OPTIMAL ALGORITHMS FOR CALCULATING THE RESPONSE OF LINEAR OSCILLATORS TO DIGITIZED GROUND ACCELERATIONS

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SUMMARY

A new family of optimal computer algorithms is presented for calculating the response (relative displacement, velocity or acceleration) of a single-degree-of-freedom linear oscillator subjected to a ground acceleration history. Each algorithm is optimal within a class of linear recursive algorithms containing a fixed number of parameters in the sense that its transfer function gives the least-squares match of the oscillator transfer function over an appropriate frequency band. The choice of which optimal algorithm to use for a particular response quantity generally involves a trade-off between accuracy and computation time.

INTRODUCTION

An important problem in earthquake engineering is to calculate efficiently and accurately the response of a single-degree-of-freedom linear oscillator subjected to a ground acceleration history which is available in digitized form. Computer algorithms to solve this problem may be used, for example, to calculate response spectra during routine processing of strong-motion accelerograms. The algorithms may also be used to compute the earthquake response of structures modelled as multi-degree-of-freedom linear systems with classical uncoupled modes.

The development of appropriate algorithms has attracted much interest over the last two decades. Initially, general methods for numerical integration of ordinary differential equations, such as the third-order Runge-Kutta scheme, were adapted to the problem. Later, more efficient techniques were developed which exploited the linearity of the equation of motion of the oscillator (e.g., Refs. 1-3). Some of these authors have utilized the fact that an algorithm to calculate the response of a linear oscillator may be viewed as a digital filter, and they have therefore used methods from filter theory to design a filter with appropriate frequency characteristics.

In this paper, a general class of recursive algorithms is considered which encompasses most of previously published algorithms. Two criteria concerning the accuracy of the algorithms are applied to select the optimum one with a specified number of parameters. Accurate algorithms are presented for the displacement, velocity and acceleration of an oscillator which involve only three, four and five multiplications per time-step respectively. More accurate algorithms are possible, although generally at the expense of increased computation time.

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THE ALGORITHMS

The problem is to calculate the solution at discrete times $t_n = n\Delta t$, $n = 0,1,\ldots, N$, of the equation of motion:

$$\ddot{x} + 2h\omega_0\dot{x} + \omega_0^2 x = -\ddot{z}(t) \tag{1}$$

where $x(0) = x_0$, $\dot{x}(0) = v_0$ are specified, the ground motion $\ddot{z}$ is known at each t_n, and the natural frequency ω_0 and damping factor h of the oscillator are specified.

Let y_n denote an approximation to a response quantity y at time t_n, which may be relative displacement x, velocity $\dot{x}$, or acceleration $\ddot{x}$, of the oscillator. The following general class of linear recursive algorithms is considered for calculating the sequence $\{y_n\}$:

$$y_n = \sum_{k=1}^{K} b_k y_{n-k} + \sum_{i=0}^{I} c_i \ddot{z}_{n-i} \tag{2}$$

The novel feature of the approach presented herein is that the b_k and c_i are chosen in such a way that there is an optimal match of the frequency characteristics of the algorithm and the oscillator, thereby producing algorithms which are in a certain sense the most accurate within the class defined by (2). These optimal algorithms prove to be distinct from any of the previously published algorithms known to the authors.

Selection of Parameters

The parameters in equation (2) are selected by imposing two conditions. The first is that the transient part of the solution of (1) due to the initial conditions is determined exactly, ignoring the finite precision of computer arithmetic. The second condition is that the transfer function of the algorithm optimally matches the transfer function of the oscillator, in a least-squares sense, over some frequency band.

The first condition means that the general free vibration solutions of the algorithm and the oscillator at times $t_n = n\Delta t$ must be equal. This is achieved by taking K = 2 and:

$$\left.\begin{aligned} b_1 &= 2\exp(-h\Omega_0)\cos\left[(1-h^2)^{\frac{1}{2}}\omega_0\right] \\ b_2 &= -\exp(-2h\Omega_0) \end{aligned}\right\} \tag{3}$$

where $\Omega_0 = \omega_0\Delta t$. With this choice, the independent solutions of the homogeneous form of equation (1), $\exp(\lambda t)$ and $\exp(\bar{\lambda} t)$ with $\lambda = -h\omega_0 + j(1-h^2)^{\frac{1}{2}}\omega_0$, are also for $t = n\Delta t$ independent solutions of the homogeneous form of equation (2). Another interpretation of this choice of b_1 and b_2 is that it ensures that the poles of the transfer function of the oscillator are also poles of the transfer function of the algorithm.

The second condition above is now imposed to determine the values of c_i, $i = 0,\ldots,I$ as those which minimize the measure-of-fit:

$$J(c_0,\ldots,c_I) = \sum_{m=0}^{M} \left|H_y(\Omega_m) - H_y^*(\Omega_m)\right|^2 \Big/ \sum_{m=0}^{M} \left|H_y(\Omega_m)\right|^2 \tag{4}$$

where $H_y(\Omega)$ is the exact transfer function for the response y and $H_y^*(\Omega)$ is the

transfer function of the algorithm, and J has been normalized to facilitate comparison of different cases. Notice that the complex-valued transfer functions are matched, so both the amplitude and phase characteristics are considered. In (4):

$$\left.\begin{aligned} H_y(\Omega) &= -(\Delta t)^2 / A(\Omega) && \text{if } y=x \\ &= -\Delta t j\Omega / A(\Omega) && \text{if } y=\dot{x} \\ &= \Omega^2 / A(\Omega) && \text{if } y=\ddot{x} \end{aligned}\right\} \tag{5}$$

and

$$H_y^*(\Omega) = \sum_{i=0}^{I} c_i \exp(-ji\Omega) / B(\Omega) \tag{6}$$

where

$$A(\Omega) = \Omega_0^2 - \Omega^2 + j2h\Omega\Omega_0 \tag{7}$$

and

$$B(\Omega) = 1 - b_1 \exp(-j\Omega) - b_2 \exp(-j2\Omega) \tag{8}$$

The nondimensional frequency $\Omega_m = \frac{m\pi}{M}$ in equation (4) ranges over a corresponding frequency band from DC to the Nyquist frequency in equal steps. If M is sufficiently large, the minimizing values become insensitive to the actual value of M and the results correspond to using integrals in equation (4). We have taken M = 200. The frequency band of 0 to π for $\Omega = 2\pi f\Delta t$, where f is the excitation frequency, is consistent with the result from sampling theory that the discrete values $\{\ddot{z}(n\Delta t)\}$ can only represent correctly the frequency content of $\ddot{z}(t)$ over the band-width from DC to the Nyquist frequency. There is therefore no point in attempting to force the algorithm to transform correctly those frequencies outside this band. Anyway, this is prevented by the periodicity of $H_y^*(\Omega)$.

The minimization of J may be performed analytically for one or two parameters. For more parameters, it is easier to determine numerically the minimizing values of the c_i by using Gaussian elimination to solve the linear system of equations which arises from the condition for the minimum of J. For either case, a FORTRAN subroutine has been written to calculate the c_i for a specified ω_0, h, Δt and I. In addition, some of the c_i, $i < I$, may be forced to be zero. For example, the optimal algorithms with only c_I nonzero were studied for various values of I and it was found that the best displacement algorithm involving only one c_i is the one optimized with respect to c_1.

Accuracy of the Algorithms

All algorithms with $I \leq 3$ were studied, including setting some of the c_i, $i < I$, to be exactly zero. Thus, the optimal algorithms were determined within each of the 7 families given by $(c_0,c_1,c_2) = (c_0,0,0)$, $(0,c_1,0)$, $(0,0,c_2)$, $(c_0,c_1,0)$, $(c_0,0,c_2)$, $(0,c_1,c_2)$, and (c_0,c_1,c_2) in full form.

The accuracy of the algorithms as measured by their match of the amplitude and phase of the oscillator transfer function deteriorates as the oscillator frequency approaches the Nyquist frequency, that is, as the number of time-steps per period decreases. For 10 time-steps per period, an accurate displacement algorithm is obtained which has only one c_i coefficient. For the same number of time-steps per period, accurate algorithms for velocity and acceleration require two and three coefficients respectively. These algorithms for displacement, velocity and acceleration have the following forms:

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + c_1 \ddot{z}_{n-1} \tag{9}$$

$$v_n = b_1 v_{n-1} + b_2 v_{n-2} + c_0^* \ddot{z}_n + c_2^* \ddot{z}_{n-2} \tag{10}$$

$$a_n = b_1 a_{n-1} + b_2 a_{n-2} + c_0' \ddot{z}_n + c_1' \ddot{z}_{n-1} + c_2' \ddot{z}_{n-2} \tag{11}$$

The good accuracy of these algorithms in the frequency domain is exhibited in Figs. 1,2. The transfer function for the acceleration algorithm (Fig.2a) is for the case of 5 time-steps per period. For 10 time-steps per period the plots of the oscillator and algorithmic transfer functions are indistinguishable.

The optimal algorithms of the form of equations (9) to (11) were observed empirically to have a time-domain error of order $(\Delta t)^2$ for forced vibrations where the solutions of (1) were available in analytical form. For free vibrations, the computed solutions are exact, except for round-off error due to the finite precision of the arithmetic operations in the computer, and also due to small errors, discussed later, in starting the algorithms when x_0 and v_0 are specified.

The rate of convergence, $(\Delta t)^n$, of the error is often used to compare the accuracy of time-stepping algorithms. This has the advantage of applying to linear and nonlinear differential equations, but it gives no indication of the relative accuracy of different algorithms which have the same rate of convergence. For linear systems, it is much more informative to compare how the amplitude and phase of each frequency component in the excitation is changed by the oscillator and by the algorithm.

On the basis of their transfer functions, Preumont (Ref. 4) has compared the accuracy for the solution of (1) of some important algorithms within the Newmark β family (Ref. 5), including the central difference, trapezoidal and Fox-Goodwin algorithms. On the same basis, he also compared the Nigam-Jennings (NJ) algorithm (Ref. 1), which is derived by assuming the excitation is linear between sampling times. The NJ algorithm, which is the current standard in the U.S. for computing response spectra of strong-motion records, proved to be the most accurate of those studied by Preumont.

The Newmark β family and the NJ algorithm have the coupled form:

$$\left.\begin{aligned} x_n &= a_{11} x_{n-1} + a_{12} v_{n-1} + b_{11} \ddot{z}_n + b_{12} \ddot{z}_{n-1} \\ v_n &= a_{21} x_{n-1} + a_{22} v_{n-1} + b_{21} \ddot{z}_n + b_{22} \ddot{z}_{n-1} \end{aligned}\right\} \tag{12}$$

However, the corresponding displacement and velocity transfer functions still have the form given in equations (6) and (8). From this fact, it is possible to derive equivalent algorithms which consist of one recursive equation for displacements and another for velocities. These uncoupled equations each have three c_i coefficients as in the acceleration algorithm (Eq. 11), except that the values of the coefficients are different. The equivalent NJ algorithms do have the same coefficients b_1 and b_2 (Eq. 3), as expected, since the basic assumption of piecewise linearity of the excitation is exact for free vibrations.

By the definition of algorithmic accuracy in terms of transfer functions, the optimal algorithms for displacement and velocity involving three coefficients c_0, c_1 and c_2 must therefore be more accurate than the NJ algorithm, which in turn is better than the Newmark β family. It turns out that the

algorithms in Eqs. (9) and (10), which combined involve only 7 multiplications per time-step, are also more accurate than the NJ algorithm which involves 8 multiplications per time-step. Their match of the amplitude of the oscillator transfer functions is superior, although the phase match is slightly poorer than that of the NJ algorithm. The excellent phase behavior of the latter is because linear interpolation of the ground motion does not alter its phase, and so the subsequent exact solution of Eq. (1) with the piecewise linear history for $\ddot{z}$ must reproduce the correct phase behavior in the response. It is only the final sampling of the response at every Δt that produces some phase error because of aliasing.

The optimal displacement algorithm with the form of Eq. (9) was also compared with one due to Ehrenberg and Hernandez (Ref. 3). The EH algorithm is derived as a covariance-invariant digital filter, which is a filter which closely reproduces the amplitude of any desired transfer function. It has the form:

$$x_n = b_1 x_{n-1} + b_2 x_{n-2} + \hat{c}_0 \ddot{z}_n + \hat{c}_1 \ddot{z}_{n-1} \qquad (13)$$

This is actually a modified form of the equation in their paper which uses $\ddot{z}_{n-1}$ and $\ddot{z}_{n-2}$ in the forcing terms. The original algorithm produces a large phase error. Changing the subscripts as in Eq. (13), but still using the original coefficients, removes a linear trend from the phase distortion without altering the amplitude behavior of the algorithm. This produces a modified algorithm which has a smaller phase error, but it is still worse than that of the optimal algorithm, Eq. (9). The amplitude of the transfer function of the EH algorithm is better, however, than that of Eq. (9). In effect, the optimal algorithm produces a very acceptable compromise between the two extremes of the NJ algorithm, on one hand, which has a smaller phase error but poorer amplitude error, and the EH algorithm on the other hand, which has a smaller amplitude error but poorer phase error.

As a final comment on the subject of algorithmic accuracy, observe that if the ground motion was truly linear between sampling times, then the NJ algorithm would give exact solutions of (1). In terms of a frequency-domain interpretation, the frequency components in the sampled ground acceleration from DC to the Nyquist frequency would be "aliased, or corrupted, by the higher frequencies in the piecewise linear acceleration in such a way to compensate exactly for the deficiencies in the frequency characteristics of the algorithm. In general, the variation of the ground acceleration between sampling times is unknown, but it certainly will not be exactly linear. It seems safer, then, to judge the accuracy of an algorithm in terms of its transfer function rather than on the basis of assumptions about the behavior of the excitation.

Numerical Efficiency

The numerical efficiency of an algorithm may be assessed on the basis of the number of multiplications required to solve the response over one period of the oscillator, if, for convenience, the time-step Δt is chosen so that an integral multiple of it equals the period. Additions may be ignored since the computer processing time for these operations is normally much smaller than for multiplications.

The numerical efficiency can be increased by using fewer time-steps per period, but obviously at the expense of accuracy in general. It is possible, however, to get increased efficiency without a loss of accuracy. For example,

the optimal displacement algorithm with 3 nonzero c_i coefficients has essentially the same accuracy with 5 time-steps per period as Eq.(9) with 10 time-steps per period (compare Fig. 1a and Fig. 2b). Thus, although the number of multiplications per time-step is increased by a factor of 5/3 over that in Eq. (9), the number of time-steps per period can be reduced by 1/2, leading to fewer arithmetic operations per period without loss of accuracy.

Usually the time-step cannot easily be varied since it is fixed by the sample interval of the available ground acceleration data. Numerical efficiency then depends on the number of multiplications per time-step, so, for example, Eq. (9) would have the best efficiency for any displacement algorithm.

Starting the Algorithms

Algorithms in the form of Eqs. (9) to (11) are not self-starting since Eq. (1) must be solved subject to the specified initial conditions, x_0 and v_0. Eqs. (9) and (10) require x_0, x_1 and v_0, v_1 respectively, while Eq. (11) requires x_0, v_0, x_1 and v_1 in order to calculate a_0 and a_1 from (1).

One solution to this problem is to calculate x_1 and v_1 by employing a self-starting algorithm, such as the trapezoidal one, whose coefficients can be evaluated efficiently. After the first time-step, a switch is made to the appropriate optimal algorithm. Another possible solution, yet to be investigated, is to determine an optimal self-starting algorithm in the form of Eqs. (12) where the a_{ij} would be chosen to give the exact free-vibration solution and the b_{ij} would be chosen by a weighted least-squares matching of both the displacement and velocity transfer functions.

Calculation of Maximum Response

Accurate computation of the oscillator response at discrete times $t_n = n\Delta t$ does not guarantee accurate determination of the true (continuous-time) peak response, since this peak will usually occur within one of the time intervals of length Δt. The maximum of the calculated discrete response will be within 5% of the true peak if at least 10 time-steps per period are used (Ref. 6). To take advantage of those algorithms, such as those corresponding to Fig. 2, which give good accuracy at times t_n using fewer time-steps than 10, the discrete solution could be recalculated with smaller Δt in the vicinity of the discrete maximum determined using the original Δt.

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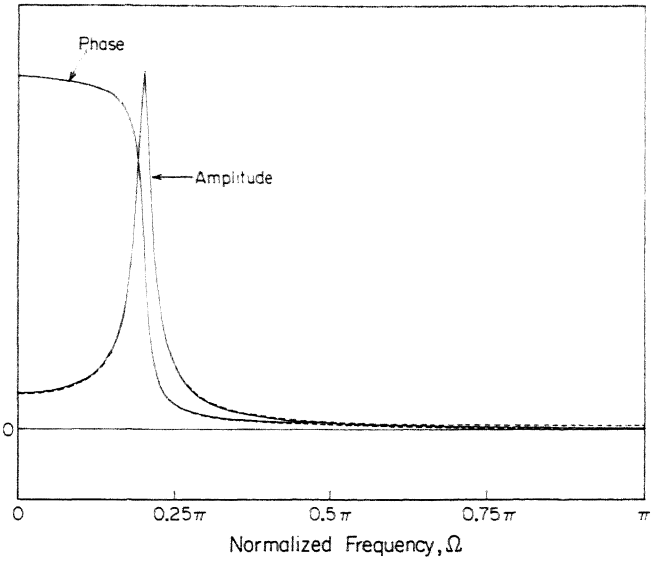
(a)



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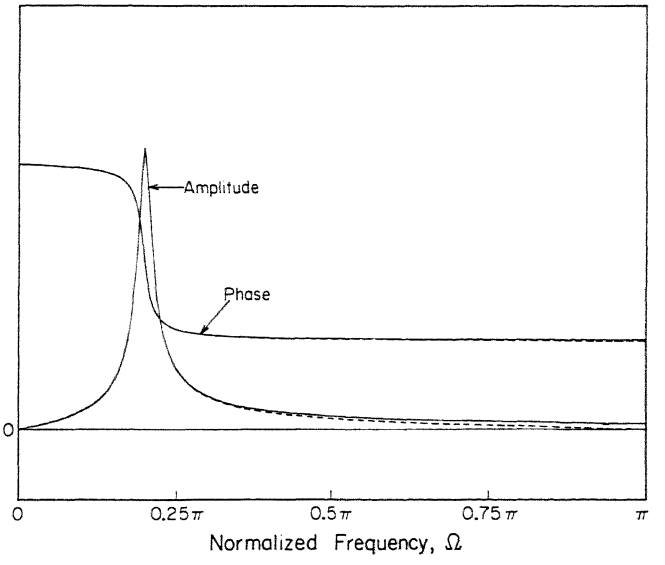
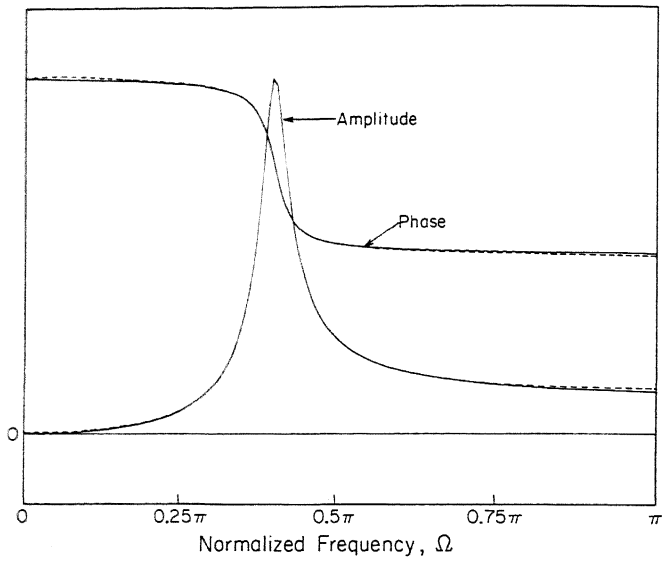


Fig. 1 Comparison of the oscillator transfer function (——) and that of the optimal algorithm (---) with the form of:

(a) Eq. 9 for displacement,
(b) Eq. 10 for velocity.

The oscillator frequency is 0.2 of the Nyquist frequency and its damping is 5%.

(a)



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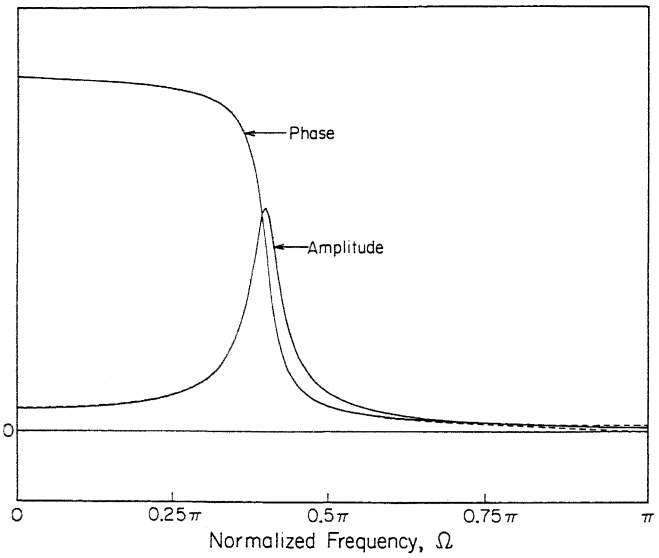

Fig. 2 Comparison of the oscillator transfer function (——) and that of the optimal algorithm (---) with the form of Eq. 11:

(a) for acceleration,
(b) for displacement.

The oscillator frequency is 0.4 of the Nyquist frequency and its damping is 5%.