Proceedings of Ninth World Conference on Earthquake Engineering
August 2-9, 1988, Tokyo-Kyoto, JAPAN (Vol.III)

5-1-5

A MODIFIED INDIRECT BOUNDARY ELEMENT METHOD TO COMPUTE THE IMPEDANCE FUNCTIONS FOR RIGID FOUNDATIONS

Hirokazu TAKEMIYA and WANG Can Yun

Department of Civil Engineering, Okayama University, Okayama, JAPAN

ABSTRACT

To compute the impedance functions for embedded foundations, an unified formulation, which covers the direct- and indirect- boundary element methods (BEMs), is presented based on the weighted residual technique. A modified indirect BEM for rigid foundations is proposed based on considering the resultant effects of the rigid body condition. Compared with the conventional BEM, this method requires no traction Green functions and much less source forces. The method is validated by comparison with the results obtained by other methods for a cylindrical foundation embedded in a layered stratum.

INTRODUCTION

By use of the substructure approach, two basic problems, involving the soil and the massless foundation, must be solved in the linear soil-structure interaction analysis. The first one corresponds to the calculation of the dynamic relationship between the external forces and the foundation response. In the case of rigid foundations this force-displacement relationship is represented by the 6×6 impedance (dynamic stiffness) matrix. The second one corresponds to the calculation of the foundation response to the seismic excitation. These two basic problems are intimately related such that the second one can be readily solved, if the solution of the first problem and the free-ground motion are known .(1)

Recently, a considerable effort has been directed toward the application of BEMs. The technique is based on representing the wave radiation field as resulting from source forces in a weighted residual sense. The advantages over other numerical methods, such as the finite element method (FEM), proceed from an implicit satisfaction of the radiation condition at infinity and a reduction of problem dimension. However, BEMs are less competitive for layered media, because the Green functions are so sophisticated that the requirement of a large number of source forces will lead to long computing time for the boundary integral.

The objective of this paper is to propose a modified indirect BEM based on considering the resultant effects of the rigid body condition, which leads to a formulation involving no traction Green functions and much less source forces. The first part of the paper deals with the formulation for embedded foundations of arbitrary shape. The second part gives the validation of the method by comparison of the results for embedded cylindrical foundations with those obtained by other methods.

STATEMENT OF THE PROBLEM

A massless rigid foundation occupies the volume V' and is perfectly bonded to the soil along the interface S. The soil, in V, is assumed to be a viscoelastic medium consisting of parallel layers overlying a half-space (refer to Fig.1). The foundation is subjected to external forces with the harmonic time dependence $e^{i\omega t}$ (since the responses are with the same time dependence, $e^{i\omega t}$ will be omitted and only the amplitudes are concerned). The external forces are represented by the generalized force $F_0=\{ F_{0x}, F_{0y}, F_{0z}, M_{0x}, M_{0y}, M_{0z} \}^T$. The subscripts x,y,z denote the direction of components. F_0 can be evaluated from the traction vector $T(x)=\{ T_x(x), T_y(x), T_z(x)\}^T$ on the interface S.

$$F_0 = \int_S \alpha(\mathbf{x})^T \overset{\nu}{T}(\mathbf{x})\ dS(\mathbf{x}) \tag{1}$$

where $\alpha(\mathbf{x})$ is designated as a rigid body motion influence matrix given by

$$\alpha(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 & -(z-z_0) & (y-y_0) \\ 0 & 1 & 0 & (z-z_0) & 0 & -(x-x_0) \\ 0 & 0 & 1 & -(y-y_0) & (x-x_0) & 0 \end{bmatrix} \tag{2}$$

The unit normal ν to the surface S is taken as positive when pointing into V'. We can see that (F_{0x}, F_{0x}, F_{0z}) and (M_{0x}, M_{0y}, M_{0z}) are the resultant forces and the resultant moments acting at a reference point (x_0, y_0, z_0) in the foundation. The foundation response is represented by the generalized displacement at the reference point $U_0 = \{ U_{0x}, U_{0y}, U_{0z}, \Theta_{0x}, \Theta_{0y}, \Theta_{0z} \}^T$, consisting of three translations (U_{0x}, U_{0y}, U_{0z}) and three rotations ($\Theta_{0x}, \Theta_{0y}, \Theta_{0z}$). The displacement field in the soil $U(\mathbf{x}) = \{ U_x(\mathbf{x}), U_y(\mathbf{x}), U_z(\mathbf{x}) \}^T$ must satisfy the Navier equations of motion in each layer, the continuity conditions across layer interfaces, the condition of vanishing tractions on the free surface of the soil, the radiation conditions at infinity and the rigid body motion condition

$$U(x) = \alpha(\mathbf{x})\ U_0 \qquad x \in S \tag{3}$$

This mixed boundary-value problem corresponds to the computation of the 6×6 impedance matrix K_0. The generalized displacement and the generalized force are linearly related by

$$F_0 = K_0\ U_0 \tag{4}$$

For the problem of symmetric geometry about the z axis, Eq.(4) can be written in the form

$$\begin{Bmatrix} F_{0x} \\ F_{0y} \\ F_{0z} \\ M_{0x}/a \\ M_{0y}/a \\ M_{0z}/a \end{Bmatrix} = Ga \begin{bmatrix} K_{HH} & 0 & 0 & 0 & K_{HM} & 0 \\ 0 & K_{HH} & 0 & -K_{HM} & 0 & 0 \\ 0 & 0 & K_{VV} & 0 & 0 & 0 \\ 0 & -K_{MH} & 0 & K_{MM} & 0 & 0 \\ K_{MH} & 0 & 0 & 0 & K_{MM} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{TT} \end{bmatrix} \begin{Bmatrix} U_{0x} \\ U_{0y} \\ U_{0z} \\ a\Theta_{0x} \\ a\Theta_{0y} \\ a\Theta_{0z} \end{Bmatrix} \tag{5}$$

in which G is a shear modulus of reference, a is a length of reference (for a cylindrical foundation a is taken as the radius), and $K_{HH}, K_{MM}, K_{VV}, K_{TT}$ and $K_{HM} = K_{MH}$ are the normalized horizontal, rocking, vertical, tortional and coupling impedance functions, respectively.

Fig. 1. Foundation-soil model and notation

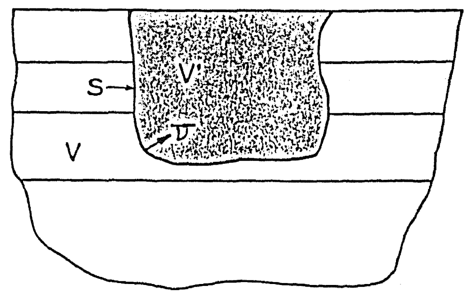
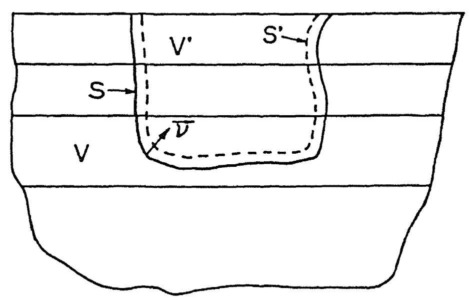


Fig. 2. Free-field and source force model



FORMULATION

We consider the free-field under strained by a source force $F(\mathbf{y}) = \{ F_x(\mathbf{y}), F_y(\mathbf{y}), F_z(\mathbf{y}) \}^T$ distributed on a surface S' within V' (refer to Fig. 2). The resulting displacement and traction fields can be obtained from

$$U^R(\mathbf{x}) = \int_{S'} G(\mathbf{x},\mathbf{y})\ F(\mathbf{y})\ dS'(\mathbf{y}) \qquad \overset{\nu}{T}{}^R(\mathbf{x}) = \int_{S'} \overset{\nu}{H}(\mathbf{x},\mathbf{y})\ F(\mathbf{y})\ dS'(\mathbf{y}) \tag{6),(7}$$

where $G(\mathbf{x},\mathbf{y})$ is the 3×3 displacement Green function matrix and $\overset{\nu}{H}(\mathbf{x},\mathbf{y})$ is the 3×3 traction Green function matrix, in which the j th columns correspond, respectively, to the displacement vector and the traction vector at $\mathbf{x}$ due to a unit harmonic force at $\mathbf{y}$ acting in j direction.

We define the difference between the actual fields $\{U(\mathbf{x}), \overset{\nu}{T}(\mathbf{x})\}$ and the resulting fields $\{U^R(\mathbf{x}), \overset{\nu}{T}{}^R(\mathbf{x})\}$ in V as the error field

$$\{\ \delta U(\mathbf{x}),\ \delta \overset{\nu}{T}(\mathbf{x})\ \} = \{\ U(\mathbf{x}) - U^R(\mathbf{x})\ ,\ \overset{\nu}{T}(\mathbf{x}) - \overset{\nu}{T}{}^R(\mathbf{x})\ \} \tag{8}$$

Obviously, if $F(\mathbf{y})$ is distributed such that $U^R(\mathbf{x})$ and/or $\overset{\nu}{T}{}^R(\mathbf{y})$ on S coincide with the original boundary conditions, the error field will be zero and exact solution can be obtained . In the numerical procedure, it is possible to satisfy boundary conditions in a weighted residual sense such that

$$\int_S W_d(\mathbf{x})^T\, \delta U(\mathbf{x})\ dS(\mathbf{x}) - \int_S W_t(\mathbf{x})^T\, \delta \overset{\nu}{T}(\mathbf{x})\ dS(\mathbf{x}) = 0 \tag{9}$$

in which the vectors $W_d(\mathbf{x})$ and $W_t(\mathbf{x})$ consist of certain weighting functions. It is desirable to minimize the error field by proper choice of these weighting functions. For the problem considered, we may achieve the formulation under three different considerations:

$$1)\ \ W_d(\mathbf{x}) = \overset{\nu}{T}(\mathbf{x}),\ \ W_t(\mathbf{x}) = U(\mathbf{x});\qquad 2)\ \ W_d(\mathbf{x}) = \overset{\nu}{T}{}^R(\mathbf{x}),\ \ \delta\overset{\nu}{T}(\mathbf{x}) = 0;\qquad 3)\ \ W_d(\mathbf{x}) = U^R(\mathbf{x}),\ \ \delta\overset{\nu}{T}(\mathbf{x}) = 0. \tag{10}$$

Substituting these conditions into Eq.(9) yields three integral equations, respectively.

$$\int_{S'}\int_S F(\mathbf{y})^T G(\mathbf{x},\mathbf{y})^T \overset{\nu}{T}(\mathbf{x})\, dS(\mathbf{x})\, dS'(\mathbf{y}) = \int_{S'}\int_S F(\mathbf{y})^T \overset{\nu}{H}(\mathbf{x},\mathbf{y})^T U(\mathbf{x})\, dS(\mathbf{x})\, dS'(\mathbf{y}) \tag{11}$$

$$\int_{S'}\int_S F(\mathbf{y})^T \overset{\nu}{H}(\mathbf{x},\mathbf{y})^T U(\mathbf{x})\, dS(\mathbf{x})\, dS'(\mathbf{y}) = \int_{S'}\int_{S'}\int_S F(\mathbf{y})^T \overset{\nu}{H}(\mathbf{x},\mathbf{y})^T G(\mathbf{x},\mathbf{y}')F(\mathbf{y}')\, dS(\mathbf{x})\, dS'(\mathbf{y}')\, dS'(\mathbf{y}) \tag{12}$$

$$\int_{S'}\int_S F(\mathbf{y})^T G(\mathbf{x},\mathbf{y})^T U(\mathbf{x})\, dS(\mathbf{x})\, dS'(\mathbf{y}) = \int_{S'}\int_{S'}\int_S F(\mathbf{y})^T G(\mathbf{x},\mathbf{y})^T G(\mathbf{x},\mathbf{y}')F(\mathbf{y}')\, dS(\mathbf{x})\, dS'(\mathbf{y}')\, dS'(\mathbf{y}) \tag{13}$$

Eqs.(11),(12) and (13) can also be derived from, the Maxwell-Betti's dynamic-reciprocity theorem, the condition of extreme value of work (Euler's equation), and the least square error technique, respectively. Eq.(11), known as the formulation of the direct BEM, gives the relation between the boundary values. On the other hand, Eqs.(12),(13), the formulations of the indirect BEM, give the source force distribution by imposing the given displacement boundary condition. Eq.(13) is preferred here, because it does not involve the traction Green function. For a rigid foundation in which $U(\mathbf{x})$ is given by Eq. (3), it is possible to write

$$F(\mathbf{y}) = \Lambda(\mathbf{y})\ U_0 \tag{14}$$

where $\Lambda(\mathbf{y})$ is a 3×6 matrix which gives the source force distributions on S' required to produce unit rigid body displacements of the surface S.

Once $\Lambda(\mathbf{y})$ has been determined, the contact traction can be obtained by substituting Eq.(14) into Eq.(7)

$$\overset{\nu}{T}(\mathbf{x}) = \int_{S'} \overset{\nu}{H}(\mathbf{x},\mathbf{y})\ \Lambda(\mathbf{y})\ dS'(\mathbf{y})\ U_0 = \overset{\nu}{\Gamma}(\mathbf{x})\ U_0 \tag{15}$$

in which $\overset{\nu}{\Gamma}(\mathbf{x})$, designated as a contact traction matrix, is a 3×6 matrix whose columns give the contact traction vectors produced by unit rigid body displacements of the foundation. Finally, substituting Eq (15) into Eq. (1) leads to the desired impedance matrix

$$K_0 = \int_S \alpha(\mathbf{x})^T\, \overset{\nu}{\Gamma}(\mathbf{x})\ dS \tag{16}$$

By the above formulation, the contact traction must be evaluated as an intermediate solution. Since the primary interest is placed on evaluating the generalized force, the computation of the unnecessary traction requires much more efforts than enough. In fact, numerical computation of tractions demands deliberate considerations. This defect may be overcome by an alternative formulation proposed here. In the view of that the generalized force must be in equilibrium with the source force on S' and the inertia within V', we can obtain the impedance matrix directly from

$$K_0 = \int_{S'} \alpha(\mathbf{y})^T\, \Lambda(\mathbf{y})\ dS'(\mathbf{y}) + \omega^2 \int_{V'}\int_{S'} \rho(\mathbf{x})\ \alpha(\mathbf{x})^T\ G(\mathbf{x},\mathbf{y})\ \Lambda(\mathbf{y})\ dS'(\mathbf{y})\ dV'(\mathbf{x}) \tag{17}$$

APPROXIMATE ANALYSIS

To simulate the radiation filed requires a large number of source forces, especially at high frequencies. This will result in not only long computing time but also a problem of divergence .[2],[3] In the case of rigid foundations which have only 6 degrees of freedom, it is possible to reduce the number of source forces by estimating the equivalent rigid body motion.

By use of the reciprocity theorem, we have

$$\int_S \overset{\nu}{\Gamma}(\mathbf{x})^T\ U^R(\mathbf{x})\ dS(\mathbf{x}) = \int_S \alpha(\mathbf{x})^T\ \overset{\nu}{T}{}^R(\mathbf{x})\ dS(\mathbf{x}) \tag{18}$$

in which $U^R(\mathbf{x})$ is a resulting displacement vector from the source forces and $\overset{\nu}{T}{}^R(\mathbf{x})$ is the corresponding traction vector. We introduce the decomposition

$$U^R(\mathbf{x}) = \alpha(\mathbf{x})\ U_0 + \overset{*}{U}(\mathbf{x}) \qquad \mathbf{x}\in S \tag{19}$$

in which U_0, designated as an equivalent rigid body motion, is a 6×1 generalized displacement, and the remaining deformation vector $\overset{*}{U}(\mathbf{x})$ satisfies the condition

$$\int_S \overset{\nu}{\Gamma}(\mathbf{x})^T\ \overset{*}{U}(\mathbf{x})\ dS(\mathbf{x}) = 0 \tag{20}$$

Then Eq.(18) reduces to

$$\int_S \overset{\nu}{\Gamma}(\mathbf{x})^T \alpha(\mathbf{x})\ dS(\mathbf{x})\ U_0 = \int_S \alpha(\mathbf{x})^T\ \overset{\nu}{T}{}^R(\mathbf{x})\ dS(\mathbf{x}) \tag{21}$$

Considering the symmetry of the impedance matrix yields

$$K_0\ U_0 = F_0 \tag{22}$$

If we have six sets of different $\{U(\mathbf{x}),T(\mathbf{x})\}$, the 6×6 impedance matrix can be evaluated by

$$K_0 = [\ \mathbf{F}_0\ ]\ [\ \mathbf{U}_0\ ]^{-1} \tag{23}$$

where $[\ \mathbf{F}_0\ ]=[\ F_0^1, F_0^2, \ldots, F_0^6]$ and $[\ \mathbf{U}_0\ ] = [\ U_0^1, U_0^2, \ldots, U_0^6\ ]$, in which U_0^i indicates one of the equivalent rigid-body motions of the six different resulting displacements and F_0^i is the corresponding generalized force.

Since the deformation $\overset{*}{U}(\mathbf{x})$ must satisfy Eq.(20), the decomposition of Eq. (19) involves the unknown contact traction matrix $\Gamma(\mathbf{x})$ and can only be solved approximately. The least square error technique is available for this purpose. *i.e.*,

$$\frac{\partial}{\partial U_0}\int_S [\ U^R(\mathbf{x})-\alpha(\mathbf{x})U_0\ ]^T\ [\ U^R(\mathbf{x})-\alpha(\mathbf{x})U_0\ ]\ dS(\mathbf{x}) = 0 \tag{24}$$

which yields

$$U_0 = H^{-1}\int_S \alpha(\mathbf{x})^T\ U^R(\mathbf{x})\ dS(\mathbf{x}) \tag{25}$$

where H is the 6×6 matrix given by

$$H = \int_S \alpha(\mathbf{x})^T\ \alpha(\mathbf{x})\ dS(\mathbf{x}) \tag{26}$$

If we choose the reference point at the centroid of the surface S, then

$$H = diag\ \{\ S, S, S, I_x, I_y, I_z\ \} \tag{27}$$

in which S and I_i denote, respectively, the area and the second moment area about i axis of the surface S. We can see that the approximate solution is based on the assumption that the equivalent rigid body motion corresponds to the weighted average displacement. This assumption is acceptable when the approximate solution $\overset{*}{U}(\mathbf{x}) = [U^R(\mathbf{x})\ -\alpha(\mathbf{x})H^{-1}\int_S\alpha(\mathbf{x})^T\ U^R(\mathbf{x})dS(\mathbf{x})\ ]$ satisfies

$$\frac{\int_S \overset{\nu}{\Gamma}(\mathbf{x})\ \overset{*}{U}(\mathbf{x})\ dS(\mathbf{x})}{\int_S \overset{\nu}{\Gamma}(\mathbf{x})\ U^R(\mathbf{x})\ dS(\mathbf{x})} \approx 0 \tag{28}$$

Eq.(28) is justified by viewing that when $\overset{\nu}{\Gamma}(\mathbf{x})$ is a relatively slowly varying function and $\overset{*}{U}(\mathbf{x})$ is rapidly oscillating, the numerator will be relatively small due to a self-cancelling on the integral path.

Now we can see that if the imposed displacement boundary condition happens to be the equivalent rigid body motion, the solutions obtained by BEMs will be correct. But this is not guaranteed when the number of sources is not sufficient enough. The results may be modified by use of Eq.(23). In this case, $[\mathbf{F}_0]$ corresponds to the solution of the impedance matrix computed from the indirect BEM, and $[\mathbf{U}_0]$ must be evaluated from the resulting displacement $U^R(\mathbf{x})$ computed from Eq.(6) after the source force distribution has been determined.

NUMERICAL COMPUTATIONS AND DISCUSSIONS

To validate the proposed method, the impedance functions for a cylindrical foundation embedded in a viscoelastic layer overlying a rigid half-space are computed. The semianalytical Green functions for ring loads obtained by E. Kausel *et al* ,[(4),(5)] have been employed. These solutions are based on a discretization of the medium in the direction of layering and a normal mode expansion analysis in the wave number domain. The integral transform from the wave number domain to the spacial domain can be carried out in a closed form. In this study, the numerical computation is conducted by discretizing all the variables involved with linear shape functions. The material damping in the soil is introduced by taking the complex $P-$ and $S-$ wave velocities $V_p(1+i\xi_p)$ and $V_s(1+i\xi_s)$ in which ξ_p and ξ_s are hysteretic damping radios. The characteristic values of the soil-foundation model are presented in Fig. 3. The impedance functions are referred to the center of the foundation and computed for a number of values of the dimensionless frequency $a_0=\omega a/V_s$ in the range of engineering interest.

The accuracy of the solutions is checked by comparison with the results obtained by the FEM with lateral transmitting boundary. The efficiency of the proposed methods is demonstrated by comparing computing time with conventional indirect BEM. Two cases with different number of source forces are considered.

First, the computation is performed for the model shown in Fig. 4(case 1), in which the number of source points, the observation points on the boundary S and the observation points within V' ($N_s, N_o{}^b, N_o{}^i$) are (9, 30, 10) , respectively. Each source point represents one ring load and observation points denote the nodal position at which the response is to be evaluated. The source surface S' is offset from the actual interface S by a distance. Considering both accuracy and economy, the observation points are densely spaced near the source while relatively sparsely far from it. The results of ($K_{HH}, K_{VV}, K_{MM}, K_{TT}, K_{MH}$) are plotted in Fig. 5. Note that the results agree quite well with those by FEM and by the conventional indirect BEM. Significant differencies appear only at high frequencies, which is believed can be reduced by increasing the source and observation points. The computing time required for the conventional indirect BEM has been reduced considerably by the proposed method, since the computation for the traction Green functions is not involved. Although the inertial force in the interior volume V' has to be considered, it dose not need much effort, since it can be sufficiently accounted for when the element size of the grid is smaller than, say, 1/10 of the wave length.

Secondly, the number of source points is reduced to two as shown in Fig. 4(case 2). Solutions obtained by the conventional indirect BEM diverge at high frequencies, while those modified by the proposed method still agree well with other results in the first case.

CONCLUSIONS

A modified indirect BEM to compute the impedance functions for embedded rigid functions is proposed and applied to a cylindrical foundation embedded in a viscoelastic layered stratum. The results are compared with those obtained by FEM and indirect BEM. Numerical computation has revealed that the conventional indirect BEM is not efficient for layered media, because it requires a large number of source forces and long computing time for evaluating the Green functions. By comparison, the proposed method requires no traction Green functions and much less source forces.

REFERENCES

1. J. E. Luco , On the relation between radiation and scattering problems for foundations embedded in a elastic half-space, Soil Dynamics and Earthquake Engineering, 1986, Vol. 5, No. 2, 97-101.

2. R. J. Apsel, Dynamic Green's functions for layered media and applications to boundary-value problems, Ph.D. Dissertation, University of California, San Diego, La Jolla, CA, 1979.

3. R. J. Apsel and J. E. Luco, Impedance functions for foundations embedded in a layered medium: an integral equation approach, Earthquake Engineering and Structural Dynamics. Vol. 15, 213-231. (1987)

4. Kausel, E., An explicit solution for the Green functions on layered media, Bull. Seism. Soc. Am. 43, 17-34. (1974)

5. G. Waas, H. R. Riggs and H. Werkle, Displacement Solutions for Dynamic loads in Transversely-Isotropic Stratified Media, Earthquake Engineering and Structural Dynamics, Vol. 13, 173-193. (1985)

6. Iguchi, M. An approximate analysis of input motions for rigid embedded foundations, Trans. Architectural Institute of Japan 1982, 57, 315, 61-75.

7. Kausel, E. and Peek, R. Boundary integral method for stratified soils, M.I.T. Research Report, Publication No. R82-50, Order No. 746, Dept. of Civil Engineering, (1982).

Fig. 3. Cylindrical foundation-soil model

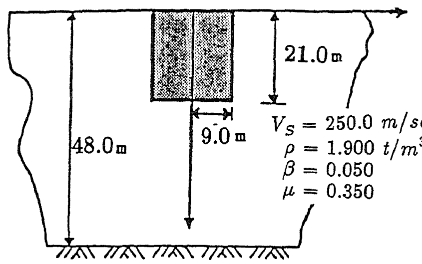


Fig. 4. Location of
× source points
○ boundary observation points
● interior observation points

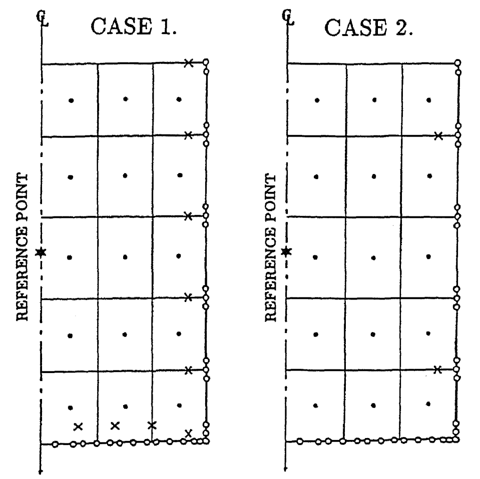


Fig. 5. Normalized impedance functions

FEM	—·—
INDIRECT BEM	
CASE (1) [N_S = 9]	○ ○ ○ ○
CASE (2) [N_S = 2]	- - - - - - - -
MODIFIED BEM	
CASE (1) [N_S = 9]	★ ★ ★ ★
CASE (2) [N_S = 2]	———