A STUDY ON OVERTURNING VIBRATION OF RIGID STRUCTURES

by

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SUMMARY

Rocking and overturning of rigid bodies are the typical phenomena to be observed during earthquake. Rigid columns to make rocking motions on a rigid base is a basic model for investigations of such phenomena. In this paper the periodic motion and it's stability of such a basic model excited by sinusoidal vibrations are examined and compared with the experiments and numerical simulations. The reduction of energy due to the impact of collisions between a column and a base will be considered as a impulsive damping ratio.

INTRODUCTION

Fig.1 shows the model to be considered, in which slip or jump are assumed not to occur and the base is excited by sinusoidal horizontal vibrations. The periodic motions of such models were studied by Muto, Sonobe and Umemura[2], but the effects of energy reduction due to collisions have not been considered. This energy reduction can be simply expressed by a impulsive damping ratio of rotating angular velocity of a column as follows:

$$\dot{\theta} \rightarrow \delta \times \dot{\theta} \text{ at } \theta=0 \quad 0<\delta \leq 1 \tag{1}$$

The values of δ may be estimated by the theories[1-3] or the measurements of free rocking motions. Now, the equation of motion of the model is exactly given by

$$\ddot{\theta}=n^2\ddot{u}\cos(\theta\pm\alpha)+n^2 g\sin(\theta\pm\alpha),\quad \theta\lessgtr 0\ ,\qquad n^2=\frac{MR}{I_G+MR^2} \tag{2}$$

But here, a more simple form by an approximation for small values of $\theta\pm\alpha$ is used.

$$\ddot{\theta}=n^2\ddot{u}+\lambda^2(\theta\pm\alpha),\quad \lambda=n\sqrt{g}\,,\ \theta\lessgtr 0 \tag{3}$$

where g is the acceleration due to gravity.

PERIODIC SOLUTIONS

The general solution of Eq.(3) for the excitation $\ddot{u}=a\sin\omega t$ can be easily obtained(Fig.2):

$$y_i=(-1)^i+A_i e^{\frac{\tau-\tau_i}{\sigma}}+B_i e^{-\frac{\tau-\tau_i}{\sigma}}-C\sin\tau\ ,\quad \tau_i\leq\tau\leq\tau_{i+1},\ i=0,1,2,\cdots \tag{4}$$

in which all variables are represented by dimensionless quantities using the transformations

$$\omega t=\tau,\ \frac{\theta}{\alpha}=y,\ \frac{\omega}{\lambda}=\sigma,\ \frac{a}{g\alpha}=k,\ \frac{k}{\sigma^2}=m\ ,\ C=\frac{k}{1+\sigma^2} \tag{5}$$

The joinning of the adjacent solutions y_i and y_{i+1} is expressed by

$$y_i(\tau_{i+1})=y_{i+1}(\tau_{i+1})=0\ ,\qquad \dot{y}_{i+1}(\tau_{i+1})=\delta\cdot\dot{y}_i(\tau_{i+1}) \tag{6}$$

Now, considering the basic type of periodic motion, in which the period between adjacent collisions is equal to a half of period of the excitation and the motion is symmetric with respect to the position $y=0$, then, the additional requirements for the periodic solutions are given by

$$\tau_i=\bar{\tau}+i\pi\ ,\qquad y_i(\tau)=-y_{i+1}(\tau+\pi),\ \tau_i\leq\tau\leq\tau_{i+1} \tag{7}$$

and then

$$A_i=(-1)^i\bar{A},\ B_i=(-1)^i\bar{B}\quad (\bar{A},\bar{B}\text{:unknown constants}) \tag{8}$$

From these equations the simultaneous equation in the unknowns $\bar{A},\bar{B}$ and $\bar{\tau}$ can be derived, and the phase difference $\bar{\tau}$ is found to satisfy

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$$P\cos\bar{\tau}+\sin\bar{\tau}=Q \quad \text{where} \quad P=r\sigma d,\ Q=\frac{rd^2}{C}=\frac{1+\sigma^2}{k}rd^2,\ r=\frac{1-\delta}{1+\delta}\ ,\ d=\tanh\frac{\pi}{2\sigma} \tag{9}$$

Therefore, the following relation is required for the possibility of periodic solutions.

$$P^2+1\geq Q^2 \quad \text{namely} \quad k\geq\frac{(1+\sigma^2)rd^2}{\sqrt{1+r^2\sigma^2d^2}} \tag{10}$$

At this time two types of periodic solution are possible corresponding to two values of $\bar{\tau}$ given by Eq.(9), and when $r\to0$, one has the same phase($\bar{\tau}=0$) as the excitation and the other the opposite($\bar{\tau}=\pi$), and here, even if $r\neq0$, these are refered to as the type of the same phase($i_p=1$) and of the opposite phase($i_p=-1$), respectively. The form of solutions can be obtained as follows:

$$(-1)^i y_i=1+\frac{d-1}{2}\left\{\frac{C(Q-i_pP\sqrt{P^2+1-Q^2})}{d(P^2+1)}+1\right\}e^{\frac{\tau-\tau_i}{\sigma}}+\frac{d+1}{2}\left\{\frac{C(Q-i_pP\sqrt{P^2+1-Q^2})}{d(P^2+1)}\right.$$
$$\left.-1\right\}e^{-\frac{\tau-\tau_i}{\sigma}}-(-1)^iC\sin\tau \qquad \tau_i\leq\tau\leq\tau_{i+1} \tag{11}$$

From the assumption on the period between collisions, these solutions must not have a zero between adjacent collisions. This condition is approximately given by using the value at $\tau=\tau_i+\pi/2$ as follows(Fig.3):

$$y_{mid}=1-\frac{1}{\cosh\frac{\pi}{2\sigma}}-C\cos\bar{\tau}>0 \tag{12}$$

Fig.4 shows the approaching of the transient motions calculated by use of Eq.(2) or Eq.(3) to the periodic solutions. It is seen that the larger the value of damping ratio, the more quickly the transient motion approaches to the periodic.

STABILITY OF SOLUTIONS

From the physical point of view it is important if the solutions are stable. For this, the small deviations of the coefficients of periodic solutions are to be considered as follows:

$$A_i=(-1)^i\bar{A}+\Delta A_i\ ,\quad B_i=(-1)^i\bar{B}+\Delta B_i\ ,\quad \tau_i=\bar{\tau}+i\pi+\Delta\tau_i\ ,\quad \Delta z_i=(-1)^i\Delta\tau_i \tag{13}$$

Using Eqs.(7),(6) and (13), and omitting the small terms higher than second order, the difference equations in ΔA_i, ΔB_i and $(-1)^i\Delta\tau_i$ can be derived. Assuming that the solutions of this equations have the form of $\Delta A_i=X\xi^i$, $\Delta B_i=Y\xi^i$, $\Delta z_i=Z\xi^i$, then, a characteristic equation can be obtained as follows:

$$\begin{vmatrix} 1 & 1 & -C\cos\bar{\tau} \\ \xi-e^{\pi/\sigma} & \xi-e^{-\pi/\sigma} & H_1 \\ \xi-\delta e^{\pi/\sigma} & -\xi+\delta e^{-\pi/\sigma} & H_2 \end{vmatrix}=0 \tag{14}$$

where

$$H_1=\frac{1}{\sigma}(e^{\pi/\sigma}\bar{A}-e^{-\pi/\sigma}\bar{B})(1+\xi)\ ,\quad H_2=\frac{\delta}{\sigma}(e^{\pi/\sigma}\bar{A}+e^{-\pi/\sigma}\bar{B})(1+\xi)+\sigma(1-\delta)\xi C\sin\bar{\tau} \tag{15}$$

Now, the stability of periodic solutions may be given by that

$$\Delta A_i \text{ etc.} \to 0 \quad \text{when } i\to\infty \tag{16}$$

Thus, the roots of Eq.(14) are all required to satisfy that $|\xi|<1$. This condition is given by the following inequalities.

$$\delta<1 \quad \text{and} \quad 0<\frac{(1+P^2)\sqrt{P^2+1-Q^2}}{(1+r^2)(1-d^2)[\sqrt{P^2+1-Q^2}-i_p(Q/P)]}<1 \tag{17}$$

If $i_p=1$, it can be seen that the above relation can't hold for all values of σ and r, namely, the periodic solution of the type of the same phase may always be unstable. On the other hand, the condition of stability for the type of the opposite phase($i_p=-1$) can be obtained as follows:

$$m < \frac{(1+\sigma^4)d}{\sigma^3 f(\sigma)} \sqrt{P^2 + (f(\sigma)-1)^2}, \quad \text{where} \quad f(\sigma) = (1+P^2) - (1+r^2)(1-d^2) \tag{18}$$

EXPERIMENTS AND NUMERICAL SIMULATIONS

Fig.5 shows a test model with the parameters δ=0.91,A=10cm,H=24cm,α=22.6^0,λ=6.73sec^{-1},M=6.14g·sec^2/cm,I_G=690g·sec^2·cm. In Fig.6, m_1 shows the lower limit of the existance for periodic motions of this model given by Ineq.(10),m_2 the upper limit of the existance for the type of the same phase by Ineq.(12), m_3 the upper limit of the stability for the type of the opposite phase by Ineq.(18), and m_0 the beginning condition of rocking motion. The model was excited by sinusoidal vibrations with constant displacement amplitude as shown in Fig.6 and frequency raising as slowly as possible, and the rotating angle of the model and the displacement of shaking table were electrically measured.

Fig.7 shows theoretical response curves obtained by Eq.(11) and the measured values, in which real and dotted lines show the stable and unstable solutions, respectively. The jump phenomena near the limit of existance or stability can be seen in the results of experiments in accordance with the theoretical curves as is shown in P (↓:stop of the motion) or Q,R (↑: overturning). Fig.8 shows the records of rocking and table displacement in P and Q.

The responses of this model were simulated by numerical method using the same parameters as shown in Fig.9. This figure also shows the jump phenomena, that is, the stop of motion near the limit of existance and the overturning near the limit of stability for periodic solutions.

The stability and jump phenomena in periodic motions as described above may be of interest from the point of view of non-linear vibration due to earthquake. The results presented here would be used for investigations of the overturning of equipments. Transient behaviours and their relation with periodic solutions would be the subject for a future study.

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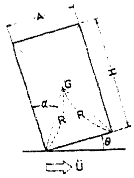
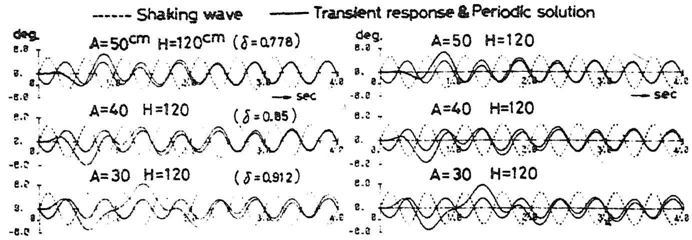


Fig.1 Rocking model



(a) Case of Eq.(3) (b) Case of Eq.(2)

Fig.4 Comparison of transient & periodic solutions

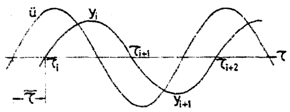


Fig.2 A diagram of rocking motion

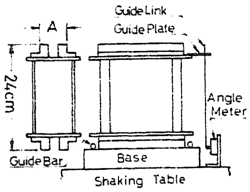


Fig.5 Test model

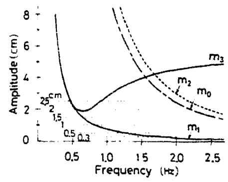


Fig.6 Stability zone of the test model

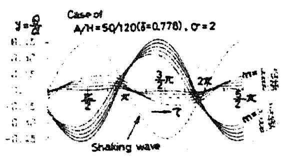
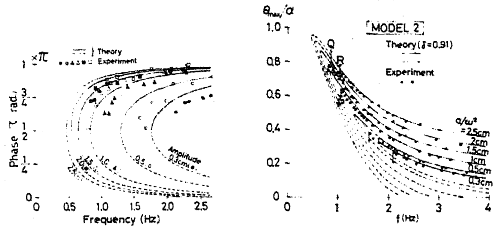
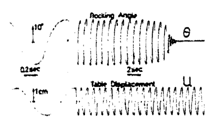


Fig.3 Two types of periodic solution

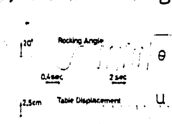


(a) Phase (b) Amplitude

Fig.7 Results of experiment



(b) At P in Fig.7 (b)



(a) At Q in Fig.7 (b)

Fig.8 Records of experiment

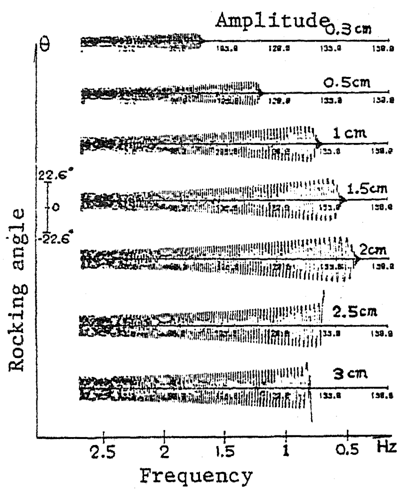


Fig.9 Calculated responses of the same model