MATEHMATICAL MODELLING OF INTERACTION BETWEEN THIN WALL CYLINDRICAL RESERVOIR AND THE FLUID

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SUMMARY

This paper presents a possible method of mathematical model formulation of the interaction between thin wall cylindrical reservoir and an ideally incompressible fluid, during horizontal and vertical ground motion. In mathematical modelling of the fluid the boundary element method was used. Basic theoretical assumptions of the boundary element method have been given. In case of an ideal nondeformability of reservoir walls it is assumed that the reservoir-fluid system moves horizontally with constant acceleration.

However, the walls of the cylinder reservoir are deformable. For this case of interaction between the fluid and reservoir body, basic theoretical principles and example have been given treated for horizontal sinusoidal ground motion. The objective of this paper is to present the principles of the boundary element method and the possibility to apply it in analysis of similar problems, as well as to provide comparison with the finite element method.

INTRODUCTION

The interaction between two different media such as water fluid and cylindrical reservoir walls has always been an interesting subject, taking into account the fact that this type of structure has increasingly been used in industry, at present. There are more methods and approaches to solve this problem. However, the application of the boundary element method in modelling of the water fluid has certain advantages, compared to other methods, having in mind that it operates with contact surfaces for which the boundary conditions of fluid motion are known. The compatibility conditions of fluid-cylindrical walls motion define the fluid-reservoir interaction in a way which is based on additional mass principles or the principles for definition of the water pressure, for each step of motion of the system. The effect of ideally rigid walls of the cylindrical reservoir and the effect of wall deformability have been shown through examples. Comparison between the B.E method of water fluid-reservoir problem and the methods based on the F.E type discretization of 3D water domain (Galerkin's method) has also been discussed in this paper.

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Introduction to the Problem, Boundary Element Equations Based on Weighted Residual Method

The equation of motion of incompressible fluid with a relatively small velocity amplitude is given in the following Laplacian form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \phi \qquad \text{... (a)}$$

where $w=w(x,y,z)$ is a water pressure function.
This equation together with the boundary conditions completely defines the problem of fluid motion. The boundary conditions are either known potentials $\bar{w}$, the so called natural boundary conditions, or known normal derivations $\partial\bar{w}/\partial n$ i.e., the so called essential boundary conditions. It is known from the mathematical analysis of the weighted residual methods that the following equation exists:

$$\int_{\Omega} L(w).F\, d\Omega = \int_{\Omega} w\, L(F)\, d\Omega + \int_{\Omega 1} L1(F)\ .\ L2(F)\, d\Omega 1 - \int_{\Omega 2} L1(w)\ .\ L2(F)\ .\ d\Omega 2 \qquad \text{... (b)}$$

where the symbols L( ), L1( ), L2( )are certain operators, for example, if we apply the operator L under the function $w=w(x,y,z)$ we can receive some other function $p=p(x,y,z)$. $F=F(x,y,z)$ is known as a weighted function. The integrals over domain Ω are 3D integrals, while integrals over domains $\Omega 1$ and $\Omega 2$ are surface integrals. In the case of the given Laplacian equation (a) and the types of the described boundary conditions, the operators L and L1 are the following:

$$L \equiv \frac{\partial^2}{\partial x^2}(\ ) + \frac{\partial^2}{\partial y^2}(\ ) + \frac{\partial^2}{\partial z^2}(\ ) \qquad L1 = \frac{\partial}{\partial n}$$

The operator L2 defines the same function as a function on which it is applied. If we select the form of the weighted function $F=F(x,y,z)$ so that the function satisfies Laplace equation (a), the volume integrals over the space domain Ω in the equation (b) will vanish and we will have the equation in which only the surface integrals over surface domains $\Omega 1$ and $\Omega 2$ exist. Fig. (1) shows the space domain Ω and surface domains $\Omega 1$ and $\Omega 2$ for which the boundary conditions are known. Function $F=F(x,y,z)$ satisfying the Laplace equation (a) is known as a fundamental solution function and for a 3D domain isotropic problems is defined as:

$$F(x,y,z) = \frac{1}{4\pi r(x,\ y,\ z)} \qquad \text{... (c)}$$

where $r=r(x,y,z)$ is the distance between some observed point "i" at the surface domains and any point over domain Ω. After integrating of the equation (b) and assuming that the fundamental function satisfies the Laplace equation over the whole domain Ω , except for a particular point "i" at the surfaces $\Omega 1$ or $\Omega 2$ in which we have unit potential, or unit derivative, the equation (b) becomes:

$$\frac{1}{2} wi + \int_{\Omega 2} w \frac{\partial F}{\partial n}\, d\Omega 2 + \int_{\Omega 1} \bar{w} \frac{\partial F}{\partial n}\, \Omega 1 = \int_{\Omega 2} \frac{\partial \bar{w}}{\partial n} F\, d\Omega 2 + \int_{\Omega 1} \frac{\partial w}{\partial n} F\, d\Omega 1 \qquad \text{... (d)}$$

The equation (d) is a fundamental relation on which the boundary element method is based.

Discretization Procedure, Typical Boundary Equation for Node "i" and Boundary Element Definition

Fig. (2) shows 3D domain Ω with surface subdomains Ω_1 and Ω_2. For the subdomain Ω_1 the natural boundary conditions are known and for subdomain Ω_2 the essential boundary conditions $\partial \bar{w}/\partial_n$ are known. Discretizing subdomains Ω_1 and Ω_2 in a form of small surface areas, so called boundary elements, we can express the typical boundary element equation for the node "i":

$$\frac{1}{2} w_i + \int_{\Omega_1} w \cdot \frac{\partial F}{\partial n} d\Omega_1 = \frac{\partial w}{\partial n} \cdot \int_{\Omega_2} F \, d\Omega_2 \qquad \dots (1)$$

where:

$F = F(x,y,z)$ is a fundamental function satisfying Laplace equation:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = -\delta_i \qquad \dots (2)$$

$\frac{\partial}{\partial n}(\ )$ is a derivative corresponding to the normal at surfaces for the i-th node. Because we have discrete form of surfaces Ω_1 and Ω_2, the integrals over subdomains Ω_1 and Ω_2 (see equation (1)) become:

$$J_1 = \int_{\Omega_1} w \frac{\partial F}{\partial n} d\Omega_1 = \sum_{L=1}^{L=NBE} \int_{\sigma} w \frac{\partial F}{\partial n} d\sigma$$

$$J_2 = \int \frac{\partial w}{\partial n} F \, d\Omega_2 = \sum_{L=1}^{L=NBE} \int_{\tau} \frac{\partial w}{\partial n} \cdot F \, d\tau \qquad \dots (3)$$

Now equation (1) can be expressed:

$$\frac{1}{2} w_i + \sum_{L=1}^{L=NBE} \int_{\sigma} w \frac{\partial F}{\partial n} d\sigma = \sum_{L=1}^{L=NBE} \int_{\sigma} \frac{\partial F}{\partial n} \cdot F \cdot d\sigma \qquad \dots (4)$$

For 3D isotropic domain the fundamental function $F=F(x,y,z)$ is given:

$$F = F(x,y,z) = \frac{1}{4\pi r} = \frac{1}{4\pi r(x,y,z)} \qquad \dots (5)$$

where $r = r(x,y,z)$ is a distance between the observed node "i" and any node "J" with the coordinates x,y,z:

$$r = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \qquad \dots (6)$$

Normal derivative can be expressed:

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial x} \cdot \cos\alpha + \frac{\partial F}{\partial y} \cdot \cos\beta + \frac{\partial F}{\partial z} \cos\gamma \qquad \dots (7)$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the normal cosines for the node "i". The partial derivatives $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ can be expressed:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{x_i - x}{4\pi r^2} \qquad \frac{\partial F}{\partial y} = \frac{y_i - y}{4\pi r^2} \qquad \frac{\partial F}{\partial z} = \frac{z_i - z}{4\pi r^2} \qquad \dots (8)$$

The typical boundary equation (4) for the node "i" is expressed as a function of the integrals over the B.E. surfaces. For the purpose of integration 8 node isoparametric B.E. is taken (see Fig. (3)). The potential w and its derivative $\partial w/\partial n$ distribution in the B.E. domain is given as:

$$w = w(\xi,\ \eta) = \sum_{k=1}^{k=MEL} wk\ .\ Nk(\xi,\ \eta)$$

$$\frac{\partial w}{\partial n} = \frac{\partial w(\xi,\eta)}{\partial n} = \sum_{k=1}^{k=MEL} \frac{\partial wk}{\partial n}\ .\ Nk(\xi,\eta) \qquad \dots (9)$$

and coordinates are expressed as:

$$x(\xi,\eta) = \sum_{k=1}^{MEL} xk\ .\ Nk(\xi,\eta) \qquad y(\xi,\eta) = \sum_{k=1}^{MEL} yk\ .\ Nk(\xi,\eta)$$
$$z(\xi,\eta) = \sum_{k=1}^{MEL} zk\ .\ Nk(\xi,\eta) \qquad \dots (10)$$

The normal cosines at the node "i" of the 8 node B.E. are defined:

$$nx = \frac{\partial y}{\partial \xi}\cdot\frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta}\cdot\frac{\partial z}{\partial \xi} \qquad ny = \frac{\partial z}{\partial \xi}\cdot\frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi}\cdot\frac{\partial z}{\partial \eta}$$
$$nz = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi} \qquad \dots (11)$$

$$n = \sqrt{nx^2+ny^2+nz^2} \qquad \cos\alpha = \frac{nx}{n} \qquad \cos\beta = \frac{ny}{n} \qquad \cos\gamma = \frac{nz}{n} \qquad \dots (12)$$

Now equation (4) can be expressed:

$$\frac{w_i}{2} + \sum_{L=1}^{NBE} \int \sum_{k=1}^{MEL} \frac{w_k\ .\ N_k}{4\pi r^2} \{ (xi-xj).\frac{nx}{n} + (yi-yj).\frac{ny}{n} + (zi-zj).\frac{nz}{n} \}.n.d\xi.d\eta =$$

$$= \sum_{L=1}^{NBE} \int \sum_{k=1}^{MEL} \frac{\partial wk}{\partial n}\ .\ Nk\ .\ \frac{n}{4\pi r}\ .\ d\xi\ .\ d\eta \qquad \dots (13)$$

For simplicity we can write:

$$H_{ki} = \int_{-1}^{1}\int_{-1}^{1} Nk\ .\ f_j\ .\ d\xi.d\eta \qquad G_{ki} = \int_{-1}^{1}\int_{-1}^{1} Nk\ .\ g_j\ .\ d\xi\ .\ d\eta$$

and:

$$f_j = \frac{1}{4\pi r^2}\{ (xi-xj).nx+(xi-yj).ny+(zi-zj).nz \} \qquad g_j = \frac{n}{4\pi r} \qquad \dots (14)$$

Finally, equation (13) becomes:

$$\frac{wi}{2} + \sum_{L=1}^{NBE} \sum_{k=1}^{MEL} wk\ .\ H_{ki} = \sum_{L=1}^{NBE} \sum_{k=1}^{MEL} \frac{\partial wk}{\partial n}\ .\ G_{ki} \qquad i = 1,N \qquad \dots (15)$$

where: NBE is a total number of boundary elements
MEL is a number of B.E. nodes. In our case MEL=8
N is a total number of boundary nodes.

we can write N equations similarly to the equation (15). Matrix expression of these equations can be written as:

$$\underline{A} \cdot \underline{w} = \underline{B} \cdot \frac{\partial w}{\partial n} \qquad \dots (16)$$

Again, we have N linear equations with N unknowns either potentials w, or derivatives $\partial w/\partial n$. In general case, the known boundary conditions can be natural values of potential $\bar{w}$ or derivatives $\partial \bar{w}/\partial n$. Having this situation, the unknown values of w or $\partial w/\partial n$ can be at the both sides of the equation (16). Rearranging the equation (16) so that the unknowns will be at the left side we have the following linear equations:

$$\underline{F} \cdot \underline{Y} = \underline{Y}_o \qquad \dots (17)$$

in which $\underline{Y}$ is unknown vector of a potential w and derivatives $\partial w/\partial n$ and has to be solved:

$$\underline{Y} = \underline{F}^{-1} \cdot \underline{Y}_o \qquad \dots (18)$$

In the case of non rigid cylindrical walls the equation of motion of the entire system (fluid and cylindrical structure) is given by:

$$\underline{M} \cdot \underline{r} + \underline{C} \cdot \underline{\dot{r}} + \underline{K} \cdot \underline{r} + \underline{W} = \underline{\phi} \qquad \dots (19)$$

where: $\underline{M}$, $\underline{K}$, $\underline{C}$ are the mass matrix, stiffness matrix of the cylindrical reservoir and damping matrix.

$\underline{W}$ is the hydrodynamic water pressure force vector.

$\underline{r}$ is the total acceleration vector.

The hydrodynamic force vector $\underline{W}$ can be defined using the following matrix operations:

$$\underline{Pn} = \underline{H}^T \cdot \underline{rn} \qquad r = \underline{\lambda} \cdot \underline{r} \quad \underline{Pn} = \underline{H}^T \lambda \cdot \underline{r} \qquad Wn = \underline{A} \cdot \underline{Pn}$$

$$\underline{W} = \underline{\lambda}^T \cdot \underline{Wn} = \underline{\lambda}^T \cdot \underline{A}\, \underline{H}^T \cdot \underline{\lambda} \cdot \underline{r} \qquad \dots (20)$$

where: $\underline{Pn}$ is a vector of the normal water pressure at the discrete nodes.

$\underline{rn}$ is a vector of the total normal accelerations at the discrete nodes.

$\underline{\lambda}$ is the cosines transform matrix

$\underline{A}$ is a tributary area matrix for the associated nodes.

H^T is the "influence" matrix. The coefficients of H matrix can be defined using the following approach:

. Using the B.E. mathematical model, and giving at the node "i" the boundary condition that $\partial w_i/\partial n = 1$, and for all other nodes $\partial w_j/\partial n = 0$ we can solve and define the partial water pressur coefficients at all nodes caused by unit normal acceleration at the node "i". The nodal water pressure values for that case define the matrix H coefficients for "i"th row. Repeating N times the solutions of the system for other nodes we can define all H matrix coefficients. The further procedure of solving the equation (19) is a standard process in which the added mass technique can be applied. The added mass matrix is defined as:

$$\underline{Ma} = \lambda^T \cdot \underline{A} \cdot \underline{H}^T \cdot \underline{\lambda} \cdot \underline{r} \qquad \dots (21)$$

Examples and Improvement of Theoretical Procedures

As an example, a cylindrical reservoir is taken with diameter D=20m and height h = 10m. The fluid in the reservoir is assumed to be incompressible. Also the reservoir walls are assumed to be rigid. The contour domains at the surface of the fluid, bottom and cylindrical wall are shown in Fig. (4). The boundary element discrete mesh is given in Fig. (5). Reservoir system, in the first case is subjected to the horizontal excitation. The boundary conditions for this case are given in Fig. (6), picture (a) and (b). The water pressure distribution at the sections 1-1, 2-2 and 3-3 are given in Fig (6), picture (c) and (d). For the case of vertical excitation the boundary conditions are given in Fig. (7), picture (a) and water pressure distribution diagram at the cylindrical wall is given in picture (b).

CONCLUSIONS

The paper presents theoretical principles of the B.E. method and provides representative examples in support of the solution consistency and accuracy of the fluid-structure interaction problem. As an example a case of rigid cylindrical wall reservoir has been treated taking that the system is moving both horizontally and vertically under constant acceleration distribution of 1g, for each direction, separately. Assumption has been made of incompressible fluid.

It is concluded that a consistent water pressure distribution exists while the results correlate to the known theoretical method results for similar contour conditions. For deformable reservoir walls, a brief presentation has been made of the principles on which the fluid-reservoir interaction is based. The added mass process has been suggested. Comparing this procedure with the F.E. procedure for solving this type of problem is concluded:

. In the B.E. method we deal with discrete system of nodes and B.E. related only to the surface domains, while the F.E. deals with space discrete fluid models. Therefore, it should be expected that the total degree of freedom of a F.E. system compared to the B.E. system is larger.

. However, the B.E. method operates with nonsymmetric matrices so that bandwidth and symmetry can not be used. The B.E. matrices are full matrices of order N x N where N is the total number of boundary nodes of the system.

. The use of B.E. method is also favourable from a viewpoint that we operate with contact areas which is of interest in the case of interaction problems.

. The definition of H-matrix terms by B.E. method is relatively simple while the computation time is reasonable.

. The basic conclusion is that the use of B.E. method for solving of interaction problems is possible and in some cases rather efficient.

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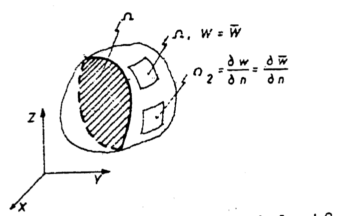

Fig. 1. 3 D domain Ω and surface subdomain Ω_1 and Ω_2 where the natural as well as essential boundary conditions are known

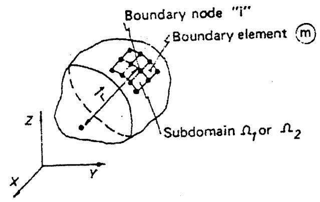

Fig. 2. Discretschem of the surface domains Ω_1 or Ω_2 typical boundary node "i" and boundary element (m)

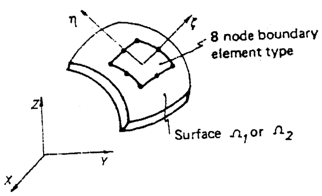

Fig. 3. The typical 8 node boundary element

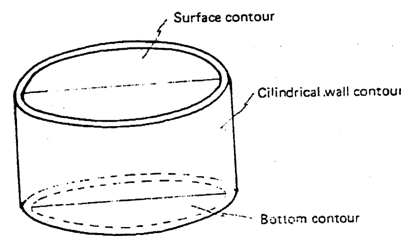

Fig. 4. The contour domain description

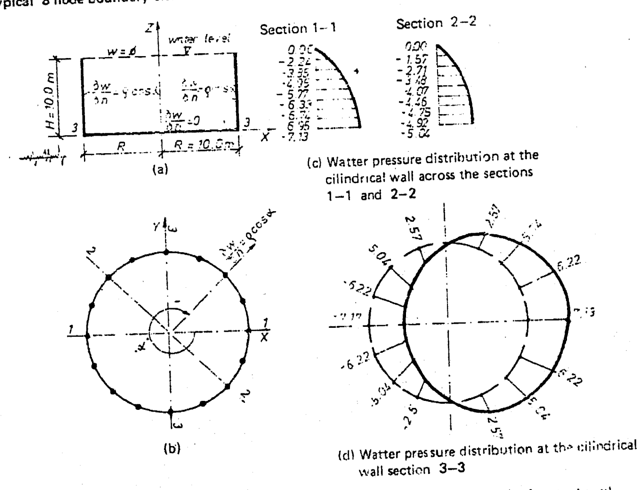

Fig. 6. Water pressure distribution at the vertical cilindrical wall due to horizontal (x–x) motion, the reservoir is assumed to be rigid. The boundary surface conditions are given in the picture (a) and (b)

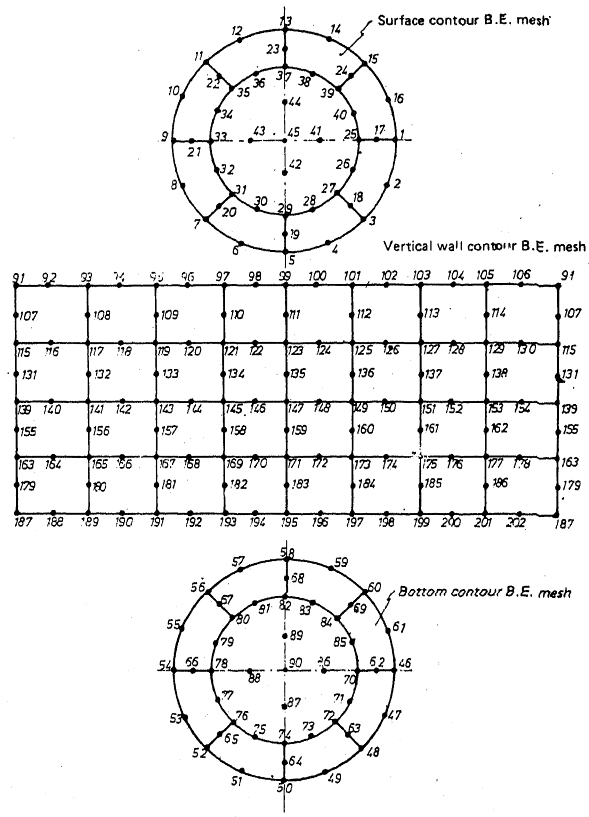

Fig. 5. Boundary element discrete mesh for the surface regions

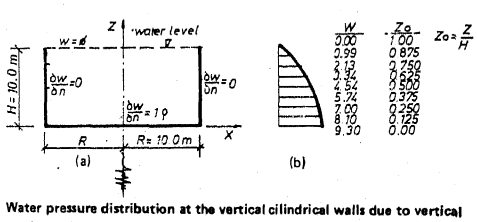

Fig. 7. Water pressure distribution at the vertical cilindrical walls due to vertical motion. The reservoir is assumed to be rigid. The boundary conditions are given in the picture (a)