A METHOD OF SEISMIC RESPONSE ANALYSIS OF HYSTERETIC STRUCTURES BASED ON STOCHASTIC DIFFERENTIAL EQUATIONS

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SUMMARY

An analytical method of probabilistic seismic response analysis of hysteretic structures is presented. The structural systems considered herein are state-dependent hysteretic systems including degrading and stiffening hysteretic characteristics. Formulations of the hysteretic systems as well as seismic excitations are presented in the form of first-order ordinary differential equations. Then, as the augmented state equations of the overall dynamic system, the Itô stochastic differential equations are derived. The structural responses including the maximum displacements and the cumulative plastic deformations are approximately estimated by solving the moment equations. The accuracy of the proposed method is verified against digital simulation.

INTRODUCTION

In evaluating the reliability of structures to seismic excitations, the structural responses representing seismic safety have to be considered from the probabilistic point of view. Experimental evidence has revealed that most structures show hysteretic behaviors with degrading or stiffening characteristics for repeated loadings. This paper deals with an analytical method of probabilistic seismic response analysis of such state-dependent hysteretic structures showing degrading and stiffening hysteretic characteristics. First, by introducing appropriate extra state variables which control the hysteresis, the state-dependent hysteretic systems are represented by a set of first-order nonlinear differential equations. Secondly, the seismic excitation is modeled as a filtered white noise which is the output of shaping filters driven by Gaussian white noise 1,2). The filter system can be also described by a set of first-order differential equations. Through such differential formulations, the state space equations of the overall system consisting of the shaping filter and the hysteretic structure are expressed in the form of the Itô stochastic differential equations. Then, the associated moment equations are derived and are approximately solved. The maximum displacements and cumulative plastic deformations which are representative measures in assessing structural reliability are also estimated. The analytical method is illustrated by examples for the bilinear hysteretic system, Kato-Akiyama's hysteretic system 3) and Clough's hysteretic system 4).

DIFFERENTIAL FORMULATION OF HYSTERESIS

The hysteretic restoring force characteristics of structures could not be in nature represented by single-valued functions in terms of displacements

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and velocities. Therefore, by introducing appropriate extra state variables the differential formulations of hysteretic characteristics have been proposed to obtain well-defined mathematical expressions in the form of nonlinear ordinary differential equations with single-valued functions 5, 6, 7). The hysteretic characteristics considered herein are bilinear hysteretic model, Kato-Akiyama's hysteretic model and Clough's hysteretic model.

The hysteretic characteristic Φ is described by

$$\Phi = rx + (1-r)z \tag{1}$$

where Φ is normalized to have a primary rigidity of unity, x is the nondimensional displacement with reference to the elastic limit deformation, z is the hysteretic component of the force, and r is the ratio of the linear part to the total force.

The differential representation of z in the bilinear hysteretic model as shown in Fig. 1 can be described by 6)

$$\dot{z} = \dot{x}\,[1 - U(\dot{x})\,U(z-1) - U(-\dot{x})\,U(-z-1)] \equiv g_z(\dot{x}, z) \tag{2}$$

where $U(x) = 1$ for $x \geqq 0$, and $= 0$ for $x < 0$.

The hysteretic model presented by Kato and Akiyama 3) shows degrading or stiffening characteristics with the cumulative plastic deformation as shown in Fig. 2. The hysteretic component z in Kato-Akiyama's model is governed by a set of differential equations 7)

$$\dot{z} = \dot{x}[1 - \bar{s}\,U(\dot{x})\,U(z-z_L^+) - \bar{s}\,U(-\dot{x})\,U(-z-z_L^-)] \equiv g_z(\dot{x}, z, \eta^+, \eta^-) \tag{3}$$

$$\dot{\eta}^+ = \bar{s}\,\dot{x}\,U(\dot{x})\,U(z-z_L^+) \equiv g_{\eta^+}(\dot{x}, z, \eta^+) \tag{4}$$

$$\dot{\eta}^- = -\bar{s}\dot{x}\,U(-\dot{x})\,U(-z-z_L^-) \equiv g_{\eta^-}(\dot{x}, z, \eta^-) \tag{5}$$

where $z_L^+ = 1 + s\eta^+/\bar{s}, \quad z_L^- = 1 + s\eta^-/\bar{s}, \quad \bar{s} = 1 - s$ (6)

s denotes a rigidity ratio associated with the hysteretic component z. Thus the state variables η^+ and η^- govern the state-dependent yield strengths z_L^+ and z_L^- of the hysteretic component z. The one-sided cumulative plastic deformations η_p^+ and η_p^- of the hysteretic system in the positive and negative directions, respectively, are related to η^+ and η^- by the equations,

$$\eta_p^+ = (1-r)\eta^+, \quad \eta_p^- = (1-r)\eta^- \tag{7}$$

In the particular case where $s = 0$, Eq. 3 reduces to Eq. 2.

Clough's hysteretic model is shown in Fig. 3. In this model, the hysteretic component z is governed by a set of differential equations 7)

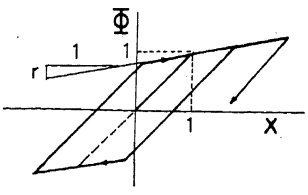


Fig. 1. Bilinear hysteretic model.

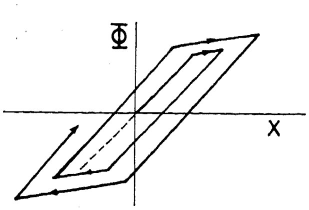


Fig. 2. Kato-Akiyama's hysteretic model.

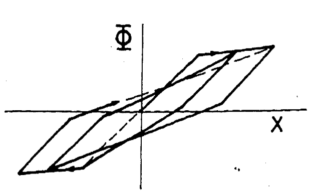


Fig. 3. Clough's hysteretic model.

$$\dot{z} = \dot{x}\,U(z)\,\{A^{+}\,U(\dot{x})[1-U(z-1)] + U(-\dot{x})\}$$
$$+ \dot{x}\,U(-z)\,\{A^{-}\,U(-\dot{x})[1-U(-z-1)] + U(\dot{x})\} \equiv g_z(x,\dot{x},z,\eta^{+},\eta^{-}) \tag{8}$$
$$\dot{\eta}^{+} = \dot{x}\,U(\dot{x})\,U(z-1) \equiv g_{\eta^{+}}(\dot{x},z) \tag{9}$$
$$\dot{\eta}^{+} = -\dot{x}\,U(-\dot{x})\,U(-z-1) \equiv g_{\eta^{-}}(\dot{x},z) \tag{10}$$

where
$$A^{+} = (1-z)/(1+\eta^{+}-x), \quad A^{-} = (1+z)/(1+\eta^{-}+x) \tag{11}$$

The stiffness degrading is governed by the state variables η^{+} and η^{-} introduced here. These state variables are closely related to the maximum and minimum displacement responses x_m^{+} and x_m^{-}, respectively, beyond the elastic limits,

$$x_m^{+} = 1 + \eta^{+}, \quad x_m^{-} = -1 - \eta^{-}, \quad x_m^{+} \geqq 1, \; x_m^{-} \leqq -1 \tag{12}$$

STOCHASTIC PROCESS MODEL FOR SEISMIC EXCITATION

On the basis that prescribed frequency characteristics of earthquakes can be produced by passing white noise through a linear shaping filter 1,2), the seismic excitation f is given by

$$f = \gamma_1 v + \gamma_2 w_2 \tag{13}$$

where, v is the output of a shaping filter driven by a Gaussian white noise. Specifically the filter is governed by the following equations:

$$v = L_2 u, \quad L_1 u = \gamma_3 w_1 \tag{14}$$

in which

$$L_1 = \sum_{i=0}^{m} a_i \frac{d^i}{dt^i}, \quad L_2 = \sum_{j=0}^{l} b_j \frac{d^j}{dt^j}, \quad a_m = b_l = 1, \quad m > l \tag{15}$$

with zero initial conditions. In the above equations w_1 and w_2 are zero-mean Gaussian white noise processes, γ_1, γ_2 and γ_3 are deterministic multipliers, and a_i and b_j, in general, time-variant coefficients. Eq. 14 is rewritten in the following matrix form:

$$_E\dot{Z} = {_E}A\,{_E}Z + {_E}N, \quad {_E}Z_{t=0} = 0 \tag{16}$$

where

$$_EZ = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ u_m \end{bmatrix}, \quad _EA = -\begin{bmatrix} 0 & -1 & 0 & \cdot\cdot & 0 \\ 0 & 0 & -1 & \cdot\cdot & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ 0 & 0 & 0 & \cdot\cdot & -1 \\ a_0 & a_1 & a_2 & \cdot\cdot & a_{m-1} \end{bmatrix}, \quad _EN = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \gamma_3 w_1 \end{bmatrix} \tag{17}$$

and $u_i = d^{i-1}u/dt^{i-1}$, $i = 1, 2, \cdot\cdot, m$

Then, Eq. 13 becomes

$$f = \gamma_1 b^{\mathrm{T}}\,{_E}Z + \gamma_2 w_2 \tag{18}$$

where
$$b^{\mathrm{T}} = [\, b_0 \;\; b_1 \;\; \cdot\cdot \;\; b_l \;\; 0 \;\; \cdot\cdot \;\; 0 \,] \tag{19}$$

STOCHASTIC DIFFERENTIAL EQUATIONS

The governing equations for a multi-degree-of-freedom hysteretic structural system are given by

$$\dot{{}_S Z} = {}_S A \, {}_S Z + {}_S G({}_S Z) + {}_{SE}A \, {}_E Z + {}_S N , \quad {}_S Z_{t=0} = 0 \tag{20}$$

where

$${}_S Z = \begin{bmatrix} x \\ \dot{x} \\ z \\ \eta \end{bmatrix}, \quad {}_S G = \begin{bmatrix} 0 \\ 0 \\ g_z \\ g_\eta \end{bmatrix}, \quad {}_{SE}A = \begin{bmatrix} 0 \\ -\gamma_1 i \, b^T \\ 0 \\ 0 \end{bmatrix}, \quad {}_S N = \begin{bmatrix} 0 \\ -\gamma_2 w_2 i \\ 0 \\ 0 \end{bmatrix}, \quad i = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$$

In the above equations, x is the displacement vector, z is the hysteretic component vector, η is the state vector which controls the state-dependent hysteresis, and g_z and g_η are single-valued nonlinear vector functions. When the input excitation to the structural system is given by Eq. 18, the overall system is governed by the augmented differential equations

$$\dot{Z} = AZ + G(Z) + N, \quad Z_{t=0} = 0 \tag{21}$$

where

$$Z = \begin{bmatrix} {}_S Z \\ {}_E Z \end{bmatrix}, \quad A = \begin{bmatrix} {}_S A & {}_{SE}A \\ 0 & {}_E A \end{bmatrix}, \quad G = \begin{bmatrix} {}_S G \\ 0 \end{bmatrix}, \quad N = \begin{bmatrix} {}_S N \\ {}_E N \end{bmatrix}$$

Since a Gaussian white noise is formally treated as the hypothetical time derivative of a Brownian motion, Eq. 21 can be rewritten by the Itô stochastic differential equations of the form:

$$dZ = F(Z)\,dt + V\,dB , \quad F(Z) = A\,Z + G(Z) , \quad Z_{t=0} = 0 \tag{22}$$

where V is the matrix of time modulations, and B is vector Brownian motion with zero-mean and diffusion intensity matrix Q, namely

$$E[\,dB] = 0, \quad E[\,dB\,dB^{\mathrm{T}}] = Q\,dt \tag{23}$$

The moments $M(l_1, l_2, \cdots, l_n)$ of the state vector Z are defined by

$$M(l_1, l_2, \cdots, l_n) = E[q] , \quad q = \Pi_{i=1}^{n} z_i^{l_i} \tag{24}$$

Then, the first-order nonlinear differential equations for the moments can be derived by using Itô's theorem 8) as follows:

$$\begin{aligned} \dot{M}(l_1, \cdots, l_n) = & \textstyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Gamma_{ij} \, l_i \, l_j \, M(l_1, \cdots, l_i - 1, \cdots, l_j - 1, \cdots, l_n) \\ & + \frac{1}{2} \textstyle\sum_{i=1}^{n} \Gamma_{ii} \, l_i (l_i - 1) M(l_1, \cdots, l_i - 2, \cdots, l_n) + \sum_{i=1}^{n} l_i E[F_i \, q / Z_i] \end{aligned} \tag{25}$$

$$M(l_1, \cdots, l_n)_{t=0} = 0, \quad M(0, \cdots, 0) = 1 \tag{26}$$

where $\Gamma = VQV^{\mathrm{T}}$, Γ_{ij} is the ij element of Γ, and F_i and Z_i are the ith components of F and Z, respectively. In particular, the mean vector $E[Z]$ and the covariance matrix K of the state vector Z are given by

$$\dot{E}[Z] = E[F(Z)] , \quad E[Z] = 0 \tag{27}$$

$$\dot{K} = E[F(Z) Z_d^{\mathrm{T}}] + E[Z_d F(Z)^{\mathrm{T}}] + \Gamma , \quad K_{t=0} = 0 \tag{28}$$

where $\quad Z_d = Z - E[Z]$

In determining the above equations, the probability density function $p(Z;t)$ is required in order to define the expectation operator E in Eqs. 25, 27 and 28. However, unlike the cases involving only linear dynamic system driven white noise, the probability density function $p(Z;t)$ is not of Gaussian and cannot be obtained precisely. Therefore an approximate probability density function is introduced in the following analysis.

STOCHASTIC RESPONSE ANALYSIS AND NUMERICAL RESULTS

An approximate non-Gaussian probability density function of a single-degree-of-freedom hysteretic system is assumed in the cases of bilinear and Clough's hysteretic models as

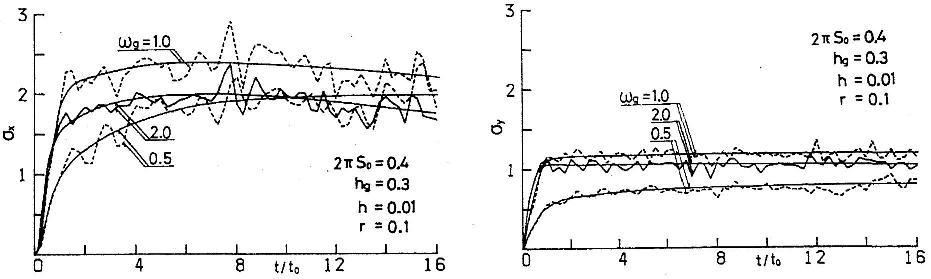
$$p(Z;t) \cong \tilde{p}(Y;t)\,\delta(\eta - E[\eta]) \tag{29}$$

$$\tilde{p}(Y;t) = [1 - U(z-1) - U(-z-1)]\,w(Y;t) + \delta(z+1)\int_{-\infty}^{-1} w(x,y,z',{}_EZ;t)dz' + \delta(z-1)\int_{1}^{\infty} w(x,y,z',{}_EZ;t)dz' \tag{30}$$

In the case of Kato-Akiyama's hysteretic model, $p(Z;t)$ is given by

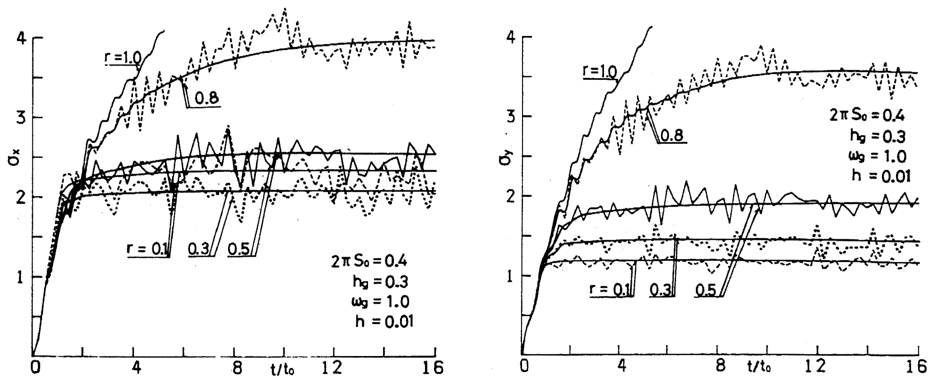
$$p(Z;t) \cong \tilde{p}(Z;t)\,\delta(\eta - E[\eta]) \tag{31}$$

$$\tilde{p}(Z;t) \cong [1 - U(z - z_L) - U(-z - z_L)]\,w(Y;t) + \frac{1}{s}U(z-z_L)\,w(x,y,\frac{z}{s} - \frac{\bar{s}}{s}z_L,{}_EZ;t) + \frac{1}{s}U(-z-z_L)\,w(x,y,\frac{z}{s} + \frac{\bar{s}}{s}z_L,{}_EZ;t) \tag{32}$$



(a) displacement response (b) velocity response

Fig. 4. Standard deviations σ_x and σ_y of bilinear hysteretic system under stationary Kanai-Tajimi filtered white noise with various ω_g values.



(a) displacement response (b) velocity response

Fig. 5. Standard deviations σ_x and σ_y of bilinear hysteretic system with various r values under stationary Kanai-Tajimi filtered white noise.

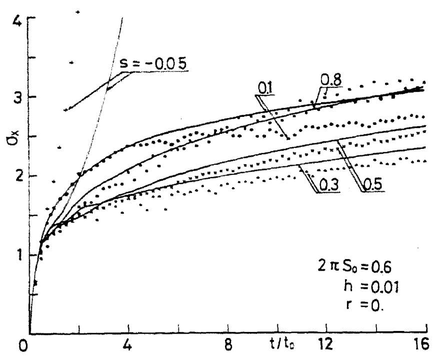
where $Z^T = [\, x\ y\ z\ \eta\ {}_E Z^T \,]$, $Y^T = [\, x\ y\ z\ {}_E Z^T \,]$, $y = \dot{x}$

Here, to simplify the problem, η^+ and η^- are represented by η, since $E[\eta^+] = E[\eta^-]$. $\phi(Y;\nu)$ is the normal probability density function with zero-means and the unknown covariance matrix ν. By using the relationship between K and ν, Eqs. 27 and 28 is transformed to simultaneous first-order nonlinear differential equations, with respect to the unknown parameter matrix ν, which can be solved numerically under nonstationary states.

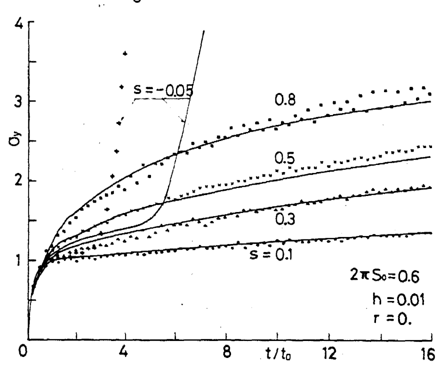
Numerical results obtained by the above-mentioned analytical method are presented, and they are compared with the results obtained by 500 samples of digital simulation. Figs. 4 and 5 show the standard deviations σ_x and σ_y of the bilinear hysteretic system subjected to the stationary filtered white noise process through the Kanai-Tajimi transfer function 1), namely

$$ {}_E Z = \begin{bmatrix} x_g \\ y_g \end{bmatrix}, \quad {}_E A = -\begin{bmatrix} 0 & 1 \\ \omega_g^2 & 2h_g\omega_g \end{bmatrix}, \quad b = \begin{bmatrix} \omega_g^2 \\ 2h_g\omega_g \end{bmatrix}, \quad \gamma_1 = U(t), \quad \gamma_2 = 0 $$

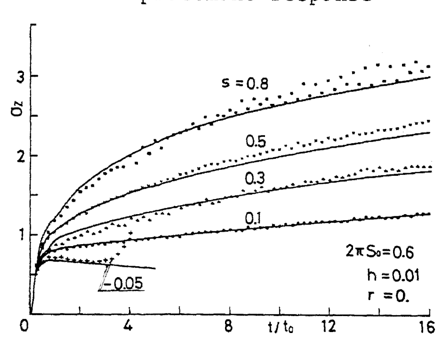
In these figures s_0 is the nondimensional spectral intensity of the white noise process, h is the critical damping ratio and t_0 is the nondimensional



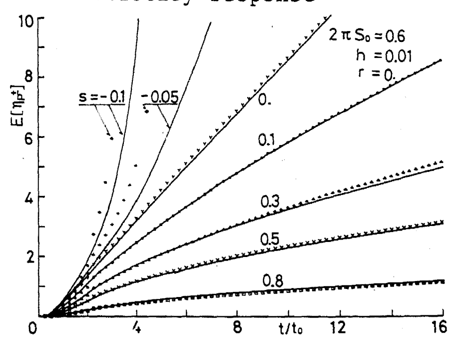
(a) standard deviation of displacement response



(b) standard deviation of velocity response



(c) standard deviation of hysteretic component



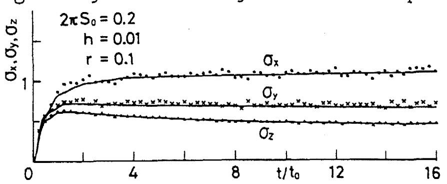
(d) mean value of one-sided cumulative plastic deformation

Fig. 6. Statistics of responses for Kato-Akiyama's hysteretic system under stationary white noise, $r = 0$.

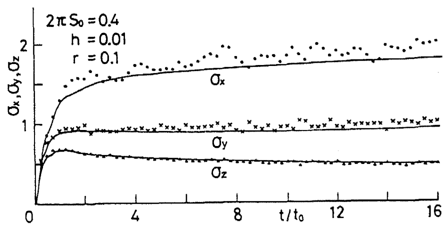
natural period of the system. The effects of the predominant frequency of excitation and of the rigidity ratio of bilinear hysteresis on the responses are shown, respectively, in Figs. 4 and 5. In particular, Fig. 4(a) illustrates a progressive growth of displacement response, due to the access of decreasing virtual natural frequency of the hysteretic system to the predominant frequency of the excitation in the case where $\omega_g < 1$.

The analytical results of Kato-Akiyama's hysteretic system subjected to a stationary Gaussian white noise with spectral intensity s_0 are shown in Figs. 6-8 for the case where $r = 0$. In Figs. 6(a), (b), (c) and (d) the standard deviations σ_x, σ_y and σ_z, and the mean value of one-sided cumulative plastic deformation are shown, respectively, for various values of the rigidity ratio.

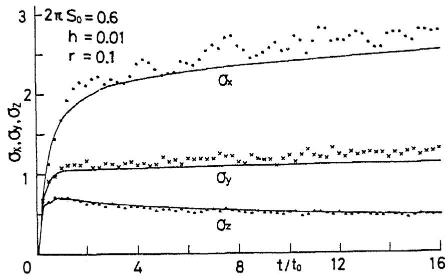
Figs. 7, 8 and 9 present the analytical results of Clough's hysteretic system subjected to stationary Gaussian white noise excitations. In Figs. 7(a), (b) and (c) the standard deviation σ_x, σ_y and σ_z are shown for the cases $2\pi s_0 = 0.2$, 0.4 and 0.6, respectively. Fig. 8 shows the mean value of η. Furthermore, the mean values of the maximum displacement response for Clough's hysteretic system are compared with those of the bilinear hysteretic



(a) $2\pi s_0 = 0.2$



(b) $2\pi s_0 = 0.4$



(c) $2\pi s_0 = 0.6$

Fig. 7. Standard deviations σ_x , σ_y and σ_z of Clough's hysteretic system under stationary white noise.
$2\pi s_0 = 0.2$, 0.4 and 0.6

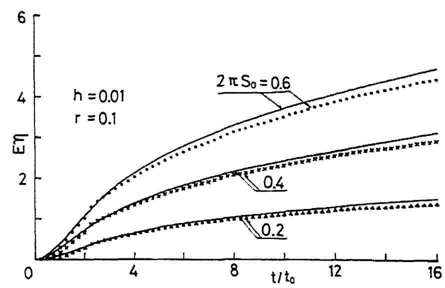


Fig. 8. Mean value $E[\eta]$ of Clough's hysteretic system.

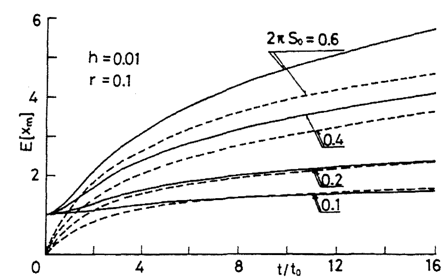


Fig. 9. Mean value of maximum displacement response.

—— Clough's system
---- bilinear system

system in Fig. 9. The maximum displacements of Clough's system are slightly less than those of the bilinear system for low values of the excitation level, but they become greater for high excitation levels. This tendencies may be attributed to the stiffness degrading of Clough's hysteretic characteristics.

The results obtained by the corresponding digital simulation are also plotted in Figs. 4-8. They are compared with the analytical results to show satisfactorily good agreement except for the cases of negative rigidity ratio.

CONCLUSIONS

The differential formulation of the hysteretic characteristics such as bilinear, Kato-Akiyama's and Clough's hysteretic models, is presented. The differential representation for the constituent elements of seismic excitation is also presented. From these formulations, an analytical method based on the stochastic differential equations is presented for computing the transient responses of the hysteretic systems under random excitations. The results obtained by the analytical method are compared with the corresponding digital simulation estimates. It is indicated that the proposed method is efficient in predicting the stochastic response properties of the hysteretic systems with degrading or stiffening characteristics. The numerical examples given herein are hysteretic systems subjected to the stationary filtered white noise or Gaussian white noise process. The present analytical method can be applied for evaluating the responses of the hysteretic systems under nonstationary excitations, if time-variant multipliers and coefficients of filter system are given depending on the nonstationarity of the intensity and spectral characteristics of excitation.

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