EXISTENCE OF CENTERS OF RESISTANCE AND TORSIONAL UNCOUPLING OF EARTHQUAKE RESPONSE OF BUILDINGS

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SUMMARY

Seismic design codes define static eccentricity as the distance between the center of mass of a story and a point on its plan called the center of resistance. This eccentricity is regarded as a measure of the torsional tendency of the structure. This paper shows, however, that the existence of centers of resistance as origins of eccentricity is restricted to a particular class of structures; for a general multistory building such concepts are phsyically meaningless.

INTRODUCTION

Current seismic design codes, such as ATC, require three-dimensional analysis for buildings classified as irregular. Classification criteria are typically based on the concept of static eccentricity, i.e., the distance between the center of mass of a story and a point on its plan called the center of resistance. Furthermore, even if a three-dimensional analysis is performed, some codes prescribe a fifty per cent increase of the static eccentricity to account for dynamic amplification. Obviously, these specifications reflect the implicit belief that a building can be induced to vibrate in a purely translational motion if the story mass is shifted to compensate for the torsional tendency of the stiffness layout.

The calculation of the center of resistance in a one-story structure is quite straightforward. The corresponding static eccentricity is a property of the structure that has a clear physical meaning, namely that for zero eccentricity the structure vibrates without torsion when excited by a translational earthquake motion. However, there does not appear to be any generally accepted extension of the concept of center of resistance to multistory buildings, except for cases where symmetry makes the problem trivial. In fact, seismic design codes require the determination of the coordinates of the centers of resistance of all stories to account for torsional effects, but very few of them give a definition of such centers or specify a procedure to compute their positions. Moreover, the most commonly used definitions, or interpretations, are inconsistent and do not necessarily relate to torsion free response (Ref. 1).

The purpose of this paper is to discuss the physical meaningfulness of the concept of center of resistance in multistory buildings, and to identify the class of structures for which it is relevant.

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THE PSEUDO-TRIDIMENSIONAL MODEL

Most buildings are structured as a set of mutually perpendicular resisting planes, connected at each story level by floor diaphragms sufficiently rigid so as to be regarded as undeformable. A suitable model for such buildings considers the resisting planes as macro-elements connected to three-degree of freedom nodes, i.e., the rigid diaphragms. For dynamic analysis, it is desirable that the diaphragm degrees of freedom chosen be the two components of displacement of its center of mass, u_i and v_i, and its rotation θ_i. Assembling the lateral stiffnesses of the component resisting planes accordingly, the following stiffness equation is obtained

$$\begin{vmatrix} \{U\} \\ \{V\} \\ \{\theta\} \end{vmatrix} = \begin{vmatrix} [K_{xx}] & [K_{xy}] & [K_{x\theta}] \\ [K_{xy}]^t & [K_{yy}] & [K_{y\theta}] \\ [K_{x\theta}]^t & [K_{y\theta}]^t & [K_{\theta\theta}] \end{vmatrix} \begin{vmatrix} \{u\} \\ \{v\} \\ \{\theta\} \end{vmatrix}$$

where the components of $\{U\}$ and $\{V\}$ are the story forces in the X and Y directions, and the components of $\{\theta\}$ the torsional moments about the center of mass of each floor. The submatrices in this equation can be found to be

$$[K_{xx}] = \sum_j^p [K_{xj}] \qquad [K_{yy}] = \sum_j^q [K_{yj}] \qquad [K_{xy}] = 0$$

$$[K_{x\theta}] = -\sum_j^p [Y_j][K_{xj}] \qquad [K_{y\theta}] = \sum_j^q [X_j][K_{yj}] \qquad (1)$$

$$[K_{\theta\theta}] = \sum_j^p [Y_j][K_{xj}][Y_j] + \sum_j^q [X_j][K_{yj}][X_j]$$

where $[K_{xj}]$ and $[K_{yj}]$ are the lateral stiffness matrices of the p resisting planes in the X direction and of the q resisting planes in the Y direction respectively; $[X_j]$ is a diagonal matrix containing the distances measured from the centers of mass of each story to the j-th Y-direction plane; $[Y_j]$ is a diagonal matrix containing the distances measured from the centers of mass of each story to the j-th X-direction plane.

It is fitting at this point to discuss the limitations of the pseudo-tridimensional model. In the first place, the rigid diaphragm assumption, though realistic in most cases, does not hold in buildings in which one of the plan dimensions is significantly larger than the other, or in which the plan is irregularly shaped. In the second place, strictly speaking, the model is good only if the resisting planes are unconnected, i.e., if no columns belong simultaneously to two resisting planes and if no beam of one plane frames into another plane. In fact, the model does not enforce axial compatibility of orthogonal elements. However, the results given by the model are correct when all planes in each direction are identical and the torsional component of the response is negligible. This is so because under such conditions, the X-direction response does not involve flexural deformation of the Y-direction frames.

For buildings in which the planes in a direction are not all equal, but not substantilly different, the response given by the pseudo-tridimensional model is a reasonable approximation to the real one, provided it involves a low level of torsion. Actually, torsion is the most important source of incompatible axial deformation of columns and torsional coupling of beams, since it involves a non-uniform deformation of the resisting planes in both directions simultaneously. This discussion leads to the conclusion that the pseudo tridimensional model, despite its limitations, is suitable for studying torsional uncoupling and the existence of centers of rigidity.

TORSIONAL UNCOUPLING

The governing equations of motion for a pseudo-tridimensional model of a building under plane seismic excitation can be written as

$$\begin{vmatrix} [M] & 0 & 0 \\ 0 & [M] & 0 \\ 0 & 0 & [J] \end{vmatrix} \begin{vmatrix} \{\ddot{u}\} \\ \{\ddot{v}\} \\ \{\ddot{\theta}\} \end{vmatrix} + \begin{vmatrix} [K_{xx}] & 0 & [K_{x\theta}] \\ 0 & [K_{yy}] & [K_{y\theta}] \\ [K_{x\theta}]^t & [K_{y\theta}]^t & [K_{\theta\theta}] \end{vmatrix} \begin{vmatrix} \{u\} \\ \{v\} \\ \{\theta\} \end{vmatrix} = -\ddot{u}_g \begin{vmatrix} \cos\alpha\,[M]\{1\} \\ \text{sen}\alpha\,[M]\{1\} \\ 0 \end{vmatrix}$$

where $[M]$ and $[J]$ are diagonal matrices containing, respectively, the mass and the centroidal moment of inertia of the stories; α is the angle formed by the excitation plane with respect to the x-axis; and $\{1\}$ represents a column vector of ones.

The structural response will ordinarily involve both displacement and rotation of all stories. Actually, for torsion-free displacement to be possible, the two following formally independent differential equations

$$[M]\{\ddot{u}\} + [K_{xx}]\{u\} = -\ddot{u}_g \cos\alpha\,[M]\{1\}$$
$$[M]\{\ddot{v}\} + [K_{yy}]\{v\} = -\ddot{u}_g \,\text{sen}\alpha\,[M]\{1\} \qquad (2)$$

must have a solution that complies with the coupling linear relationship

$$[K_{x\theta}]^t\{u\} + [K_{y\theta}]^t\{v\} = 0$$

Except for a very particular case, i.e., a structural system in which $[K_{xx}] = [K_{yy}]$ and $[K_{x\theta}] = \beta\,[K_{y\theta}]$ under an excitation in the direction $\alpha = \tan^{-1}(-\beta)$, the conditions are met only if

$$[K_{x\theta}] = [K_{y\theta}] = 0 \qquad (3)$$

so that the coupling equation vanishes altogether.

Certainly, this is also a particular case of structural stiffness layout, but it corresponds to a type of building that will have a rotation-free response for any excitation that does not involve torsion. Furthermore, since the matrices $[K_{x\theta}]$ and $[K_{y\theta}]$ are dependent on the positions of the centers of mass of the stories, it might be possible that Eq. 3 could be satisfied for any structure if suitable centers of mass were specified, i.e., it might be possible to define a mass distribution layout that would "comprensate" the torsional tendencies of the stiffness layout.

A change of position of the centers of mass, within the context of the assemblage of the stiffness matrix, can be regarded as the coordinate transformation

$$\{u\} = \{\bar{u}\} + [\eta]\{\theta\} \qquad \{v\} = \{\bar{v}\} - [\zeta]\{\theta\} \tag{4}$$

in which $[\eta]$ and $[\zeta]$ are diagonal matrices that contain as elements the differences

$$\zeta_i = \bar{x}_i^* - x_i^* \qquad \eta_i = \bar{y}_i^* - y_i^* \tag{5}$$

of coordinates between the new center of mass, $(\bar{x}_i^*, \bar{y}_i^*)$, and the corresponding old one, (x_i^*, y_i^*). The coordinates of the centers of mass of all stories are referred to a common origin that will be referenced under that name in what follows.

The mass matrix is formally invariant under this transformation. Three of the component submatrices of the stiffness matrix, namely, $[K_{xx}]$, $[K_{yy}]$ and $[K_{xy}]$ do not change, while the remaining three submatrices are modified as follows

$$[\bar{K}_{x\theta}] = [K_{x\theta}] + [\eta][K_{xx}] \qquad [\bar{K}_{y\theta}] = [K_{y\theta}] - [\zeta][K_{yy}]$$

$$[\bar{K}_{\theta\theta}] = [K_{\theta\theta}] + [\eta][K_{xx}][\eta] + [\zeta][K_{yy}][\zeta] + [\eta][K_{x\theta}] + [K_{x\theta}]^t[\eta]$$

$$- [\zeta][K_{y\theta}] - [K_{y\theta}]^t[\zeta]$$

Imposing the condition that both transformed coupling matrices $[\bar{K}_{x\theta}]$ and $[\bar{K}_{y\theta}]$ be equal to zero, leads to the expressions

$$[\eta] = -[K_{x\theta}][K_{xx}]^{-1} \qquad [\zeta] = [K_{y\theta}][K_{yy}]^{-1} \tag{6}$$

that would render the solution of the problem provided the matrix multiplications involved actually produce diagonal matrices as results. In a general structure this does not occur, so that inertial "compensation" of the torsional tendency of the stiffness layout is possible in a particular class of structures only.

In order to study the characteristics of the "compensable" class of structures, it is convenient to examine the expressions given by Eqs. 1. The matrix $[Y_j]$ that appears in the definition of $[K_{x\theta}]$ can be written as

$$[Y_j] = \tilde{y}_j [1] - [Y^*]$$

in which $\tilde{y}_j$ is the coordinate of the j-th X-direction plane with respect to the common origin; $[1]$ is the unit matrix; and $[Y^*]$ is a diagonal matrix containing the coordinates y_i^*. Hence, $[K_{x\theta}]$ can be written as

$$[K_{x\theta}] = -\sum_j^p \tilde{y}_j [K_{xj}] + [Y^*] \sum_j^p [K_{xj}]$$

or in the form

$$[K_{x\theta}] = -[\tilde{K}_{x\theta}] + [Y^*][K_{xx}]$$

where

$$[\tilde{K}_{x\theta}] = \sum_j^p \tilde{y} [K_{xj}]$$

Matrix $[\tilde{K}_{x\theta}]$ is independent of the position of the centers of mass. Furthermore, since it is obtained as a linear combination of the symmetric stiffness matrices of the resisting planes, it is itself symmetric. In terms of this new matrix, the vanishing condition for the coupling matrix $[K_{x\theta}]$ can be expressed as

$$([\eta] + [Y^*])[K_{xx}] = [\tilde{K}_{x\theta}]$$

An important implication of this relationship is that the product of a diagonal matrix, $[\zeta] + [Y^*]$, and a symmetric matrix, $[K_{xx}]$, is supposed to yield a symmetric matrix, $[\tilde{K}_{x\theta}]$, also. This is the center of the discussion, since this condition requires that the diagonal matrix be proportional to the unit matrix, i.e.,

$$[\eta] + [Y^*] = y_c [1] \tag{7}$$

Two conclusions can be drawn from this result. In the first place, it identifies torsionally compensable systems as those having a stiffness layout such that

$$[\tilde{K}_{x\theta}] = y_c [K_{xx}] \tag{8}$$

i.e., the coupling matrix $[\tilde{K}_{x\theta}]$ is proportional to the translational stiffness matrix $[K_{xx}]$. In the second place, since Eq. 7 can be rewritten as

$$(\bar{y}_i^* - y_i^*) + y_i^* = y_c \qquad (i = 1, 2, ..., n)$$

it shows that the compensating centers of mass of all stories have the same coordinate $\bar{y}_i^* = y_c$.

An identical argumentation will lead to the conclusion that torsional compensation requires that the Y-direction stiffness layout be such that

$$[\tilde{K}_{y\theta}] = x_c [K_{yy}] \tag{9}$$

in which $[\tilde{K}_{y\theta}]$ is defined as

$$[\tilde{K}_{y\theta}] = \sum_{j}^{q} \tilde{x}_j \, [K_{yj}]$$

where $\tilde{x}_j$ is the coordinate of the j-th Y-direction plane with respect to the common origin. It will also follow that the compensating centers of mass have the same coordinate $\bar{x}_i^* = x_c$.

As a final remark, it is worth to note that if compensating centers of mass do exist, they lie in a vertical line, since they all have the same coordinates. The identification of compensating centers of mass with centers of rigidity, is quite obvious.

ASSESSMENT OF TORSIONAL COMPENSABILITY

From the above it is concluded that a structure is compensable in the X-direction if the linear relationship represented by Eq. 8 is satisfied. The most obvious case in which the linear relationship holds is symmetry. A second case is that of resisting planes with proportional stiffness matrices. Additionally, given the linearity of the condition, a structure with two subgroups of resisting planes independently compensable, will be compensable as a whole if both subgroups have a coincident line of centers of resistence.

Most structures will not be compensable, and therefore, no centers of resistance can be defined from a dynamic response point of view. However, it is reasonable to think that there may exist structures that barely fail to qualify as compensable, for which Eqs. 8 and 9 are satisfied in an approximate sense. An indication of the deviation from compensability is obtained by examining the elements of the matrix $[D]$ given by

$$[D] = [\tilde{K}_{x\theta}]\,[K_{xx}]^{-1}$$

Of course, in a truly compensable case, $[D]$ is a diagonal matrix and all the elements of its diagonal are equal. In nearly compensable cases, the position of the origins of eccentricity could be estimated as the average of the diagonal elements, thus minimizing the quadratic error of the diagonal terms. However, such a procedure partially ignores the torsional properties of the system that certainly affect compensability.

A second approach to the assessment of torsional compensability is to compare the real modes of vibration with those of an hypothetical uncoupled structure. Assuming that the Y-direction is compensated, so as to limit the discussion to the x-θ coupling, the normal modes of the uncoupled structure are solutions of the eigenvalue problems

$$[K_{xx}]\{\psi_x\} = \omega_x^2\,[M]\{\psi_x\} \qquad\qquad [K_{\theta\theta}]\{\psi_\theta\} = \omega_\theta^2\,[J]\{\psi_\theta\}$$

in terms of which the real modes can be written as

$$\{\phi_x\}_i = \begin{vmatrix} \Sigma\, a_{ij} \{\psi_x\}_j \\ \\ \Sigma\, c_{ik} \{\psi_\theta\}_k \end{vmatrix} \qquad \{\phi_\theta\}_i = \begin{vmatrix} \Sigma\, c'_{ik} \{\psi_x\}_k \\ \\ \Sigma\, b_{ij} \{\psi_\theta\}_j \end{vmatrix}$$

where the $\{\phi_x\}$ are predominantly translational modes and the $\{\phi_\theta\}$ are predominantly rotational modes. In a compensated system, $[a_{ij}]$ and $[b_{ij}]$ are equal to the unit matrix while $[c_{ik}]$ and $[c'_{ik}]$ are null. The unitary character of the first two matrices is remarkably stable, the last two depart rather rapidly from nullity when the compensated system is perturbated by altering its sttiffness properties.

From a practical point of view, the above approach has the disadvantage of requiring the computation of the normal modes of the coupled system. An alternative approach can be developed by making use of the perturbation theory presented by Kan and Chopra in Ref. 2., where the following second-order approximation coefficients are given

$$a_{ij} = b_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad c_{ik} = -c_{ki} = \frac{\{\psi_x\}_i^t\,[K_{x\theta}]\,\{\psi_\theta\}_k}{\omega_{xi}^2 - \omega_{\theta k}^2} \tag{10}$$

This approximations are very accurate for nearly compensable structures. This suggests the possibility of estimating a sort of "approximate origins of eccentricity" in nearly compensable structures. Such origins, chosen all in the same vertical line, can be determined making c_{11} given by Eq. 10 equal to zero, in an attempt to minimize the torsional tendency of the structure by uncoupling the two fundamental modes of vibration. The common coordinate y_a of the approximate origin of eccentricity is given by

$$y_a = -\frac{\{\psi_x\}_1^t\,[\tilde{K}_{x\theta}]\,\{\psi_\theta\}_1}{\{\psi_x\}_1^t\,[K_{xx}]\,\{\psi_\theta\}_1}$$

It must be noted, though, that $\{\psi_\theta\}_1$ depends on the value of y_a. Hence, an iterative scheme must be followed solving for the first mode of the eigen value problem

$$([K_{\theta\theta}] + y_a^2\,[K_{xx}])\{\psi_\theta\} = \omega_\theta^2\,[J]\,\{\psi_\theta\}$$

In several numerical examples performed, it was found that the c_{ik} coefficients for nearly compensable cases with the centers of mass placed in the approximate origin of eccentricity, were indeed very close to zero. In fact, they were smaller by an order of magnitude than those obtained by placing the centers of mass in points recommended by some codes to be used as centers of resistance.

SUMMARY AND CONCLUSIONS

The most relevant features of the torsional phenomenon in multistory buildings discussed in this paper are:

a) The definition of centers of resistance as origins for measuring eccentricity must be associated to the possibility of torsion-free dynamic reponse. From a physical point of view, centers of resistance have a sensible meaning only if they represent a set of points such that if the centers of mass coincide with them, the structure will have a purely translational motion as a response to any torsion-free seismic excitation.

b) Centers of resistance must be such that if the eccentricity is zero in all stories, the modes of vibration of the building uncouple into purely translational modes and purely torsional modes.

c) Centers of resistance do not exist in general. Caution should be used in dealing with the available definitions of centers of resistance. In particular, the "centers of shear" of ideal shear type buildings are not to be regarded automatically as centers of resistance.

d) When centers of resistance do exist, they all lie in a vertical line. Torsion-free vibration is achieved when all story centers of mass lie in that same line.

e) Structures for which centers of resistance do exist, i.e., structures susceptible of dynamic compensation, have stiffness layouts such that the torsional matrices are proportional to the corresponding translational stiffness matrix.

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