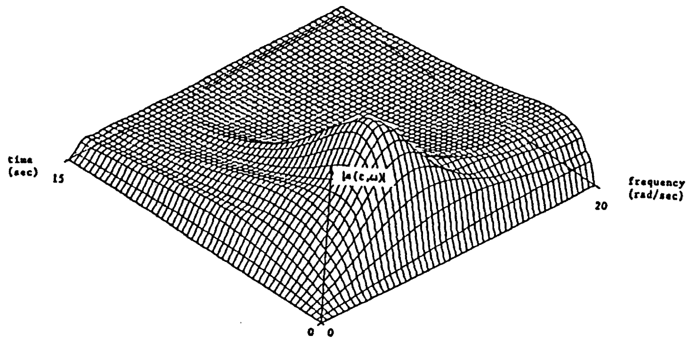

FIG. 2: Shape of Modulating Function $|a(t, \omega)|$ of the Evolutionary Process modeling Earthquake Excitation

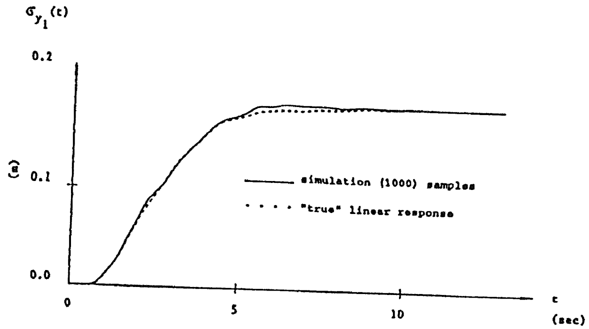

FIG. 3: Standard Deviation of the Displacement Response (Drift) of the Coulomb Slip System

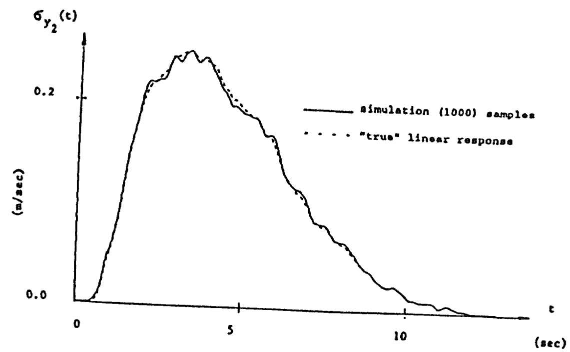

FIG. 4: Standard Deviation of the Velocity Response of the Coulomb Slip System