VIBRATIONS OF A SEMI-INFINITE ELASTIC MEDIUM DUE TO BURIED SOURCES

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SUMMARY

This paper is concerned with the computational procedure of displacements of a semi-infinite elastic medium due to harmonic loads distributing vertically over horizontal rectangular and circular planes in the interior of the medium. In calculating displacements, the authors propose the matrix-multiplication method taking advantage of the vector-matrix form which expresses the general solutions of displacements and stresses being used in the derivation of dispersion function of a semi-infinite elastic multi-layered media. As numerical examples, the displacements of the centre of the rectangular and circular planes are calculated changing the shape of distribution of the loads.

INTRODUCTION

Many works of wave propagation theory due to a harmonic load on the surface or in the interior of a semi-infinite elastic medium have been done in the field concerning earthquake wave propagations (Ref. 1). However, so far not many works have been done on the vibration problems of the interior of three-dimensional elastic medium with a free surface, because of the difficulty of mathematical formulations. Pekeris (Ref. 2) obtained time responses of surface displacements due to a buried pulse of Heaviside step function by means of Cagniard method, and Awojobi et al. (Ref. 3) obtained the solutions for the case of a vertical impulsive force of Delta function. In both works, some efforts were done in Laplace inverse method which transforms the displacements expressed in the imaginary space of Laplace transformation with respect to time into the real space. On the other hand, for a harmonic vibration of a semi-infinite elastic medium due to a concentrated force, Matsuoka et al. (Ref. 4) used the superposition method of displacement potential functions and fundamental solutions in an infinite elastic medium by means of the principle of mirror image. It is the same method as used for Mindlin's solutions. The authors (Ref. 5) proposed a method for obtaining displacements by means of the matrix-multiplication, taking advantage of the vector-matrix form which expresses the general solutions of displacements and stresses being used in the derivation of dispersion function of a semi-infinite elastic multi-layered media.

In this paper, the vertical displacements due to harmonic loads distributing over a rectangular and a circular planes in the interior of a semi-infinite elastic medium are derived by means of authors' proposing method (Ref. 5).

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The method proposed herein can be extended to the interior vibration problems either of an elastic stratum over a semi-infinite elastic rigid medium or of a semi-infinite elastic multi-layered media.

FORMULATION PROCEDURE OF DISPLACEMENTS

As shown in Fig. 1, the authors assume a semi-infinite elastic medium to be a provisional two-layered media with the same mechanical properties for each layer, and consider that a harmonic distributed load is applied on the boundary plane between the first layer and the second one.

In a Cartesian co-ordinates (x, y,z), eliminating time term $e^{i\omega t}$, a displacement and stress vector of the m-th layer (m=1,2) may be expressed as follows:

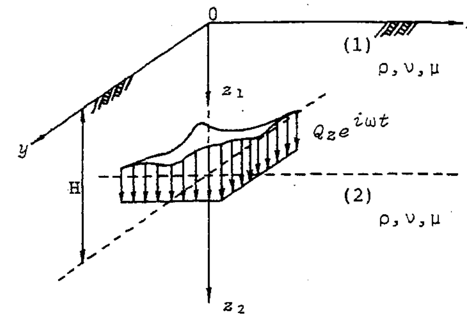

Fig. 1 Analytical model and co-ordinates

$$\frac{1}{exp\{i(\xi_1 x+\xi_2 y)\}}\left\{-\frac{u_x}{i\xi_1},\ -\frac{u_y}{i\xi_2},\ u_z,\ \frac{\tau_{zx}}{i\xi_1\mu_1},\ \frac{\tau_{yz}}{i\xi_2\mu_1},\ -\frac{\sigma_z}{\mu_1}\right\}_m^T$$

$$= [D]_m\{C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6\}_m^T \qquad (1)$$

where,

$$[D]_m =$$

$$\begin{bmatrix} A_1 & A_2 & \beta_m B_1 & -\beta_m B_2 & 0 & 0 \\ A_1 & A_2 & 0 & 0 & \beta_m B_1 & -\beta_m B_2 \\ \alpha_m A_1 & -\alpha_m A_2 & \xi_1^2 B_1 & \xi_1^2 B_2 & \xi_2^2 B_1 & \xi_2^2 B_2 \\ 2\alpha_m\mu_m^* A_1 & -2\alpha_m\mu_m^* A_2 & (\xi_1^2+\beta_m^2)\mu_m^* B_1 & (\xi_1^2+\beta_m^2)\mu_m^* B_2 & \xi_2^2\mu_m^* B_1 & \xi_2^2\mu_m^* B_2 \\ 2\alpha_m\mu_m^* A_1 & -2\alpha_m\mu_m^* A_2 & \xi_1^2\mu_m^* B_1 & \xi_1^2\mu_m^* B_2 & (\xi_2^2+\beta_m^2)\mu_m^* B_1 & (\xi_2^2+\beta_m^2)\mu_m^* B_2 \\ (k^2+\beta_m^2)\mu_m^* A_1 & (k^2+\beta_m^2)\mu_m^* A_2 & 2\xi_1^2\beta_m\mu_m^* B_1 & -2\xi_1^2\beta_m\mu_m^* B_2 & 2\xi_2^2\beta_m\mu_m^* B_1 & -2\xi_2^2\beta_m\mu_m^* B_2 \end{bmatrix}$$

$A_1=exp(-\alpha_m z_m)$, $A_2=exp(\alpha_m z_m)$, $B_1=exp(-\alpha_m z_m)$, $B_2=exp(\alpha_m z_m)$,

$\alpha_m^2=k^2-k_{pm}^2$, $\beta_m^2=k^2-k_{sm}^2$, $k^2=\xi_1^2+\xi_2^2$, $k_{pm}=\omega/V_{pm}$, $k_{sm}=\omega/V_{sm}$, $\mu_m^*=\mu_m/\mu_1$

μ_1 and μ_m^* are shear moduli of the first layer and the m-th layer, respectively. V_{pm} and V_{sm} are the dilatational wave velocity and the distorsional wave velocity in the m-th layer, respectively. ω is circular frequency of wave motion, and C_1- C_6 are unknown coefficients.

As a matter of convenience on mathematical formulation, Eq.(1) may be rewritten as the following equation in matrix form:

$$\{V\}_m = [D]_m\{C\}_m \qquad (2)$$

where, $\{V\}_m$ and $\{C\}_m$ are a displacement-stress vector and an unknown coefficient vector of the m-th layer, respectively. Moreover, $[D]_m$ in the upperest edge ($z_m=0$) and $[D]_m$ in the lowest edge ($z_m=H_m$) of the m-th layer, respectively, are expressed as the following matrix forms:

$$[E]_m = [D]_{m,z_m=0} \quad , \quad [F]_m = [D]_{m,z_m=H_m} \tag{3}$$

Next, derive the displacements for an analytical model shown in Fig. 1. The following equations are obtained from Eqs.(2) and (3):

$$\{C\}_1 = [E]_1^{-1}\{V\}_{1,z_1=0} \tag{4}$$

$$\{V\}_{1,z_1=H} = [F]_1\{C\}_1 = [F]_1[E]_1^{-1}\{V\}_{1,z_1=0} \tag{5}$$

$$\{C\}_2 = [E]_2^{-1}\{V\}_{2,z_2=0} \tag{6}$$

The boundary conditions of the surface and the provisional boundary plane are given as follows:

$$\{\sigma\}_{1,z_1=0} = 0 \tag{7}$$

$$\{V\}_{2,z_2=0} - \{V\}_{1,z_1=H} = \{0,\sigma^*\}^T \tag{8}$$

where, $\{\sigma\}$ and $\{\sigma^*\}$ are a stress vector and a known stress vector on the provisional boundary plane, respectively. Considering these boundary conditions, the following equations may be obtained from Eqs.(4) and (6):

$$\{C\}_1 = [E]_1^{-1}\{\delta,0\}^T_{1,z_1=0} \tag{9}$$

$$\{C\}_2 = [E]_2^{-1}\{0,\sigma^*\}^T + [E]_2^{-1}[F]_1[E]_1^{-1}\{\delta,0\}^T_{1,z_1=0} \tag{10}$$

where, $\{\delta\}$ is a displacement vector. Dividing a coefficient vector $\{C\}_2$ in the Eq.(10) between $\{C^+\}_2$ and $\{C^-\}_2$ for the ascending and descending wave components, respectively. $\{C^+\}_2$ may be zero because the ascending wave component cannot exist in the second layer becoming a semi-infinite elastic medium. Consequently, Eq.(10) may be rewritten as follows:

$$\begin{Bmatrix} C^- \\ 0 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \delta \\ 0 \end{Bmatrix}_{1,z_1=0} + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{Bmatrix} 0 \\ \sigma^* \end{Bmatrix} \tag{11}$$

where, K_{ij} and L_{ij} (i,j=1,2) are sub-matrices of $[K]=[E]_2^{-1}[F]_1[E]_1^{-1}$ and $[L]=[E]_2^{-1}$, respectively. From Eq.(11), an unknown displacemet vector on the surface may be obtained as follows:

$$\{\delta\}_{1,z_1=0} = -[K_{21}]^{-1}[L_{22}]\{\sigma^*\} \tag{12}$$

By substituting $\{C\}_1$ and $\{C\}_2$, which are obtained by substituting Eq.(12) into Eqs.(9) and (10), into Eq.(2), the displacements of the first and the second layers can be obtained for vibration conditions, respectively.

Each known stress vector $\{\sigma^*\}$ in Eq.(8) for " rigid ", " uniform " and " parabolic " loads distributing over a rectangular and a circular planes, respectively, may be given as follows:

For a harmonic load distributing over a rectangular plane ($2a \times 2b$),

$$\sigma_z = \begin{cases} -\dfrac{Q_z}{\pi ab}\dfrac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+1/2)\Gamma(q+1/2)}\left(1-\dfrac{x^2}{a^2}\right)^{p-1/2}\left(1-\dfrac{y^2}{b^2}\right)^{q-1/2} & |x|\leq a,\ |y|\leq b \\ 0 & |x|>a,\ |y|>b \end{cases} \tag{13}$$

$$\tau_{zx} = 0,\ \tau_{yz} = 0 \qquad \text{for all } x\ ,\ y$$

where, Q_z is an amplitude of resultant force distributing over a rectangular plane, a and b are half widths of a rectangular plane for x and y directions, respectively, p and q are constants determining the shape of distribution of the loads, and $\Gamma(\ )$ is Gamma function.

By the application of Double Fourier Integral formula, the following equations can be obtained:

$$\sigma_z^* = -\frac{Q_z}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{2^{p+q}}{(\xi_1 a)^p(\xi_2 b)^q}\Gamma(p+1)\Gamma(q+1)J_p(\xi_1 a)J_q(\xi_2 b)\cdot d\xi_1 d\xi_2 \tag{14}$$

$$\tau_{zx}^* = 0,\ \tau_{yz}^* = 0$$

where, $J_p(\ )$ and $J_q(\ )$ are the p-th and q-th order Bessel s function of the first kind, respectively.

Performing the following variable transformation

$$\xi_1 = k\cos\theta\ ,\quad \xi_2 = k\sin\theta\ ,\quad d\xi_1 d\xi_2 = kdkd\theta$$

to the displacement solution (Eq.(2)) obtained from a known stress vector given by Eq.(14), decomposing the matrix-multiplication and shortening the range of θ integration, the vertical displacements due to harmonic loads distributing over a rectangular plane may be obtained as follows:

(1) The displacement due to a " rectangular rigid distributed " load (as shown in Fig. 3) ($p=q=0$)

$$u_z = -\frac{Q_z}{2\pi^2\mu k_S^2}\int_0^{\pi/2}\int_0^{\infty}\frac{kW(k)}{\beta F(k)}\cos(kx\cos\theta)\cos(ky\sin\theta)\cdot J_0(ka\cos\theta)J_0(kb\sin\theta)\,dkd\theta \tag{15}$$

(2) The displacement due to a " rectangular uniform distributed " load ($p=q=1/2$)

$$u_z = -\frac{Q_z}{2\pi^2\mu k_S^2 ab}\int_0^{\pi/2}\int_0^{\infty}\frac{W(k)}{k\beta F(k)}\frac{\cos(kx\cos\theta)\cos(ky\sin\theta)}{\cos\theta\sin\theta}\cdot\sin(ka\cos\theta)\sin(kb\sin\theta)\,dkd\theta \tag{16}$$

(3) The displacement due to a " rectangular parabolic distributed " load ($p=q=3/2$)

$$u_z = -\frac{Q_z}{2\pi^2\mu k_S^2 a^2 b^2}\int_0^{\pi/2}\int_0^{\infty}\frac{W(k)}{k^3\beta F(k)}\frac{\cos(kx\cos\theta)\cos(ky\sin\theta)}{\cos\theta\sin\theta}$$

$$\cdot\left\{\frac{\sin(ka\cos\theta)}{ka\cos\theta}-\cos(ka\cos\theta)\right\}\left\{\frac{\sin(kb\sin\theta)}{kb\sin\theta}-\cos(kb\sin\theta)\right\}dk\,d\theta \tag{17}$$

On the other hand, the displacements for " circular distributed " loads may be obtained by deriving the equations corresponding to Eq.(1) in a cylindrical coordinates(r,θ,z) and by formulating the similar equation to the one prescribed.

A known stress vector $\{\sigma^*\}$ in Eq.(8) for each " circular distributed " load may be given as follows:

$$\sigma_z = \begin{cases} -\dfrac{Q_z}{\pi r_0^2}\,p\left(1-\dfrac{r^2}{r_0^2}\right)^{p-1} & |r|\leq r_0 \\ 0 & |r|>r_0 \end{cases} \tag{18}$$

$$\tau_{zr} = 0 \qquad \text{for all } r$$

where, r_0 is a radius of circular $(=2\sqrt{\frac{ab}{\pi}})$ equivalent to a rectangular area, and p is a constant determining the shape of load distribution over a circular plane.

By the application of Fourier-Bessel Integral formula to above equations, the following equations can be obtained:

$$\sigma_z^* = -\frac{Q_z}{\pi}\int_0^{\infty}\frac{2^{p-1}p\Gamma(p)}{(kr_0)^p}J_p(kr_0)\,kJ_0(kr)\,dk \tag{19}$$

$$\tau_{zr}^* = 0$$

Then, the vertical displacements due to harmonic loads distributing over a circular plane can be obtained as follows:

(4) The displacement due to a " circular rigid distributed " load (p=1/2)

$$u_z = -\frac{Q_z}{4\pi\mu k_S^2 r_0}\int_0^{\infty}\frac{W(k)}{\beta F(k)}\sin(kr_0)J_0(kr)\,dk \tag{20}$$

(5) The displacement due to a " circular uniform distributed " load (p=1)

$$u_z = -\frac{Q_z}{2\pi\mu k_S^2 r_0}\int_0^{\infty}\frac{W(k)}{\beta F(k)}J_1(kr_0)J_0(kr)\,dk \tag{21}$$

(6) The displacement due to a " circular parabolic distributed " load (p=2)

$$u_z = -\frac{2Q_z}{\pi\mu k_S^2 r_0^2}\int_0^{\infty}\frac{W(k)}{k\beta F(k)}J_2(kr_0)J_0(kr)\,dk \tag{22}$$

where, the symbols used in Eqs.(15)-(17) and Eqs.(20)-(22) are given by decomposing the matrix-multiplication as follows:

$$F(k) = (2k^2-k_S^2)^2-4k^2\alpha\beta$$

$$W(k)=\alpha\beta\{(2k^2-k_s^2)\,exp(-\alpha H)-2k^2 exp(-\beta H)\}\{(2k^2-k_s^2)\,exp(-\alpha z)-2k^2 exp(-\beta z)\}$$

$$+k^2\{(2k^2-k_s^2)\,exp(-\beta H)-2\alpha\beta exp(-\alpha H)\}\{(2k^2-k_s^2)\,exp(-\beta z)-2\alpha\beta exp(-\alpha z)\}$$

$$+F(k)\,[\alpha\beta exp\{\pm\alpha(z-H)\}-k^2 exp\{\pm\beta(z-H)\}]$$

$$\alpha=\sqrt{k^2-k_p^2}\,,\quad \beta=\sqrt{k^2-k_s^2}\,,\quad k_p=\omega/Vs\,,\quad k_s=\omega/Vs$$

where, the positive and negative signs of the exponential term are for the cases that the value z is smaller (higher than the loading plane), and larger (lower than the loading plane) than the depth of the loading plane, respectively.

Only the vertical displacements due to harmonic distributed loads applying vertically are shown in this paper. However, the displacements for loads applying horizontally may be obtained by the same procedure as used in vertical vibration problem (Ref. 5).

NUMERICAL RESULTS AND DISCUSSIONS

The numerical results of displacements described in previous section are expressed in the real and imaginary parts of the following non-dimensional coefficient term:

$$u_z = -\frac{Q_z e^{i\omega t}}{\mu a}\ (\ f_1 + i f_2\ ) \qquad (23)$$

In order to compare the displacements due to " rectangular distributed " load with the one due to " circular distributed " load, the authors introduce a circular radius equivalent to a rectangular area.

The calculation conditions are as follows:

loading plane	:	square ($a=b$)
" loading depth " ratio H/a	:	0 , 2 and 5
Poisson's ratio ν	:	1/3

Figs. 2, 3 and 4 show the comparisons among the displacements at the centre of loading plane due to " rigid ", " uniform " and " parabolic distributed " loads changing the H/a ratio from 0 to 2 and to 5, respectively. It is seen that, in the case of " uniform distributed " load, the real and imaginary parts of the centre displacement due to a " square distributed " load show a similar result close to those due to a " circular distributed " one being independent of the ratio H/a. On the other hand, it is of interest to know that, in both cases of " parabolic " and " rigid distributed " loads, the influences owing to the difference between " square " and " circular " loading planes appear in both real and imaginary parts of centre displacements. The centre displacements due to " parabolic " loads distributing over " square " and " circular " planes show inverse behaviours to the ones due to " rigid " loads. It may be understood in both cases that the differences between the behaviours of centre displacements due to " square " and " circular " loads become smaller with an increase of the ratio H/a.

Fig. 5 shows the centre displacement due to a " circular uniform distributed " load by changing the H/a ratio. For reference, the centre displacement due to a " circular uniform distributed " load in an infinite elastic medium is also shown in this figure. In this figure, in the case of H/a=5, the centre displacement due to a " circular " load in a semi-infinite elastic medium is

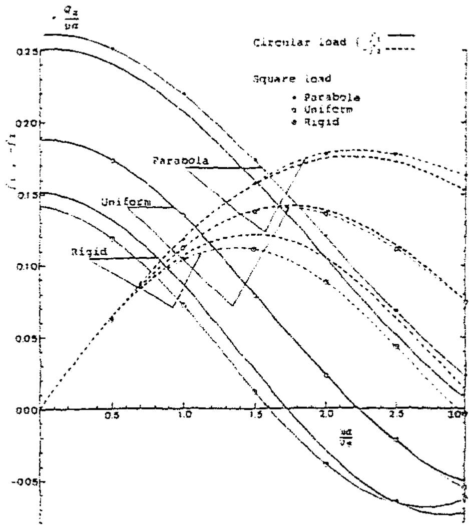

Fig. 2 Centre displacements due to distributed loads (H/a=0)

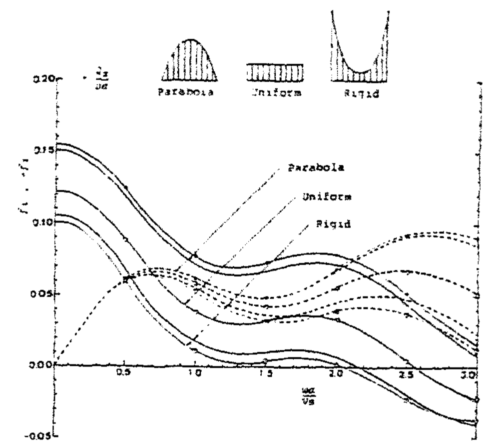

Fig. 3 Centre displacements due to distributed loads (H/a=2)

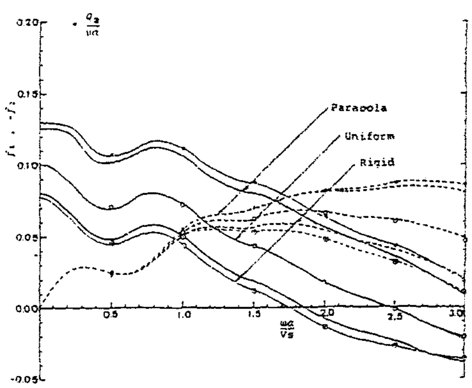

Fig. 4 Centre displacements due to distributed loads (H/a=5)

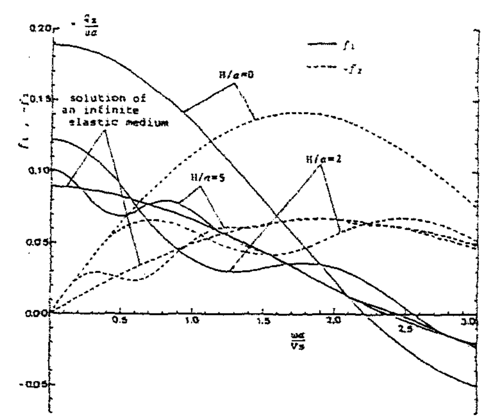

Fig. 5 Centre displacement due to " circular uniform distributed " load (H/a= 0, 2 and 5)

almost similar to that in an infinite elastic medium in the range of non-dimensional frequency $\omega a/Vs>1.5$, that is, $\omega H/Vs>7.5$. Judging from the ratio of shear wave length L to the ratio H/a, this range may be $L/H<0.84$. Similarly, in the case of $H/a=2$, it is apparent that the centre displacement due to a " circular distributed " load in a semi-infinite elastic medium tends to similar to the one in an infinite elastic medium with an increase of $\omega a/Vs$ ($\omega H/Vs$) (with a decrease of L/H). Accordingly, it may be understood that the centre displacement due to a distributed load in the interior of a semi-infinite elastic medium coincides with the one due to a distributed load on an infinite elastic medium, provided that the shear wave length is much smaller than the loading depth (in the case of the ratio L/H being small).

CONCLUSIONS

The calculation procedure of the displacements due to harmonic " rectangular " and " circular " loads applying in the interior of a semi-infinite elastic medium is reported in this paper, and as numerical examples, the centre displacements of " square " and " circular " planes, due to harmonic distributed loads were obtained for " rigid ", " uniform " and " parabolic " shapes of distribution of the loads. Although this paper does not show any stresses due to distributed loads, they can be easily calculated from Eq.(2). Moreover, as previously described, the method proposed in this paper has an advantage to be extended to the vibration problem either of an elastic stratum over a semi-infinite rigid medium or of a semi-infinite elstic multi-layered media, only by the matrix-multiplication.

It is noteworthy that the vibration problem in the interior of a semi-infinite elastic medium may be one of fundamental researches in investigating the dynamic-load-displacement behaviour of anchors embedded in the ground. Because, as far as the authors know, few works kave been done so far on the study of the dynamic-load-displacement behaviour of an anchor embedded in a semi-infinite elastic medium.

REFERENCES

1) Ewing, W. M., W.S. Jardetzky and F. Press : Elastic Waves in Layered Media, McGraw-Hill, 1957.
2) Pekeris, C. C. : The Seismic Buried Pulse, Proceedings of The National Academy of Society of the U.S.A., Vol.41, pp.629-639, 1955.
3) Awojobi, A. O. and O. A. Sobayo : Ground Vibration due to Seismic Detonation of a Buried Source, Earthquake Engineering and Structural Dynamics, Vol.5, pp.131-143, 1977.
4) Matsuoka, O. and K. Yahata : The Solution Subjected to the Harmonic Point Force in the Interior of a Semi-Inifinte Elastic Medium, Proceedings of the 5th Japan Earthquake Engineering Symposium, pp.425-432, 1978.
5) Takatani, T., Y. Kitamura and S. Sakurai : Vibrations of an Elastic Half-Space due to Buried Sources, Proceedings of the 6th Japan Earthquake Engineering Symposium, pp.1609-1616, 1982.