SIMPLIFIED DYNAMIC ANALYSIS OF TRUNCATE CONICAL WATER TANKS

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SUMMARY

The truncate conical shape is widely used for the suspended concrete water tanks. The seismic performance of such a kind of reservoir has therefore to be determined. By following Hausner's by now classical approach in the present paper the fundamental period and the convective pressure are calculated by means of a variational technique, with some experimental checks.
A method is then proposed, following the Jacobsen theory, to evaluate the impulsive components of pressure.

INTRODUCTION

By following Housner's by now classical approach (Ref. 1) it is possible to analyze the dynamic response of a rigid-walled truncate-conical reservoir, subjected to seismic actions as a combination of two different and substantially uncorrelated effects (Ref. 2): the convective force which is above all due to the fundamental sloshing of the liquid in the tank; and the impulsive reaction generated by the accelerated walls of the containers.

CONVECTIVE FORCE

The shape and period of the fundamental mode

This can be calculated by means of a simplified model. The following approximations are adopted:
a) undeformable container walls
b) uncomprensible and not viscous fluid
c) small amplitude oscillations
d) by referring to the figures 1 and 2 and indicating by $\dot{u}$, $\dot{v}$ and $\dot{w}$ the velocity components respectively according to the x, y and z axis, all the

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points of the fluid with the same coordinates x,y have the same velocity components $\dot{u}$, $\dot{v}$.

e) an ideal flat horizontal surface in the fluid stays flat during the motion.

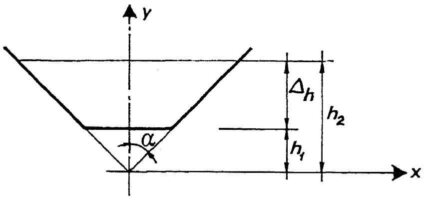


Fig. 1

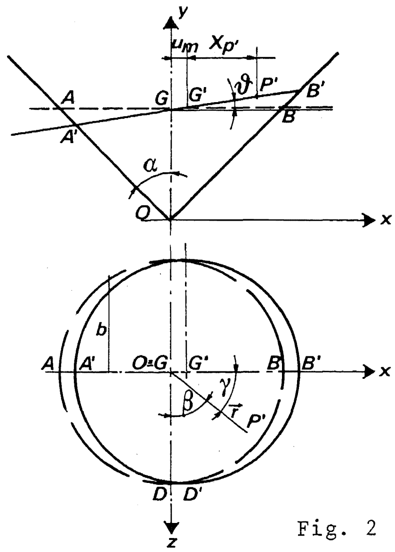


Fig. 2

The continuity and uncompressibility conditions lead to the following equations (Ref. 1) (Ref. 3).

$$\dot{v} = x\,\dot{\vartheta} \qquad 1)$$

$$\dot{u} = \dot{u}_m + \dot{u}_o = -\frac{1}{b}\frac{\partial\dot{\vartheta}}{\partial y}\int_{-R}^{x} x\,b\,dx + R\,tg\alpha\,\dot{\vartheta} \qquad 2)$$

$$\dot{w} = z\,\frac{b'}{b^2}\,\frac{\partial\dot{\vartheta}}{\partial y}\cdot\int_{-R}^{x} x\,b\,dx \qquad \text{where } b' = \frac{\partial b}{\partial x} \qquad 3)$$

It should than be possible to determine the function:

$$\vartheta = \vartheta\,(y,t)$$

A variational approach can be basedon the Hamilton principle.
The value of potential energy is:

$$V_1 = \frac{1}{2}\varrho\,g\,\vartheta_{h_2}^2\,I_z \qquad 4)$$

where ϑ_{h_2} is the rotation of the free surface

I_z is the momentum of inertia of the free surface.

The kinetic energy is explained in the following manner:

$$T = \frac{1}{2}\varrho \int_V (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, dv = \frac{1}{2}\varrho \int_{h_1}^{h_2} \left\{ \Lambda_1 \dot{\vartheta}^2 + \Lambda_2 \dot{\vartheta} \frac{\partial \dot{\vartheta}}{\partial y} + \Lambda_3 \left(\frac{\partial \dot{\vartheta}}{\partial y}\right)^2 \right\} dy \qquad 5)$$

where Λ_1, Λ_2 and Λ_3 are functions of the variable radius $R = y\,\mathrm{tg}\alpha$, being α the opening of the cone (Ref. 3).

By setting null the variation of the energy function in the time interval Δt the whole problem is reduced to the solution of the following two equations:

$$\frac{2}{27}\, y^2 \frac{\partial^2 \vartheta}{\partial y^2} + \frac{4}{9}\, y \frac{\partial \vartheta}{\partial y} - k\vartheta = 0 \qquad 6)$$

$$\frac{2}{27}\, h_2^2 \left(\frac{\partial \ddot{\vartheta}}{\partial y}\right)_{h_2} + \frac{h_2}{4} \ddot{\vartheta}_{h_2} + \frac{1}{4} g \frac{1}{\mathrm{tg}^2 \alpha}\, \vartheta_{h_2} = 0 \qquad 7)$$

The solution of 6) is an algebraic function:

$$\vartheta = \vartheta_{h_2} \left[C_1 \left(\frac{y}{h_2}\right)^{F_1} + C_2 \left(\frac{y}{h_2}\right)^{F_2} \right] \qquad 8)$$

where $\left.\begin{matrix} F_1 \\ F_2 \end{matrix}\right\} = \frac{5}{2}\left[-1 \pm \sqrt{1 + 2{,}16\, k} \right]$

and $k = \dfrac{1 - \mathrm{tg}^2\alpha}{4\, \mathrm{tg}^2 \alpha}$

The coefficients C_1 and C_2 are deduced by putting:

$$\vartheta_{(y = h_1)} = 0 \qquad \qquad \vartheta_{(y = h_2)} = \vartheta_{h_2}$$

We therefore have:

$$C_1 = \frac{1}{1 - \left(\frac{h_1}{h_2}\right)^{(F_1 - F_2)}} \qquad \qquad C_2 = -\left(\frac{h_1}{h_2}\right)^{(F_1 - F_2)} \cdot C_1 \qquad 9)$$

In most cases $C_1 \simeq 1$ and $C_2 \simeq 0$; then the equation 8) can generally be written in the approximate form:

$$\vartheta \cong \vartheta_{h_2} \cdot \left(\frac{y}{h_2}\right)^{F_1} \qquad 10)$$

without any relevant error.

Substituting 8) in 7) we obtain:

$$\vartheta_{h_2} = \vartheta_{o\,h_2} (\sin \omega t + \Phi)$$

$$\omega = \sqrt{\frac{g}{h_2 \, tg^2\alpha \left[\frac{8}{27} \left(F_1 C_1 + F_2 C_2 \right) + 1 \right]}} \qquad 11)$$

$$T_h \simeq 2\pi \, tg\alpha \sqrt{\left(\frac{8}{27} F_1 + 1 + \dots \right) \cdot \frac{h_2}{g}} \qquad 12)$$

The natural period, calculated in this manner, has been compared with the experimental results obtained by letting the water oscillate freely in three reduced models with different α .

Repeated tests were performed for each model using different water level heights. The tests are described in table P1.

TABLE P1 .

α	Δh [mm]	h_1 [mm]	h_2 [mm]	Natural period (sec) experimental values	Natural period (sec) calculated values
30°	180	217	397	0.87	0.84
	360	217	577	1.03	1.01
	540	217	757	1.16	1.16
45°	130	125	255	1.05	1.02
	260	125	385	1.26	1.25
	390	125	515	1.46	1.44
60°	80	72	152	1.30	1.25
	160	72	232	1.52	1.55
	240	72	312	1.76	1.79

CONVECTIVE PRESSURES

With higher simplicity and safety (Ref. 1) the variation of the convective pressure p_w versus γ (fig. 2) can be expressed by a co-sinusoidal low; i.e.:

$$p_w = p_o \cos\gamma$$

where p_o is p_w in the x, y plane.

By integrating along x starting with x = o we obtain:

$$p_o = -\varrho \left(- \int_0^R \frac{Q}{b}\, dx\right) \frac{\partial \ddot{\vartheta}}{\partial y} + \varrho \ddot{\vartheta} R^2 tg\alpha \qquad 13)$$

with $Q = \int_{-R}^{x} x\, b dx$

As $\ddot{\vartheta} = -\omega^2 \vartheta = -\omega^2 \vartheta_{h_2} \left[\left(\frac{y}{h_2}\right)^{F_1} - \mu^{(F_1 - F_2)} \left(\frac{y}{h_2}\right)^{F_2}\right]$

it follows:

$$p_o = \varrho . h_2^2 . tg^3\alpha \;\; \omega^2 \vartheta_{h_2} . \frac{1}{9}\left[(9 + 2F_1)\left(\frac{y}{h_2}\right)^{(F_1+2)} - (9 + 2F_2)\left(\frac{y}{h_2}\right)^{(F_2+2)}\right] \qquad 14)$$

The resultant force RH is

$$RH = \int_o^{2\pi}\int_{h_1}^{h_2} p_w . y . tg\alpha \; . \cos\gamma \; . dy . d\gamma \qquad 15)$$

In the following formulas it will appear the function A(i) of the integer parameter i:

$$A(i) = \frac{9 + 2F_1}{i + F_1}\left[1 - \mu^{(F_1+i)}\right] - \mu^{(F_1 - F_2)} \; . \; \frac{9 + 2F_2}{i + F_2}\left[1 - \mu^{(F_2+i)}\right] \qquad 16)$$

If i = 5 and $\alpha \to \pi/4$ the second term in the second member becomes:

$$\ldots - \mu^{-F_2} . \ln \frac{1}{\mu} \qquad \left(\mu = \frac{h_1}{h_2}\right)$$

approaching zero as $\mu \to o$; anyway it can be generally omitted.

$$RH = \frac{4}{9}\varrho I_{zh_2} \; . \; \omega^2 \vartheta_{h_2} \; . \; A(4)$$

The convective moment referred at the top of cone O is:

$$M = M_1 + M_2 + M_3 \qquad 19)$$

$$M_1 = \int_{s_1} p_w \cos\alpha \;\; ds = \int_o^{2\pi}\int_{h_1}^{h_2} p_w y^2 tg\alpha \cos\gamma \; d\gamma \; dy = \frac{4}{9}\varrho I_{zh_2} \omega^2 \vartheta_{h_2} . h_2 . A(5)$$

$$M_2 = \int_{s_1} p_w \sin\alpha \, ds = \int_o^{2\pi}\int_{h_1}^{h_2} p_w \; y^2 tg^3\alpha \;\; \cos\gamma \; d\gamma dy = \frac{4}{9}\varrho I_{zh_2} \omega^2 \vartheta_{h_2} . h_2 . tg^2\alpha A(5)$$

$$M_3 = \int_{s_2} p_w \, ds = I_{zh_1} \, p_o(h_1) / (\mu . h_2 . tg\alpha)$$

s_1 and s_2 indicate respectively the lateral surface and the flat bottom. In order to analyze the overall dynamic response one can represent the moving liquid with an oscillating mass m_1 sustained by a spring K.

$$m_1 = \frac{RH^2}{\omega^2 V_1} \qquad K = \frac{RH^2}{V_1}$$

where V_1 is given by 4)
The height over 0 of m_1 is

$$h_{m1} = \frac{M}{RH} = \frac{h_2(1+tg^2\alpha).A(5) + \frac{1}{8}\mu^{(F_1+2)}(F_1-F_2)}{A(4)}$$

Table P2 shows the calculated values of the oscillating mass and its height over the top of the cone.
Three cones with different α are shown: $\mu = \frac{h_1}{h_2}$ is assumed 0,3 and $h_2 = 10$ m.

Opening of the cone (α)	30°	45°	60°
Oscillating mass [kg]	$188 . 10^3$	$778 . 10^3$	2.784 . 10
h_{m1} [m]	11.3	17.0	35.5
Oscillating mass/real mass	0.55	0.76	0.91

THE IMPULSIVE EFFECT

It can be assumed that the motion of the container is an horizontal translation.
The velocity of the walls, therefore, is equal at every point in the same time.
The pressure inside the liquid is expressed by:

$$p = \varrho \frac{\partial U}{\partial t} \qquad 20)$$

$U(r,y,\gamma,t)$ in the cylindric cohordinates with the origin at the top of the cone, is the potential of the velocity field, governed by the Laplace equation:

$$\Delta^2 U = 0$$

and by the following conditions on the boundary (Ref. 4)

$U_{(y=h_2)} = 0$ (free surface flat and no pressure on it)

$\left(\frac{\partial U}{\partial y}\right)_{(y=h_1)} = 0$ (no vertical velocity at the bottom)

$$\left(\frac{\partial U}{\partial r}\cos\alpha - \frac{\partial U}{\partial y}\sin\alpha\right)_{(r=y\,tg\alpha)} = -\dot{u}_n \cos\alpha\cos\gamma \qquad 21)$$

(the wall and the nearest layer of fluid have the same velocity $\dot{u}_n$ in the direction orthogonal to the lateral surface).

The solution is:

$$U = U_0(r,y,\gamma)\ .\ f(t) = f(t)\cos\gamma \sum_{m=1}^{\infty} A_m \cos\left[K_m(y-h_1)\right] . I_1(K_m r)$$

where $K_m = \frac{\pi}{2}\frac{1}{h_2-h_1}$ and I_1 indicates a modified Bessel function of the first kind of order 1.

f(t) is the velocity of the container. With the Ritz-Galerkin method, by putting $\bar{U}$ = finite expansion at the n^{th} term approximating U, we have

$$\frac{\partial}{\partial A_m}\left(\frac{\partial \bar{U}}{\partial r}\cos\alpha - \frac{\partial \bar{U}}{\partial y}\sin\alpha + \dot{u}_n\cos\alpha\right)^2 = 0$$

The unknown coefficients A_m are determined by solving the linear system

$$\sum_j a(i,j)\ .\ A(j) = b(i)$$

where $a(i,j) = \int_{h_1}^{h_2} K_i K_j G_i G_j\, dy$

$$G_s = \cos K_s(y-h_1)\ \frac{I_1'(K_s y\,tg\alpha)}{K_s}\ \sin\left[K_s(y-h_1)\right] . I_1(K_s y\,tg\alpha)$$

and $b(i) = -\varrho\dot{f}(t)\int_{h_1}^{h_2} K_i G_i\, dy$

The function $\dot{f}(t)$ is the acceleration of the container.

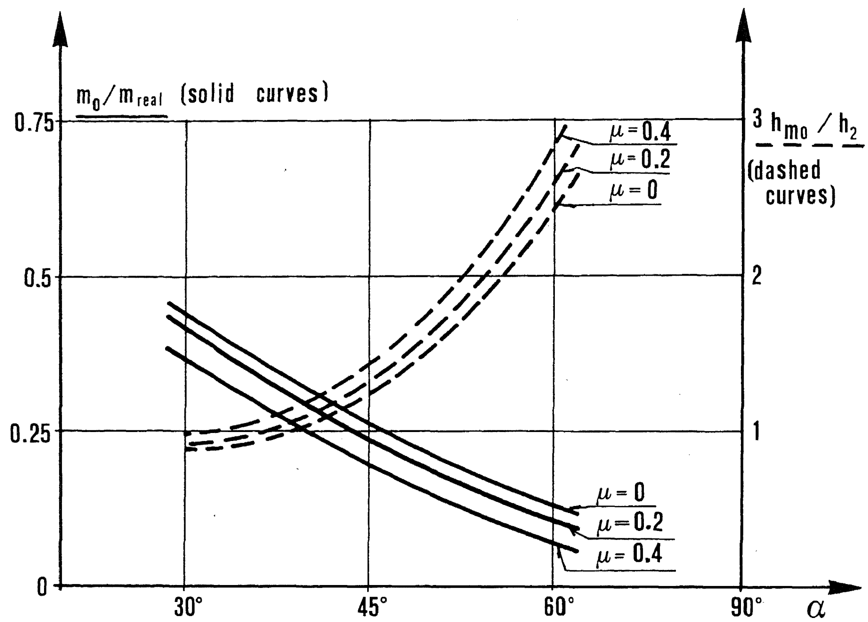
The pressure is

$$p_{wimp} = \left[\sum_{m=1}^{\infty} A_m \cos K_m(y-h_1)\ I_1(K_m y\,tg\alpha)\right] .\ \varrho\,.\dot{f}(t)\cos\gamma \qquad 22)$$

The resultant RHi and resultant moment Mi are deduced from the expression 22) by means of the same integration procedure used for the convective pressures. In order to perform the overal analysis of the structure, the impulsive response of the water can be replaced by a concentrate-rigidly fixed mass having the value $m_0 = \frac{RHi}{\dot{f}(t)}$

The diagram P3 shows the ratios m_0 versus real mass and vertical displacement of the m_0 versus h_2 for different angles α and different μ.

Diagram 1



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