FREQUENCY-DOMAIN ANALYSIS OF ENERGY INPUT MADE BY EARTHQUAKES

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SUMMARY

This paper presents a simple method to estimate the statistical parameters of the work done by the seismic load acting on a structure, namely the energy input, from the stochastic model parameters of the ground acceleration. The method is based on a frequency-domain relationship between the energy input and the Fourier square amplitude spectrum of the ground acceleration.

First the mean and the variance formulas for the linear SDOF system under classical artificial earthquakes are obtained, and then it is shown they are also applicable to the elastoplastic SDOF system using a simple equivalent-linearization technique.

INTRODUCTION

It is not ecconomical that buildings are designed to resist elastically the strongest-type earthquake, which may or may not be encountered in their life time. In such cases structural engineers can not help expecting the energy dissipation capacity of the building in inelastic range.

Consider a SDOF system, the dynamic behavior of which is governed by

$$m\ddot{x} + c\dot{x} + Q(x,\dot{x},t) = -m\ddot{y} \quad (1)$$

where

m	:	mass	t	:	time from the rest
x	:	relative displacement	$\cdot$	:	time derivative
$\ddot{y}$	:	ground acceleration	Q	:	restoring force
c	:	damping coefficient			

In this paper the work done by the effective excitation term $-m\ddot{y}$ is regarded as the energy input exerted by the earthquake. A typical time history of the energy input and its components are shown in Fig.1. The cumulative plastic strain energy E_p and the energy dissipation due to viscous damping E_h are both monotonically increasing quantities, and the energy input E_I as the load effect can be represented at its final state:

$$E_I = -m\int_{-\infty}^{\infty}\ddot{y}(t)\dot{x}(t)dt \quad (2)$$

Structural engineers should provide the building with sufficient strength and deformability, so that it can dissipate the above quantity E_I exerted by the design earthquake with no significant damage. This type of energy-based limit state design was classicaly initiated by Housner (Ref.1), and applied to the design of steel structures by Kato and Akiyama (Ref.2), both in a deterministic approach. There are still much difficulties to construct an appropriate stochastic model of excitations and structural capacities, but it is

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important to estimate at least the order of reliability in the energy-based limit state, as shown in Fig.2. For this purpose this paper investigate a practical method to estimate the fundamental statistical parameters of the energy input under a certain stochastic model of excitations.

FREQUENCY-DOMAIN EXPRESSION OF ENERGY INPUT

Mosts of the properties of the design earthquake are often described in frequency-domain. Corresponding to this, the frequency-domain expression of the energy input is convienient in the following analysis, and studied in this paragraph. The Fourier's identity used herein has the following form:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \ , \quad x(t) = (1/2\pi)\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \qquad (3)$$

where ω : circular frequency $\qquad$ j : imaginary unit

In this paper the above relationship is simply expressed by $x(t) \leftrightarrow X(\omega)$. The Pranscherel's theorem, known as the power theorem, is expressed by

$$\int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt = (1/2\pi)\int_{-\infty}^{\infty} X_1(\omega)X_2^*(\omega)d\omega \qquad (4)$$

where $\quad x_1(t) \leftrightarrow X_1(\omega)$, $\quad x_2(t) \leftrightarrow X_2(\omega)$

Z^* : conjugate of the complex number or function Z

Substituting $\dot{x}(t)$ and $\ddot{y}(t)$ in eq.(2) into eq.(4) leads to

$$E_I = -m(1/2\pi)\int_{-\infty}^{\infty} \dot{X}(\omega)\ddot{Y}^*(\omega)d\omega \qquad (5)$$

where $\quad \dot{x}(t) \leftrightarrow \dot{X}(\omega)$, $\quad \ddot{y}(t) \leftrightarrow \ddot{Y}(\omega)$

Introducing the concepts of the system function, or the complex frequency response function $H(\omega)$, and considering $\dot{x}(t)$ and $\ddot{y}(t)$ are real functions, eq.(5) results in

$$E_I = \int_{-\infty}^{\infty} W(\omega)\ddot{Y}(\omega)\ddot{Y}^*(\omega)d\omega \qquad (6)$$

after introducing

$$W(\omega) = -m\cdot \mathrm{Real}[H(\omega)]/2\pi \qquad (7)$$

where $\quad$ Real[Z] : real part of the complex number or function Z

$H(\omega)$: complex frequency response function by which $\dot{X}(\omega)$ can be obtained from $\ddot{Y}(\omega)$

The function $W(\omega)$ plays a roll as a gate for the original energy components which are expressed by the Fourier square amplitude spectrum, or the energy spectrum, of the ground acceleration. In this sense the function $W(\omega)$ is named in this paper as the energy admittance of the structural system. The energy admittance of the linear vibrational system is time-invariant and depends only on the parameters of the vibrational system. The energy admittance of the elastic SDOF system with viscous damping is given by

$$W(\omega) = \frac{mh\omega o\omega^2}{\pi[(\omega^2-\omega o^2) + 4h^2\omega o^2\omega^2]} \qquad (8)$$

where ωo : natural circular frequency
h : damping constant

The properties of the energy admittance were investigated by Takizawa (Ref.3) and by Lyon (Ref.4). The important property about the area of the energy admittance is

$$\int_{-\infty}^{\infty} W(\omega)d\omega = m/2 \qquad (9)$$

The configuration of the frequency-domain analysis of the energy input is schematically illustrated in Fig.3.

STATISTICAL PARAMETERS OF ENERGY INPUT TO LINEAR SDOF SYSTEM

The most popular non-stationary stochastic model of the ground acceleration is the filtered white noize type shown in Fig.4. In this classical procedure of generating artificial earthquakes, the simulated sample function of Gaussian white noize is first modified by the deterministic shape function a(t), and then the filtering is carried out using the deterministic filter, the Fourier transformation of which is denoted by $F(\omega)$. In this paragraph the mean and the variance formulas of the energy input to the elastic SDOF system with viscous damping are obtained under the above type of stochastic model of the ground acceleration.

MEAN

The operation of mathematical expectation, denoted by E[ ], is applied to the both sides of eq.(6), and we obtain

$$E[E_I] = \int_{-\infty}^{\infty} W(\omega)E[\ddot{Y}(\omega)\ddot{Y}^*(\omega)]d\omega \qquad (10)$$

The expected Fourier square amplitude of the above type of stochastic excitation model is given by (see Appendix 1)

$$E[\ddot{Y}(\omega)\ddot{Y}^*(\omega)] = 2\pi SoF(\omega)F^*(\omega) \int_{-\infty}^{\infty}[a(t)]^2dt \qquad (11)$$

where So : power spectral density of Gaussian white noize

Substituting eq.(11) into eq.(10), we obtain

$$E[E_I] = 2\pi So \int_{-\infty}^{\infty}[a(t)]^2dt\cdot \int_{-\infty}^{\infty}W(\omega)F(\omega)F^*(\omega)d\omega \qquad (12)$$

The integral $\int W(\omega)F(\omega)F^*(\omega)d\omega$ can be calculated systematically with the availability of the Hurwitz's determinant. The expected energy input to viscously damped SDOF linear system under three types of filter (Refs.5,6,7) are shown on Table 1. When the band-width of the energy admittance is narrow enough and the function $F(\omega)F^*(\omega)$ is slowly varing, the following approximation is admissible:

$$E[E_I] = \pi mSoF(\omega o)F^*(\omega o) \int_{-\infty}^{\infty}[a(t)]^2dt \qquad (13)$$

VARIANCE

The operation of variance, denoted by V[ ], is applied to the both sides of eq.(6), and we obtain

$$V[E_I] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\omega_1)W(\omega_2)CV_{\ddot{Y}\ddot{Y}*}(\omega_1,\omega_2)d\omega_1 d\omega_2 \quad (14)$$

after introducing

$$CV_{\ddot{Y}\ddot{Y}*}(\omega_1,\omega_2) = E[\ddot{Y}(\omega_1)\ddot{Y}*(\omega_1)\ddot{Y}(\omega_2)\ddot{Y}*(\omega_2)] - E[\ddot{Y}(\omega_1)\ddot{Y}*(\omega_1)]E[\ddot{Y}(\omega_2)\ddot{Y}*(\omega_2)] \quad (15)$$

As for the stochastic model shown in Fig.4, $CV_{\ddot{Y}\ddot{Y}*}(\omega_1,\omega_2)$ is given by

$$CV_{\ddot{Y}\ddot{Y}*}(\omega_1,\omega_2) = 4\pi^2 So^2 F(\omega_1)F*(\omega_1)F(\omega_2)F*(\omega_2) \cdot [A_2(\omega_1+\omega_2)A_2*(\omega_1+\omega_2) + A_2(\omega_1-\omega_2)A_2*(\omega_1-\omega_2)] \quad (16)$$

(see Appendix 2)

where $[a(t)]^2 \leftrightarrow A_2(\omega)$

Substituting eq.(16) into eq.(14), and considering $W(\omega)$ is an even function of ω, leads to

$$V[E_I] = 8\pi^2 So^2 \int_{-\infty}^{\infty} U(\omega)A_2(\omega)A_2*(\omega)d\omega \quad (17)$$

after introducing

$$U(\omega) = \int_{-\infty}^{\infty} W(\omega_1)W(\omega_1+\omega)F(\omega_1)F(\omega_1+\omega)F*(\omega_1)F*(\omega_1+\omega)d\omega_1 \quad (18)$$

It is possible to calculate eq.(17) directly, but the results have generally very complicated forms. Then we use the narrow-band approximation similar to eq.(13), and the function $U(\omega)$ is approximated by

$$U(\omega) \simeq \frac{[F(\omega o)F*(\omega o)]^2 m^2 h\omega o}{4\pi(1-h^2)}\left[\frac{(3-4h^2)\omega^2 - 4\omega o^2}{(4\omega o^2-\omega^2)^2+16h^2\omega o^2\omega^2} + \frac{1}{4h^2\omega o^2+\omega^2}\right] \quad (19)$$

Using the above approximation, eqs.(13), and (17), the coefficient of variation, denoted by c.o.v.[ ], of the enrgy input is expressed by

$$c.o.v.[E_I] = \sqrt{\frac{2h\omega o}{\pi(1-h^2)}\int_{-\infty}^{\infty}\left[\frac{(3-4h^2)\omega^2 - 4\omega o^2}{(4\omega o^2-\omega^2)^2+16h^2\omega o^2\omega^2} + \frac{1}{4h^2\omega o^2+\omega^2}\right]A_2(\omega)A_2*(\omega)d\omega} \Big/ \int_{-\infty}^{\infty}[a(t)]^2 dt \quad (20)$$

The above formula gives an exact solution in the case of no filter, that is, $F(\omega)F*(\omega)=1$. As for two types of shape function (Refs.7 and 8), eq.(20) was estimated and shown on Table 2.

EXAMPLE 1

Using the type A filter (see Fig.5) and the type I shape function (see Fig.6), the Monte Carlo simulation was carried out with 400 samples per each cases. The results are compared with the prediction by eqs.(12) and (20), as shown in Fig.7. The parameters used in the simulation were as follows:

Type A filter	: $\omega g = 5\pi$ (sec^{-1})	$hg = 0.3$	
Type I shape function	: $a_1 = 2/3\sqrt{3}$ (sec^{-1})	$c = 1/3$	(sec^{-1})
	that is, $\int_{-\infty}^{\infty}[a(t)]^2 dt = 1$		(sec)
Vibrational system	: $\omega o = k\pi$	$k = 1,2,\ldots,10$	(sec^{-1})
	$h = 0.02$ and 0.2		

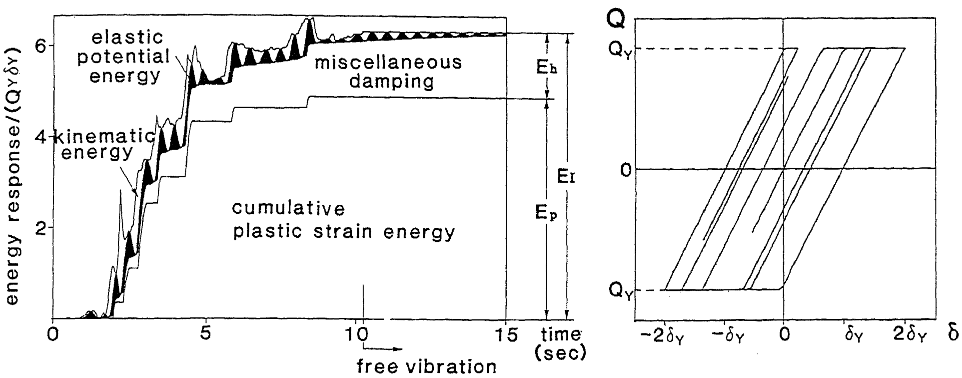

Fig.1 Energy Responses of an Elastoplastic SDOF System

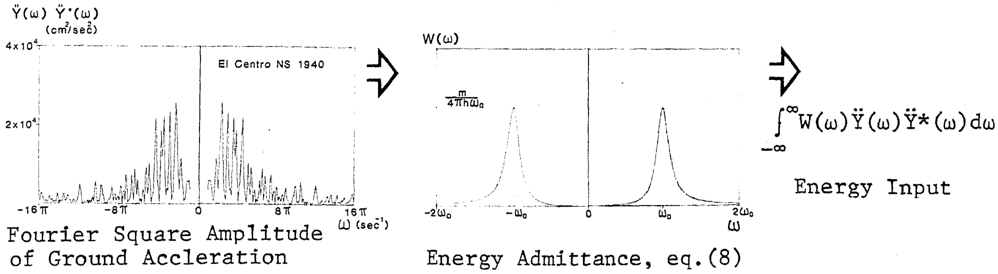

Fig.3 Frequency-domain Analysis of Energy Input Made by an Earthquake

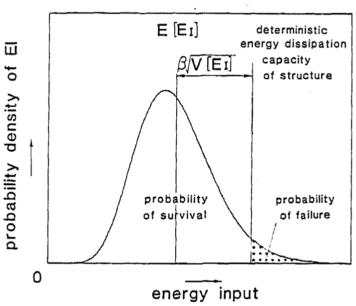

Fig.2 Idealized Reliability Problem

Table 1 Expected Energy Input to Viscously Damped Linear System

CODE	$F(\omega)F^*(\omega)$	$E[E_I] / [m\pi S_0 \cdot \int_{-\infty}^{\infty} [a(t)]^2 dt]$
A	$\frac{\omega_g^4}{(\omega_g^2-\omega^2)^2+4h_g^2\omega_g^2\omega^2}$	$\frac{\omega_g^3(\omega_0 h+\omega_g h_g)}{h_g[(\omega_g^2-\omega_0^2)^2+4\omega_g\omega_0 h_g h(\omega_0^2+\omega_g^2)+4\omega_g^2\omega_0^2(h_g^2+h^2)]}$
B	$\frac{\omega_g^4+4h_g^2\omega_g^2\omega^2}{(\omega_g^2-\omega^2)^2+4h_g^2\omega_g^2\omega^2}$	$\frac{\omega_g^2[\omega_g(\omega_0 h+\omega_g h_g)+4\omega_0 h_g^2(\omega_0 h_g+\omega_g h)]}{h_g[(\omega_g^2-\omega_0^2)^2+4\omega_g\omega_0 h_g h(\omega_0^2+\omega_g^2)+4\omega_g^2\omega_0^2(h_g^2+h^2)]}$
C	$\frac{\alpha\omega_g(\alpha^2+\omega_g^2+\omega^2)}{(\alpha^2+\omega_g^2-\omega^2)^2+4\alpha^2\omega^2}$	$\frac{\omega_g[\alpha(\omega_g^2+\alpha^2+\omega_0^2)+2\omega_0 h(\alpha^2+\omega_g^2)]}{(\alpha^2+\omega_g^2-\omega_0^2)^2+4\omega_0(\alpha+\omega_0 h)[\alpha\omega_0+h(\alpha^2+\omega_g^2)]}$
D	1 (No filter)	1

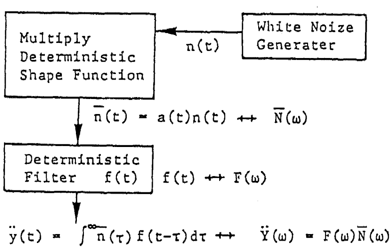

Fig.4 Non-stationary Filtered White Noize Model

Table 2 Coefficient of Variation of Energy Input to Viscously Damped Linear System

CODE	a(t)	c.o.v.$[E_I]$ $(\sqrt{V[E_I]} / E[E_I])$
I	$a_1 \cdot t\, e^{-ct}$	Square root of $\frac{c}{8(1-h^2)} \cdot \left[\frac{8c^2+9c\omega_0 h+3\omega_0^2h^2}{(c+\omega_0 h)^3} + \frac{7c^5(1-2h^2)+c^3[c(1-2h^2)+\omega_0 h](c+2\omega_0 h)-\omega_0 h(3\omega_0^2+8c^2+6ch\omega_0)\cdot(2ch+\omega_0)^2}{(c^2+\omega_0^2+2c\omega_0 h)^3}\right]$
II	$a_2 \cdot e^{-ct}$	$\sqrt{\frac{c}{1-h^2} \cdot \left[\frac{1}{c+\omega_0 h} + \frac{c(1-2h^2)-\omega_0 h}{c^2+\omega_0^2+2c\omega_0 h}\right]}$

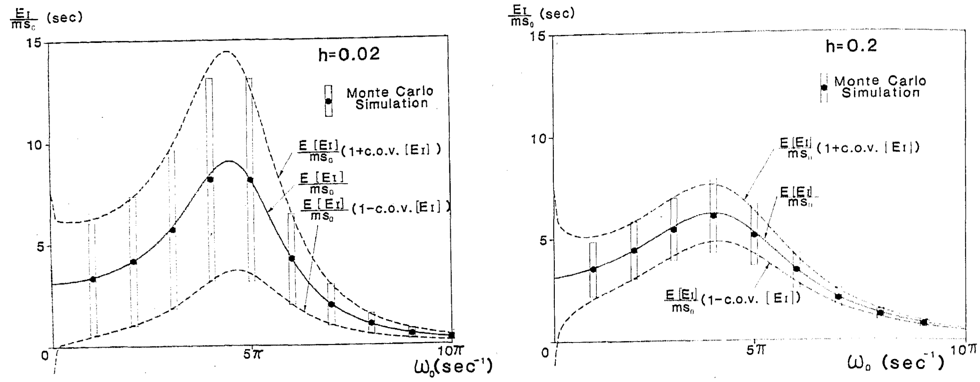

Fig.7 Example 1 —— Energy Input to Viscously Damped Linear SDOF System

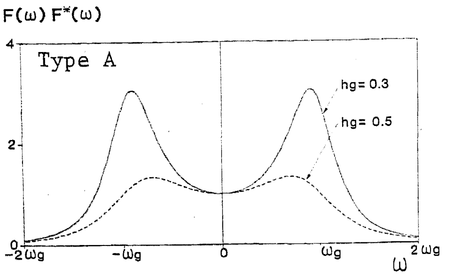

Fig.5 Deterministic Filter

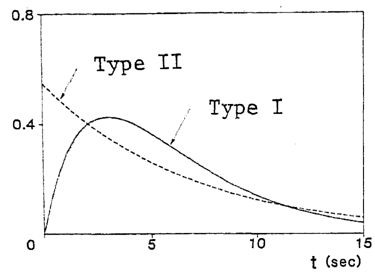

Fig.6 Shape Function

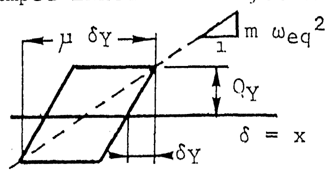

Fig.8 Elastoplastic System

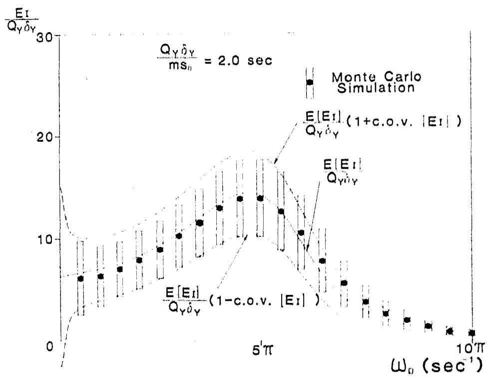

Fig.9 Example 2 —— Energy Input to Elastoplastic SDOF System

APPLICATION TO HYSTERETIC SDOF SYSTEM

Various types of the equivalent linearization technique for hysteretic vibrational systems were investigated by many researchers. Any techniques proposed can make the formulas described in the preceding paragraph applicable to hysteretic systems, so long as the first and the second moments are dealt with. In this paper no attempts are made to discuss which techniques are preferrable, and the power-balance type techniques (Refs. 9 and 10) are chosen only because the energy concepts are used.

EQUIVALENT LINEARIZATION PROCEDURE

The equivalent linearization procedure for the elastoplastic SDOF system with viscous damping was carried out iteratively in the following way:

1. The initial values of the equivalent natural circular frequency ω_{eq} and the equivalent damping constant h_{eq} were set to ω_0 and h, respectively.
2. The expected energy input exerted during the major portion of the earthquakes, from the time t_1 to t_2, was calculated by eq.(12) with the last values of ω_{eq} and h_{eq} as follows:

$$E[E_{It_1t_2}] = E[E_I]\cdot \int_{t_1}^{t_2}[a(t)]^2 dt \Big/ \int_{-\infty}^{\infty}[a(t)]^2 dt \qquad (21)$$

3. The following assumptions were made in order to improve the values, ω_{eq} and h_{eq} . Assumed that the component of the frequency ω_{eq} is very dominant in the response, the expected energy input per unit cycle e_I is approximated by

$$e_I = E[E_{It_1t_2}]\cdot(2\pi/\omega_{eq})/(t_2-t_1) \qquad (22)$$

The energy-balance equation per unit cycle is expressed by

$$e_I = e_P + e_h = e_{eq} \qquad (23)$$

where e_I, e_P, and e_h denote expected values of E_I, E_P, and E_h per unit cycle, respectively, and e_{eq} denotes the energy dissipation due to equivalent damping per unit cycle. Assumed that the response is harmonic-like, e_P, e_h, and e_{eq} are related to the local amplitude $\mu\delta_Y$ shown in Fig.8 as:

$$e_P=2Q_Y\delta_Y(\mu-2),\quad e_h=(\pi/2)h\mu^2Q_Y\delta_Y(\omega_{eq}/\omega_0),\quad e_{eq}=(\pi/2)h_{eq}\mu^2Q_Y\delta_Y \qquad (24)$$

Using the secant modulus method

$$\omega_{eq} = \omega_0\sqrt{2/\mu} \qquad (25)$$

The improved values of ω_{eq} and h_{eq} can be obtained by solving eqs.(22) to (25).

4. Using the improved parameters the procedure 2 is consecutively repeated until the value of e_I is converged. After the convergence, $E[E_I]$ and c.o.v.$[E_I]$ were calculated by eqs.(12) and (20) with the final parameters.

EXAMPLE 2

The same filter and the same shape function as Example 1 were used in the simulation of this paragraph. The original damping constant h is set to 0.02 and yield energy $Q_Y\delta_Y$ is set to the same value, $mS_0\cdot(2.0$ sec), in each elasto-plastic system. The prediction made by the above procedure and the simulated results are compared in Fig.9.

CONCLUDING REMARKS

1. The Fourier square amplitude spectrum of the ground acceleration can be transformed by the energy admittance into the energy input to the structure subjected to the earthquake.
2. The exact formula to estimate the mean, and an approximate formula to estimate the coefficient of variation, of the energy input to viscously damped linear system under classical stochastic models of the ground acceleration were obtained.
3. It is found that the above formulas are also applicable to the elasto-plastic system using an appropriate equivalent linearization technique. This provides a practical means to evaluate the reliability of hysteretic structures.

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APPENDIX 1 Expectaion of $\ddot{Y}(\omega)\ddot{Y}^*(\omega)$

$$E[\ddot{Y}(\omega)\ddot{Y}^*(\omega)]=F(\omega)F^*(\omega)\iint a(t_1)a(t_2)e^{j\omega(t_1-t_2)}E[n(t_1)n(t_2)]dt_1dt_2 \quad (A1)$$

Considering n(t) is white noize, $E[n(t)n(t+\tau)]=2\pi So\delta(\tau)$ (A2)

where So : constant power spectral density $\delta(\tau)$: unit impulse function

Substituting (A2) into (A1) leads to eq.(11) in the text.

APPENDIX 2 Covariance of $\ddot{Y}(\omega)\ddot{Y}^*(\omega)$

$$CV_{\ddot{Y}\ddot{Y}^*}(\omega_1,\omega_2)=F(\omega_1)F^*(\omega_1)F(\omega_2)F^*(\omega_2)\iiiint a(t_1)a(t_2)a(t_3)a(t_4)e^{j\omega_1(t_1-t_2)}$$
$$\cdot e^{j\omega_2(t_3-t_4)}E[n(t_1)n(t_2)n(t_3)n(t_4)]dt_1dt_2dt_3dt_4$$
$$-E[\ddot{Y}(\omega_1)\ddot{Y}^*(\omega_1)]E[\ddot{Y}(\omega_2)\ddot{Y}^*(\omega_2)] \quad (A3)$$

Considering n(t) is Gaussian,

$$E[n(t_1)n(t_2)n(t_3)n(t_4)]=E[n(t_1)n(t_2)]E[n(t_3)n(t_4)]+E[n(t_1)n(t_3)]E[n(t_2)n(t_4)]$$
$$+E[n(t_1)n(t_4)]E[n(t_3)n(t_2)] \quad (A4)$$

Substituting (A4), (A2), and eq.(11) into (A3) leads to eq.(16) in the text.

REFERENCES

1. Housner, G.W., "Limit Design of Structures to Resist Earthquakes," 1WCEE, 1956.
2. Kato, B., Akiyama, H., "Seismic Design of Steel Buildings," ASCE, ST8, August 1982.
3. Takizawa, H., "Energy Response Spectra of Earthquake Motion," Proc. 14th Symp. Natural Disaster Sci., August 1977, (in Japanese).
4. Lyon, R.H., Statistical Energy Analysis of Dynamical Systems: Theory and Applications, MIT Press, 1975.
5. Amin, M., Ang, H.S., "Non-stationary Stochastic Model of Earthquake Motions," ASCE, EM2, April 1968.
6. Tajimi, H., "Statistical Method of Determining the Maximum Responses of Building During an Earthquake," 2WCEE, 1960.
7. Bolotin, V.V., "Statistical Theory of the Aseismic Design of Structures," 2WCEE, 1960.
8. Bogdanoff, J.L., Goldberg, J.E., Bernard, M.C., "Response of a Single-Structure to a Random Earthquake-Type Disturbance," Bull. Seismo. Soc. America, Vol.51, No.2, April 1961.
9. Dean Karnopp, "Power Balance Method for Non-linear Random Vibration," Trans. ASME, march 1967.
10. Hong, G., Tanaka, H., "Hysteretic Energy Dissipation of Single-Degree-of-Freedom System Subjected to White Noize Excitation," Trans. Arch. Inst. Jap., No.270, August 1975, (in Japanese).