EFFECT OF MODAL COUPLING ON DYNAMIC RESPONSE

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SUMMARY

The accuracy of approximate modal analysis is investigated for simple structures which do not have uncoupled normal modes. It is shown that the errors of neglecting modal coupling are small in most instances, but are very significant in some particular situations. Primary emphasis is given to mean squared response values as computed from random vibration of the coupled and uncoupled structural models, and limited results are presented for deterministic response to a simple pulse of base acceleration. The possibility of major effects of modal coupling is seen to apply to both stochastic and deterministic situations. The results are found to be particularly relevant to equivalent linearization analysis of yielding structures.

INTRODUCTION

Modal analysis is a common technique for studying both deterministic and probabilistic dynamics of structural models. It is well known, of course, that the usual method is strictly applicable only for a system with damping such that there are uncoupled normal modes (Ref. 1). In many situations, though, one may choose to completely ignore this uncoupling condition on the grounds of not having any very specific information about the proper form for the damping matrix. In this situation it is common to simply choose a damping value for each mode, based on past experience. The modes are then treated as uncoupled, and no explicit damping matrix is used.

Sometimes a damping matrix is known, and it does not give uncoupled modes. That is, the transformation of variables which simultaneously diagonalizes the mass and stiffness matrices does not also diagonalize the damping matrix. There are known methods for solving a system of coupled equations, but these are generally less efficient than modal analysis. Thus, one may choose to seek an approximate solution by replacing the nondiagonal transformed damping matrix by a substitute diagonal matrix, then using modal analysis. The simplest method of doing this diagonalization is to completely neglect the off-diagonal terms, but more elaborate methods may also be used. For stochastic analysis, in particular, a diagonalization procedure has been suggested whereby the modal damping values are chosen in such a way as to attempt to account for the energy dissipation due to off-diagonal terms in the transformed matrix (Ref. 2).

The following is an analysis of the type and magnitude of errors that may result from approximating a particular type of coupled-mode system by an uncoupled system. First, the system is analyzed when the excitation is a random process. It is shown that the errors in response prediction are usually small, but can be very significant in some situations. A deterministic analysis is given for two situations where the stochastic analysis indicates a major effect of modal coupling. The results confirm the finding that the

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response of an uncoupled model can be significantly different from that of the original system with coupled modes. It is also shown that equivalent linearization analysis of a yielding structure is an important application of the present work. In particular, such linearization can lead to models for which modal coupling cannot be ignored without introducing major errors.

RANDOM VIBRATION

Figure 1 shows a mechanical model for the simple linear two-degree-of-freedom (2DF) system which will be studied here. Note that the mass and stiffness of the system are symmetric, while the damping is applied only to the x_1 motion. This model has been chosen in order to have a very simple system with a particularly large degree of modal coupling.

The equations of motion for the system can be written as:

$$m\ddot{x}_1 + c\dot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = -m\ddot{z} \tag{1}$$

$$m\ddot{x}_2 - k_2x_1 + (k_1 + k_2)x_2 = -m\ddot{z} \tag{2}$$

where dots denote derivatives with respect to time. The x_1 and x_2 are measured relative to the base, and the base acceleration $\ddot{z}$ is applied to both ends of the model. This base acceleration is taken to be a stationary white noise stochastic process with zero mean and power spectral density S_o.

Exact Solution

Simultaneous equations governing the covariances of stationary response can be obtained by multiplying eqs. 1 and 2 each by x_1, x_2, $\dot{x}_1$ and $\dot{x}_2$ and taking the expectation (statistical average) of each. This procedure is simply a component form of the usual approach for finding the response covariances from a Lyapunov matrix equation (Refs. 3 and 4). For this 2DF system the component form is quite simple. The simultaneous equations are easily solved to give

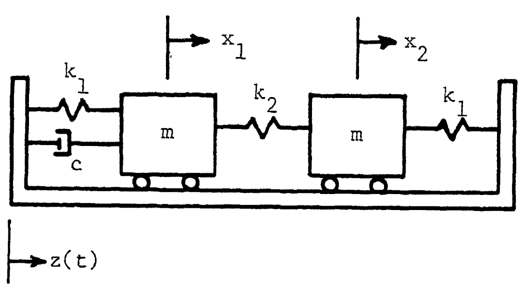

Fig. 1. Mechanical Model

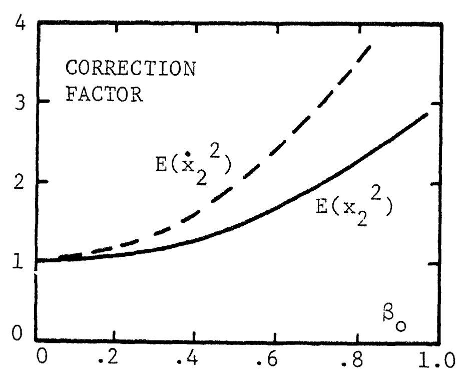

Fig. 2. Mean Squared Response

$$E(x_1^2) = E(x_1 x_2) = 2\pi m^2 S_o/(k_1 c) \tag{3}$$

$$E(x_2^2) = \frac{2\pi m^2 S_o}{k_1 c}\left(1 + \frac{k_1 c^2}{2k_2^2 m}\right) \tag{4}$$

$$E(\dot{x}_1 x_2) = -E(\dot{x}_2 x_1) = \pi m S_o/k_2 \tag{5}$$

$$E(\dot{x}_1^2) = E(\dot{x}_1 \dot{x}_2) = 2\pi m S_o/c \tag{6}$$

$$E(\dot{x}_2^2) = \frac{2\pi m S_o}{c}\left(1 + \frac{(k_1 + k_2)c^2}{2k_2^2 m}\right) \tag{7}$$

Approximate Modal Solution

The approach is to write the response vector as a linear combination of the two modes of the undamped system. For this system with symmetric mass and stiffness, this gives

$$x_1 = y_1 + y_2 \tag{8}$$

$$x_2 = y_1 - y_2 \tag{9}$$

Substituting this into eqs. 1 and 2 gives

$$m\ddot{y}_1 + (c/2)\dot{y}_1 + (c/2)\dot{y}_2 + k_1 y_1 = -m\ddot{z} \tag{10}$$

$$m\ddot{y}_2 + (c/2)\dot{y}_1 + (c/2)\dot{y}_2 + (k_1 + 2k_2)y_2 = 0 \tag{11}$$

Note that eqs. 10 and 11 are very much coupled. The so-called off-diagonal damping terms ($\dot{y}_2$ in eq. 10 and $\dot{y}_1$ in eq. 11) have the same coefficients as the on-diagonal terms.

The simplest approximate solution of eqs. 10 and 11 is to completely neglect the off-diagonal coupling terms. The resulting single-degree-of-freedom (SDF) form of eq. 11 is homogeneous, so gives $y_2(t)$ as identically zero. Thus, eqs. 8 and 9 give $x_1 = x_2 = y_1$, and solving the SDF form of eq. 10 gives

$$E(x_1^2) = E(x_1 x_2) = E(x_2^2) = 2\pi m^2 S_o/(k_1 c) \tag{12}$$

$$E(\dot{x}_1^2) = E(\dot{x}_1 \dot{x}_2) = E(\dot{x}_2^2) = 2\pi m S_o/c \tag{13}$$

$$E(\dot{x}_1 x_2) = E(\dot{x}_2 x_1) = 0 \tag{14}$$

Penzien, Kaul and Berge (Ref. 2) have presented a general procedure for replacing a coupled damping matrix by an "equivalent" uncoupled (i.e., diagonal) matrix C*. The jth term of the diagonal matrix is given by

$$c^*_{jj} = \sum_k \frac{c_{jk}E(\dot{y}_j\dot{y}_k)}{E(\dot{y}_j^{\,2})} \tag{15}$$

where c_{jk} is a component of the coupled damping matrix. It can be shown that eq. 15 gives the same energy dissipation rate for the coupled and uncoupled systems if they have the same response velocities. Note that eq. 15 defines the substitute damping parameters in terms of the response levels. Thus, iteration is generally required to simultaneously find the c^*_{jj} values and the response covariances. A simple initial guess for beginning the iteration is to neglect the off-diagonal terms c_{ij}. Applying the above procedure to the system of eqs. 10 and 11 does not give any improvement. As noted above, neglecting the coupling process gives y_2 (and therefore $\dot{y}_2$) identically zero, so that eq. 15 converges immediately to $c^*_{11} = c_{11} = c/2$. The equation for c^*_{22} is undefined (zero over zero), but this is irrelevant since the uncoupled equation for $y_2(t)$ has no excitation. Thus the results in eqs. 26-28 are completely compatible with the technique of Ref. 2. Using the C^* matrix may approximately correct for modal-coupling damping terms in some situations, but it provides no correction at all for this symmetric problem.

Effects of Modal Coupling

Of the eight response quantities given in eqs. 3-7, four agree identically with the results from the approximate modal solution (eqs. 12-14). The final terms in eqs. 4 and 7 represent important effects of modal coupling. The mean squared values of both displacement and velocity are underestimated if modal coupling is neglected. Ignoring these terms may cause insignificant errors in many situations, since they are of the order of $c^2/(km)$, but in other situations the errors may be substantial. In order to present the coupling effect in somewhat more detail, let

$$\beta_o^{\,2} = c^2/(4k_1m) \tag{16}$$

Note that β_o is the fraction of critical viscous damping in the SDF oscillator formed by considering only the left-hand portion of Fig. 1 (i.e., for $k_2 = 0$). For the special case of $k_2 = k_1$, Fig. 2 shows the correction factors of eqs. 4 and 7. Correction factor, here, means the ratio of the exact mean squared response to that obtained from the uncoupled equations.

An even more important effect of the modal coupling is seen if one considers the mean squared distortion in the k_2 spring, $E[(x_2-x_1)^2]$. Symmetry makes this term identically zero for the uncoupled approximation, but it is non-zero for the exact solution. Even though this distortion is of the order of damping squared, ignoring it gives an infinite percentage error.

DETERMINISTIC COMPARISONS

The analysis given here is limited to two particular situations in which modal coupling was shown to be quite significant by the above random vibration analysis. In particular, $k_1 = k_2$ as in Fig. 2 and the β_o of eq. 16 is chosen as 0.5 and 1.0. The particular dynamic situation considered is the response to a Dirac delta pulse of ground acceleration. Thus the equations to be solved are simply the homogeneous forms of Eqs. 1 and 2, with initial conditions of $\dot{x}_1 = \dot{x}_2 = 1$. Note that $x_1(t)$ and $x_2(t)$ are then the impulse response functions used in writing responses to other excitations by Duhamel convolution integrals.

One way to find the exact solution for this homogeneous system involves first rewriting it as a first-order matrix equation

$$m\dot{Z} + AZ = 0 \tag{17}$$

where Z is a vector with components x_1, x_2, $\dot{x}_1$ and $\dot{x}_2$, and

$$A = \begin{bmatrix} 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \\ k_1 + k_2 & -k_2 & c & 0 \\ -k_2 & k_1 + k_2 & 0 & 0 \end{bmatrix} \tag{18}$$

One can then write the solution as a linear combination of the four complex eigen solutions, with the coefficients chosen to satisfy the proper initial conditions. The solutions can be simplified to the following form

$$\omega_o x_j(t) = (A_j \cos\omega_1 t + B_j \sin\omega_1 t)\exp(-\beta_1\omega_o t)$$
$$+ (-A_j \cos\omega_2 t + C_j \sin\omega_2 t)\exp(-\beta_2\omega_o t) \quad \text{for } j = 1,2 \tag{19}$$

where $\omega_o = (k/m)^{\frac{1}{2}}$ is the frequency of the fundamental mode of the uncoupled system. For the system with $\beta_o = 0.5$, one obtains: $\omega_1 = 1.033\omega_o$, $\omega_2 = 1.601\omega_o$, $\beta_1 = 0.291$, $\beta_2 = 0.208$, $A_1 = -0.236$, $B_1 = 1.327$, $C_1 = -0.244$, $A_2 = 0.400$, $B_2 = 1.067$, and $C_2 = -0.043$. For $\beta_o = 1.0$ the values are: $\omega_1 = 0.832\omega_o$, $\omega_2 = 1.439\omega_o$, $\beta_1 = 0.862$, $\beta_2 = 0.138$, $A_1 = 0.225$, $B_1 = 1.785$, $C_1 = -0.224$, $A_2 = 0.482$, $B_2 = 0.533$, $C_2 = 0.630$.

Next consider the uncoupled approximate solution obtained by ignoring the off-diagonal damping terms. As in the random situation only the fundamental mode is excited, and this mode has undamped frequency ω_o and damping factor $\beta_o/2$. Thus the impulse response may be written as

$$x_1(t) = x_2(t) = p_o^{-1} \sin(p_o t)\exp(-\beta_o\omega_o t/2) \tag{20}$$

with $p_o = 0.968\omega_o$ for $\beta_o = 0.5$, and $0.866\omega_o$ for $\beta_o = 1.0$.

Figure 3 compares the exact x_1 and x_2 responses with the result from the uncoupled approximation. Obviously the modal coupling does have significant effects in the two situations shown, particularly for $\beta_o = 1.0$. Consider the amplitudes of x_1 and x_2 during the first "cycle" of response, i.e., until x(t) again equals zero with positive slope. The amplitude of x_2 during this cycle is significantly larger than for the uncoupled approximation. The amplitude of x_1, on the other hand, is smaller than the approximation during the first half-cycle, but exceeds the approximation during the second half-cycle. This seems to agree qualitatively with the result in random vibration that the approximation is good (exact, in fact) for $E(x_1^2)$ but underpredicts $E(x_2^2)$.

One may also note from Fig. 3 that increasing β_o from 0.5 to 1.0 has almost no effect on the amplitude of $x_2(t)$ during the first cycle, although the period is reduced by the increased damping. The increased damping does significantly reduce the first-cycle amplitude of $x_1(t)$. Finally, note that for $\beta_o = 1.0$, the exact solutions decay much less rapidly with time, than does the uncoupled approximation. Clearly this is due to the fact that β_2 is much smaller than $\beta_o/2$ in this situation.

Note, also, that the uncoupled approximation again gives no distortion in the k_2 spring of the model. Figure 3, though, shows that this distortion may, in fact, be quite substantial. This is consistent with the failure of the uncoupled approximation to predict the values of this distortion in random vibration.

EQUIVALENT LINEARIZATION

The random vibration analysis showed that the error of neglecting modal coupling varies like β_o^2. For most structural damping values this error can generally be ignored. One situation in which large damping values may occur, though, is when a spring and a dashpot are substituted for a yielding element in equivalent linearization. For example, for a nearly elastoplastic SDF oscillator one reasonable choice for the substitute linear system has a damping factor as large as $\beta_o = 0.53$ (Ref. 5). (The stiffness in this substitute system is only about 10% of the preyielding stiffness.) Thus, if yielding occurred in only one element of a 2DF model, then use of this substitute model could lead to a linear system like that treated above with $\beta_o = 0.5$. Recall that eq. 12 gave the exact value of $E(x_2^2)$ as 50% larger than the value from the uncoupled approximation in that case. Also, the mean squared distortion between the masses in that situation is about 50% of that for x_1, compared to zero for the uncoupled approximation.

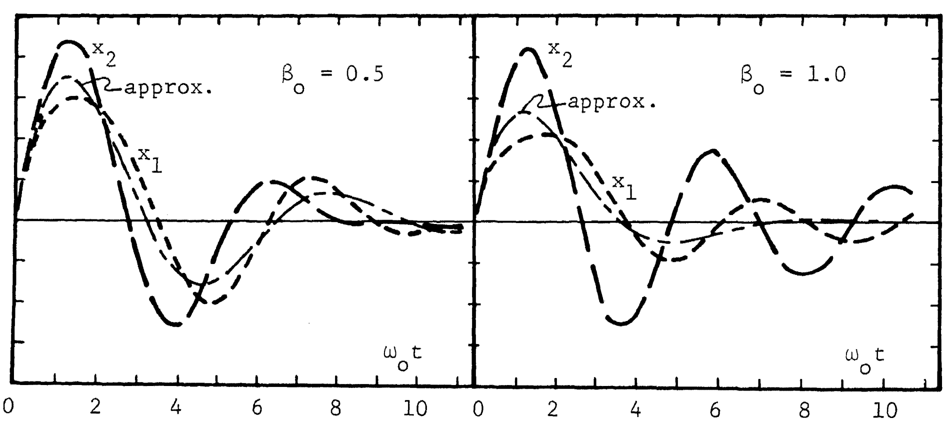

Fig. 3. Deterministic Response

As mentioned at the outset, the model of Fig. 1 was chosen to emphasize the effect of modal coupling. In particular, modal coupling provides the only excitation of the higher mode, since the system is symmetric except for the damping. However, modal coupling has also been shown to be important in another equivalent linearization situation which lacks this symmetry. Brinkmann and Jan (Refs. 6 and 7) considered a linearization similar to that described above for a model representing a two-story building with yielding only in the bottom story. For this unsymmetric model even the uncoupled mode approximation predicts the occurrence of distortion in the top story. However, the mean squared value predicted by the approximation was sometimes found to be only about 50% of that from exact analysis (Ref. 7).

CONCLUSIONS

Neglecting modal coupling has caused errors of the order of damping squared for some mean squared responses of the 2DF model considered here.

This error may be particularly significant when the model has modes which are excited only by modal coupling.

The uncoupling procedure suggested by Penzien et al. is the same as completely neglecting modal coupling for some models.

The deterministic analysis has also confirmed the importance of modal coupling.

Equivalent linearization analysis of a yielding structure can lead to a model with damping so large that modal coupling cannot be ignored without introducing major errors.

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