# RIGID BODY MECHANISMS IN STRUCTURAL DYNAMICS

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## SUMMARY

The paper examines certain aspects of rigid body mechanisms in structural dynamics by modelling one precast concrete building system as an assembly of two-dimensional rigid prisms subjected to self-imposed kinematic constraints. The corresponding nonlinear equations of motion have coefficients with step discontinuities, are generally nonautonomous, and in effect comprise $2^N$ sets of N governing equations for an N degree of freedom model. The system is closely related to the rocking block or stacked block problem.

Free and forced vibration solutions are presented for various systems. The systems of equations are poorly conditioned, in that small changes in input and/or geometry can create large changes in system response. However, by examining limiting cases of survival and failure, some apparent trends in system behavior have been discovered.

## INTRODUCTION

The phenomenon of separation, or lift-off between a structure and its base during earthquakes, has been observed to occur for various types of structures. Examples include light building equipment, precast concrete panels, minarets, nuclear shielding blocks and tombstones. There also exists at least one precast concrete building system, employing slender columns with negligible beam-column moment resistance, which would behave essentially as an assembly of constrained rigid bodies under dynamic excitation. A displaced two story model is shown in Figure 1. This structural idealization is based upon conceptualization of a precast concrete building system in which the beam-column joints are weakly connected. Under horizontal excitation and assuming that the mass of the columns (of height h and width b) is small compared to that of the floors, the structure would move as one rigid body until the horizontal ground accelerations exceeded g tan $\alpha$, where $\alpha$ = arctan b/h. Beyond this level of excitation, separation between columns and girders would occur, and rocking of the columns would ensue. In such a model the failure condition is quite clear: collapse occurs if the rotation $\theta$ exceeds the slenderness ratio $\alpha$.

A related problem is the tipping of rigid blocks. Ishiyama (Ref. 1) cites 30 references dating from 1881. Recent authors include Housner (Ref. 2) Aslam et al.

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(Ref. 3), and Yim et al. (Ref. 4). Results for rigid bodies on flexible foundations have been presented by Psycharis and Jennings (Ref. 5) and Yim and Chopra (Ref. 6). Stacked blocks forming the configuration for nuclear fuel assemblies have been investigated by Lee (Ref. 7) and Ikoshima and Nakazawa (Ref. 8).

In this paper we treat a problem that helps in understanding some new aspects of rigid body behavior in structural dynamics. Specifically, we will show that, although the structural response can often be poorly conditioned, some meaningful deterministic trends exist.

## PROBLEM FORMULATION

### Single DOF Structure

This section examines the limiting case of a single story structure with one generalized coordinate. This problem has been studied by a number of researchers including Housner (Ref. 2), Yim, et al (Ref. 4), Aslam et al (Ref. 3) and Ishiyama (Ref. 1). The equations of motion are well-known and are written compactly here:

$$\ddot{\theta} - p^2 \sin(\theta - \alpha \operatorname{sgn} \theta) = -p^2 \gamma(t) \cos(\theta + \alpha \operatorname{sgn} \theta) \tag{1}$$

where $p^2 = g/R$, $R^2 = h^2 + b^2$, $\ddot{x}_g$ = horizontal ground acceleration, $\gamma(t) = \ddot{x}_g/g$, and $\operatorname{sgn}(\theta) = 2H(\theta)-1$, where $H(\theta)$ is the Heavyside step function. Note that Eq. 1 actually represents two distinct equations: one for $\theta < 0$ and another for $\theta > 0$. However, for $\theta = 0$, the equations do not govern. The rotation and velocity will remain zero unless $|\gamma(t)| > \tan \alpha$. Once the threshold acceleration is reached, the sign of $\theta$ will be opposite to that of the acceleration pulse and Eq. 1 governs. Herein energy loss at impact will be taken into account by considering a drop of the impact velocity. This approach has been shown by others (Refs. 1-4) to lead to satisfactory results.

It is instructive to examine the condition of zero external force and small angle approximations.[1] By letting $\cos\theta \simeq 1$ and $\sin\theta \simeq \theta$, Eq. 1 becomes:

$$\ddot{\theta} - p^2\theta = -\alpha p^2 \operatorname{sgn} \theta \tag{2}$$

where $\alpha = b/h$. Note that Eq. 2 is still nonlinear in that the sign of the right hand changes with the sign of $\theta$. More significantly, the coefficient of the $\theta$ term is negative, yielding a solution of the form

$$\theta(t) = A_1 e^{-pt} + A_2 e^{pt} + \alpha \operatorname{sgn} \theta \tag{3}$$

This solution is aperiodic. However, if the structure is released from an initial rotation (smaller in magnitude than $\alpha$) it will return to zero with some velocity, and subsequent solution for $\theta$ will demonstrate an alternating rotation history. A periodic

[1] It can be shown, in fact, that $\alpha$ can be made arbitrarily small by scaling it to the peak acceleration of the forcing function.

rocking is clearly predicted, but not by an inherent periodicity in the general solution. Housner (Ref. 2) showed, in fact, that the "period" is highly amplitude dependent, with the frequency approaching infinity with smaller vibrations.

## Multi-DOF Structure

In an N-story structure the equations of motion can be written using Lagrange's equations on the appropriate energy quantities. The total potential energy can be obtained with little difficulty. The kinetic energy, however, is a rather complex function of $\{\theta\}$ and $\{\dot{\theta}\}$. Cox (Ref. 9) derived the appropriate functions; the corresponding equations of motion in matrix form are as follows:

$$[A(\theta)]\ \{\ddot{\theta}\} + [B(\theta)]\ \{\dot{\theta}^2\} + p^2[C(\theta)] = -p^2\gamma_o\ f(t)[D(\theta)] \tag{4}$$

where $\gamma_o$ = maximum horizontal ground acceleration and f(t) is a prescribed base acceleration. For each i and j in the range of 1 to N:

$$A_{ii} = M_i$$
$$A_{ij} = A_{ji} = M_i \cos S_{ij}$$
$$B_{ii} = 0$$
$$B_{ij} = -B_{ji} = M_i \sin S_{ij}$$
$$C_i = M_i\ [\sin(\alpha \text{ sgn } \theta_i - \theta_i)]$$
$$D_i = M_i\ [\cos(\alpha \text{ sgn } \theta_i - \theta_i)]$$
$$M_i = N + 1 - i$$
$$S_{ij} = (\theta_j - \theta_i) + \alpha[\text{sgn}\theta_i - \text{sgn } \theta_j]$$

There are $2^N$ different sets of nonlinear equations, one set for every permutation in sign of $\{\theta\}$. As in Eq. 1, Eqs. 4 apply only when all $\theta_i$ are non-zero. If an individual story is initially closed, it will remain in that condition until inertial overturning moments exceed the restoring moments.

In this paper we study in detail the response of the two-story structure shown in Fig. 1. In order to gain some physical insight into the problem, we analyze the free vibrations of the system before proceeding to study its response to earthquake excitation. Assuming that rotations are small, the equations of motion for free vibrations are:

$$2\ddot{\theta}_1 + \ddot{\theta}_2 - 2p^2\theta_1 = -2\alpha p^2\ \text{sgn}\theta_1 \tag{5}$$
$$\ddot{\theta}_1 + \ddot{\theta}_2 - p^2\theta_2 = -\alpha p^2\ \text{sgn}\theta_2$$

Eqs. 5 represent four different systems of equations; one system for each permutation in sgn of $\theta_1$, $\theta_2$. Even though the complete system is nonlinear in $\theta_1$ and $\theta_2$ because of the presence of the sign function, each equation in (5) is linear for a given sign permutation of $\theta_1$, $\theta_2$. Accordingly, the solution may be written as a succession of terms of the form:

$$\{\theta(t)\} = [G]\ \{e^{rt}\} + \left\{ \begin{matrix} \alpha \operatorname{sgn} \theta_1 \\ \alpha \operatorname{sgn} \theta_2 \end{matrix} \right\} \qquad (6)$$

in which [G] is a two by four matrix of constant coefficients determined from the initial conditions and $\{e^{r_i}\}$ denotes a column vector with elements $e^{\pm r_i t}$ where $r_i$, i = 1,2, are the roots of the frequency equation.

In constructing the complete solution to the free vibration problem, the rotations and velocities at the end of a cycle corresponding to a change in toe position are used as initial conditions for the next cycle. These velocities also define the upcoming sign of $\theta_1$ and $\theta_2$ and, therefore, the appropriate set of equations to be used. A similar approach is used for analyzing the forced motion of the system under seismic excitation. Since the equations of motion under ground excitation are taken to be piecewise linear, it is possible, in general, to solve them exactly at each step for a given ground acceleration, $\ddot{x}_g$. In this study, we assume that $\ddot{x}_g$ is piecewise linear.

One could also attempt to solve the non-linear equation (Eq. 4) directly using numerical methods. Success is achieved by explicitly writing the accelerations as functions of the lower order terms by inverting $[A(\theta)]$ functionally, thus employing standard numerical techniques for first order differential equations and avoiding any approximation of the second derivative. This is possible because matrix $[A(\theta)]$ in Eq. 4 is positive-definite and hence invertible.

## PRELIMINARY RESPONSE OBSERVATIONS

Equations 3 and 6 show that the solution to the linearized problem is exponential in nature. This provides a theoretical explanation for the poorly conditioned behavior observed in numerical examples (Refs. 3,4,9,10,11) in which a small change in an input variable produces a large change in the resulting response.

A typical time history obtained using Eq. 6 is shown in Fig. 2. The structure, with $\alpha$ = 0.1, p = 1.79 rad/sec, is subjected to initial conditions $\theta_1$ = 0.055 radian while $\theta_2$ = 0.002 radian. At 0.76 seconds the first story goes through its first impact, and from that time on the second story displays much greater motion than the first. This is a whipping effect, and at 3.6 seconds $\theta_2$ reaches 0.0988 radians while $\theta_1$ is less than 0.005 radians. As it happens, the structure in Fig. 2 is very close to failure. While increasing the initial rotation may cause failure, it is also possible that failure could be reached by decreasing the starting rotation,or by changing $\alpha$ only infinitesimally.

The unavoidable effects of poor conditioning can become dramatic in forced 2DOF systems. For instance, one particular structure was subjected to the EW component of the Taft accelerogram (13 July, 1952) and survived in the absence of energy loss. When re-analyzed with damping it failed. The presence of damping changed the timing of the toe impacts, and created (at a certain point in the record) a set of initial conditions which were susceptible to the excitation at that instant. Comparable time histories of the second story are shown in Fig. 3.

It must be pointed out that these conditions result from the physical behavior of the system and not from any modelling inaccuracies or solution errors. A more "precise" analysis, e.g., numerical integration of (Eqs. 4), has little significance, because it suffers from poor conditioning as well. For an elastic (or even inelastic) structure an insightful "solution" also provides information on the response of similar but non-identical structures and loadings. That is not the case for the system under study; thus we seek alternative ways of representing whatever "information" we may extract.

## FAILURE ANALYSES

In view of the poorly conditioned nature of the system considered, it seems appropriate to establish only whether or not a system will fail under a given excitation. Figure 4 depicts three limiting conditions for a single story structure subjected to scaled records of the Taft earthquake. Frequency[2] is represented on the abscissa and maximum forcing intensity, normalized by $\alpha$, on the ordinate. One limiting condition is simply "no motion" in which the maximum ground acceleration does not exceed $\alpha g$. Next, two "failure" conditions were obtained. For any point on the abscissa, analyses of the time response were performed going up the ordinate axis. The "possible failure" curve is reached when failure is first detected. As the intensity is increased, there is a mix of failure and survival experiences until an apparent regime of failure only is detected. This defines the "certain failure" curve beyond which failure always occurs.

The appearance of an indeterminate zone reflects the poorly conditioned nature of the response. In practical terms the "possible failure" curve must represent the failure condition upon which design guidelines are to be drafted. Clearly, the location of the limiting curves is a function of the particular excitation.

Figure 5 is a comparable "failure" plot for a two story structure. As might be expected, it also reflects the poorly conditioned response. The plot shows the two story structure to be "weaker" than the single story structure (failure with smaller intensity). This "weakness" is due to the whipping effect, i.e., the apparent concentration of energy in the second story.

## CONCLUSIONS

The rigid body system considered herein is characterized by its poorly conditioned nature. Single time history analyses in themselves have little meaning, because one observed response has almost no value in predicting any other response. However, there are regions in which safety and failure are certain, as demonstrated by the limiting curves. The absolute safe condition is one in which the energy entering the system can be limited to less than that needed to overturn the top story. Work is currently underway to evaluate bounding principles based on that limit.

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[2] In a strict sense p is only proportional to the pseudo natural frequency.

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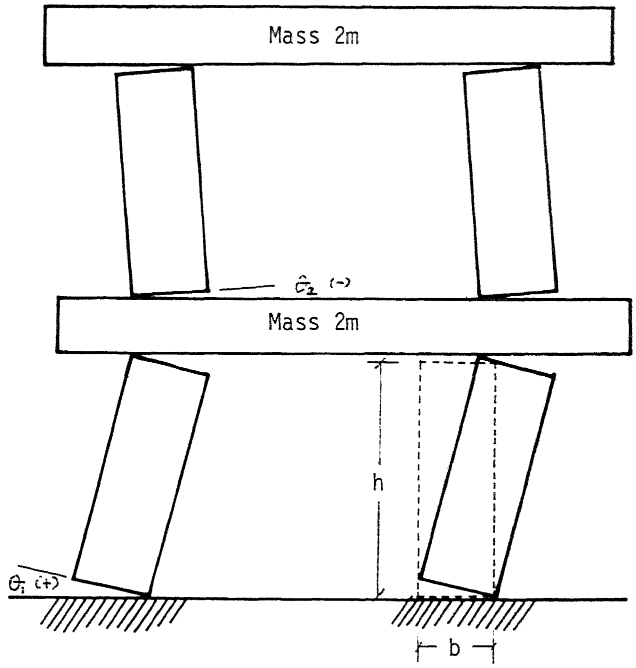


Figure 1. Two-Story Structure

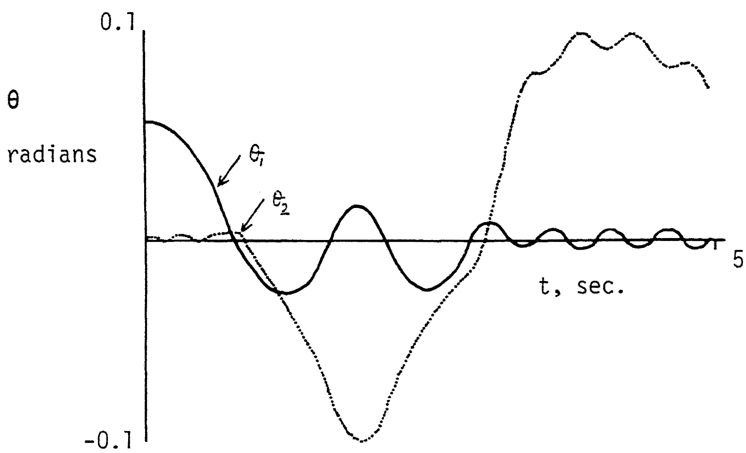


Figure 2. Time History, Free Vibration

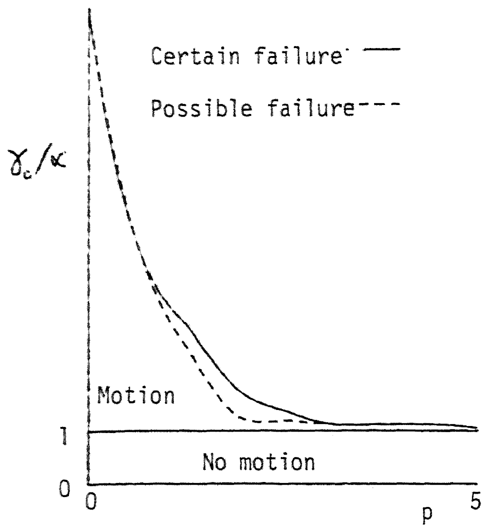


Figure 3. Failure Plot, One DOF

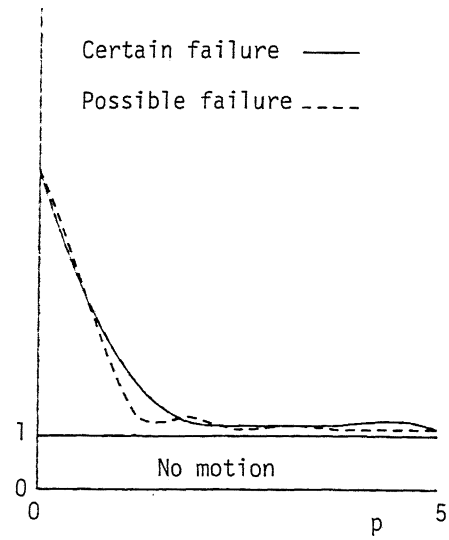


Figure 4. Failure Plot, Two DOF

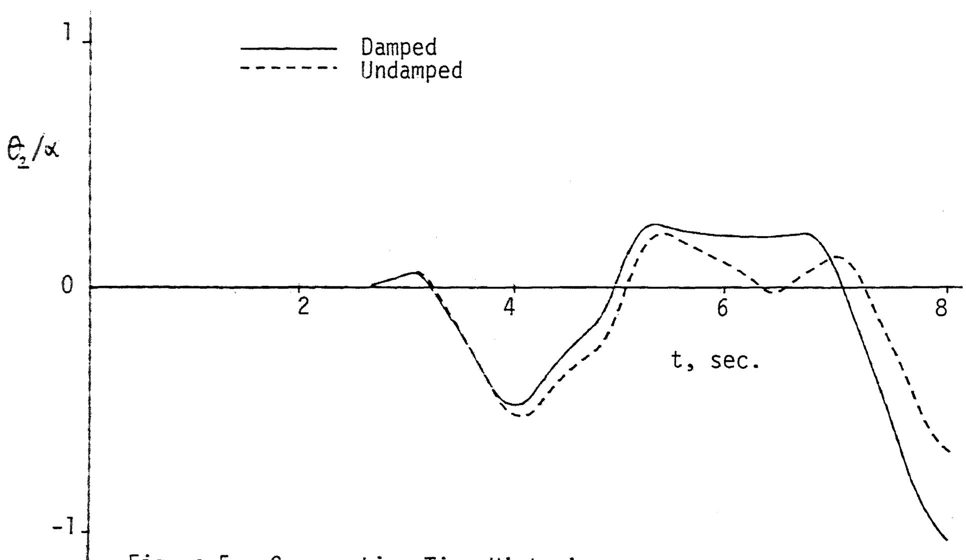


Figure 5. Comparative Time Histories