SEISMIC RESPONSE OF STRUCTURES WITH RANDOM PARAMETERS

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SUMMARY

Mass, stiffness and damping characteristics of structures are uncertain quantities and thus are random variables. To evaluate the effect of uncertainties of these structural parameters on the structural design response and acceleration floor response spectra, analytical and simulation approaches are presented. In the analytical approach, the nonlinear response expression is approximated by a Taylor series in which terms up to second order derivatives are retained. An analytical - cum - simulation approach is described which can be used to establish validity of the analytical approach. A comparison of floor response spectra results obtained by the two approaches point to a need for further investigation for development of the analytical approaches by inclusion of higher order terms in the series.

INTRODUCTION

In order to establish a reliable seismic design response value for a structure and seismic design input for supported subsystems it is important to consider the effect of uncertainties of structural mass, stiffness and damping values on the response. Especially, the effect of these uncertainties on the seismic input for the secondary systems (floor response spectra) can be quite significant because of amplification effect of the structure on a ground motion. Some impirical approaches are commonly used to include this effect (such as widening of floor response spectra peaks). In the past, simulation approaches have been undertaken to investigate this effect; see ref. 1. Herein, an analytical approach is being proposed to incorporate these parameteric uncertainties in the design response. Also, a modified analytical - cum - simulation approach has been described for verification of the analytical approach.

ANALYSIS

Response of Primary Structure

Seismic design response in a structural member can be expressed in terms of dynamic characteristics of the structure like frequency, mode shapes and participation factors and seismic inputs like ground response spectra as follows (2):

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$$R_q^2 = \sum_{j=1}^{N} q_j^2\gamma_j^2R^2(\omega_j)/\omega_j^4 + 2\sum_{j=1}^{N}\sum_{k=j+1}^{N} q_jq_k\gamma_j\gamma_k$$

$$[(\overline{A}_{jk} + \overline{B}_{jk})R^2(\omega_j)/\omega_j^4 + (\overline{C}_{jk} + \overline{D}_{jk})R^2(\omega_k)/\omega_k^4] \quad (1)$$

in which ω_j is the jth natural frequency; $R(\omega_j)$ is the ground response spectrum value at frequency ω_j and corresponding modal damping β_j; γ_j is a modal participation factor defined as $\phi_j^{\prime}[M]\{r\}$ in terms of mass normalized relative displacement modeshape ϕ_j, mass matrix [M] and influence vector $\{r\}$; $\{q_j\}$ is the jth mode shapes of the response quantity q which is related to the relative displacement modeshape ϕ_j through a transformation matrix as $\{q_j\} = [T]\{\phi_j\}$; factors A_{jk}, B_{jk}, etc., depend upon the frequency ratio = ω_j/ω_k and modal damping coefficients. These factors are explicitly defined in ref. 2. The inclusion of the double summation term in Eq. 1 may be important whenever two structural frequencies are close to each other. It is desired to obtain the variability (expressed as variance or standard deviation) of the response due to variability in the structural mass, stiffness and damping characteristics. To obtain this variability either the Monte Carlo type simulation approach or analytical approach can be used.

In the analytical approach, the statistical moments of the response, R_q, are obtained in terms of statistical moments of the basic variables. Each term in Eq. 1 depends upon the variables which affect the mass, damping and stiffness characteristics of the structure. The variables which effect mass and stiffness could be taken as variables like mass density, dimension and elastic modulus. Or to simplify, one may like to assign variabilities to the mass and stiffness matrices of the elements or a group of elements. Such an approach may be desirable when the structural model to be analyzed consists of widely different element characteristics. For example in a combined soil-structure model, different random variables may be assigned to soil, concrete or steel elemental mass and stiffness matrices. Herein, however a simplified case is described where it is assumed that the system mass and stiffness matrices are random variables such that

$$[M] = [\overline{M}]U_m, \quad [K] = [\overline{K}]U_k \quad (2)$$

in which $[\overline{M}]$ and $[\overline{K}]$ are the mean mass and stiffness matrices, and U_m and U_k are the random variables with unit means and σ_m and σ_k as their standard deviations. This assumes that the mass and stiffness characteristics of each element are perfectly and positively correlated. It is also assumed that U_m and U_k are independent, although a conspicuous dependence between these variables could be incorporated with some modifications.

Similarly for damping, each modal damping could be assumed to be a random variable. However, herein it is assumed that damping values in all modes are the same and they are characterized by a single random variable β with the mean value of $\overline{\beta}$ and the standard deviation of σ_β.

The response R_q defined by Eq. 1 can now be expressed in terms of a series from which mean and variance of R_q can be obtained as follows

$$E[R_q] \cong g(\overline{U}_1,\overline{U}_2,\overline{U}_3) + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial^2 g}{\partial U_i^2}\right)\sigma_i^2 \quad (3)$$

$$\mathrm{var}\ [R_q] \cong \frac{1}{4\bar{g}^2} \sum_i^3 \left(\frac{\partial f}{\partial U_i}\right)^2 \sigma_i^2 - \frac{1}{16\bar{g}^2} \left[\sum_i^3 \left\{\frac{\partial^2 f}{\partial U_i^2} - \frac{1}{2\bar{g}^2} \left(\frac{\partial f}{\partial U_i}\right)^2\right\}\sigma_i^2\right]^2 \quad (4)$$

Herein, only second order terms in the series have been retained. In Eqs. 3 and 4, $g \equiv R_q$, $f \equiv R_q^2$ = right hand side of Eq. 1 and $U_1 \equiv U_m$, $U_2 \equiv U_k$ and $U_3 \equiv \beta$. Thus $\sigma_1 = \sigma_m$, $\sigma_2 = \sigma_k$ and $\sigma_3 = \sigma_\beta$. All derivatives in this equation and subsequent analysis are obtained at the mean values of U_is. In cases where frequencies are close and the double summation terms contribute significantly to the response, it may be necessary to retain the second order terms in Eqs. 3 and 4; otherwise these may be dropped.

To use Eqs. 3 and 4 it is required to obtain the derivatives of the functions g and f with respect to U_i's. These functions depend upon q_j, γ_j, and ω_j. Thus it is necessary to obtain the partial derivatives of these quantities with respect to U_i's. These derivatives can be obtained as described by Fox and Kapoor (3), as

$$\frac{\partial \omega_j}{\partial U_j} = \frac{1}{2\omega_j} \{\phi_j\}' \left[\frac{\partial [K]}{\partial U_i} - \lambda_j \frac{\partial [M]}{\partial U_i}\right]\{\phi_j\}$$

$$\frac{\partial}{\partial U_i}\{\phi_j\} = -\frac{1}{2}\left[\{\phi_j\}' \frac{\partial [M]}{\partial U_i}\{\phi_j\}\right)\{\phi_j\} + \sum_{\substack{k=1\\k\neq j}}^{N} \frac{\{\phi_k\}' \left[\frac{\partial [K]}{\partial U_i} - \omega_j^2 \frac{\partial [M]}{\partial U_i}\right]\{\phi_j\}}{(\omega_j^2 - \omega_k^2)}\{\phi_k\} \quad (5)$$

A prime over a vector represents its transpose. In terms of these derivatives the derivatives for the modeshape of a response quantity $\{q_j\}$ and the participation factor γ_j can be defined as follows:

$$\frac{\partial}{\partial U_i}\{q_j\} = [T]\frac{\partial}{\partial U_i}\{\phi_j\} + \left(\frac{\partial}{\partial U_i}[T]\right)\{\phi_j\}$$

$$\frac{\partial \gamma_j}{\partial U_i} = \left(\frac{\partial}{\partial U_i}\{\phi_j\}'\right)[M]\{r\} + \{\phi_j\}'\frac{\partial [M]}{\partial U_i}\{r\} \quad (6)$$

For the variables U_m and U_k defined by Eq. 2, Eqs. 5 and 6 reduce to the following:

$$\frac{\partial \omega_i}{\partial U_m} = -\frac{\omega_i}{2}, \quad \frac{\partial \omega_i}{\partial U_k} = \frac{\omega_i}{2}, \quad \frac{\partial}{\partial U_m}\{\phi_j\} = -\frac{1}{2}\{\phi_j\}$$

$$\frac{\partial}{\partial U_k}\{\phi_j\} = 0, \quad \frac{\partial \gamma_j}{\partial U_m} = \frac{\gamma_j}{2}, \quad \frac{\partial \gamma_j}{\partial U_k} = 0 \quad (7)$$

$$\frac{\partial}{\partial U_m}\{q_k\} = -\frac{1}{2}\{q_k\}, \quad \frac{\partial}{\partial U_k}\{q_k\} = \{q_k\}$$

In the derivation of these equations the orthonormality of modes with respect to [K] and [M] matrices has been exploited. These partial derivatives with respect to the damping coefficient variables are, of course, zero if proportional damping assumption has been made.

The derivatives of response spectrum values are also required, and these can be shown to be:

$$\frac{\partial R(\omega_j)}{\partial U_m} = -\frac{\omega_j}{2}\frac{\partial R(\omega_j)}{\partial \omega_j}, \qquad \frac{\partial R(\omega_j)}{\partial U_k} = +\frac{\omega_j}{2}\frac{\partial R(\omega_j)}{\partial U_k} \tag{8}$$

in which $\frac{\partial R(\omega_j)}{\partial \omega_j}$ can be obtained from the response spectra curves provided as design input. For Newmark type of ground response spectra these derivatives and derivatives with respect to damping coefficients can be obtained from the equations provided in Ref. 4.

Factors A_j, B_j etc. are defined in terms of frequency ratio r_j and damping coefficients, and thus are affected by U_k, U_m and β. Their derivatives can be obtained from expressions defined in ref. 2. The derivative expressions are involved, but do not present any computational difficulty.

It is of interest to note that in view of Eqs. 7 the partial derivatives of $(q_j^2\phi_j^2)$ and $(q_j q_k \phi_j \phi_k)$ are zero with respect to U_m, U_k and β. This simplifies the expression of the derivative of R_q as follows:

$$\begin{aligned}\frac{\partial R_q^2}{\partial U_m} = &- \sum_{j=1}^{n} 2\gamma_j^2 q_j^2 \omega_j \frac{\partial}{\partial \omega_j}(R^2(\omega_j)/\omega_j^4) \\ &+ 2\sum_{j\neq k}\sum \gamma_j\gamma_k q_j q_k\Big[\frac{\partial(A_{jk}+B_{jk})}{\partial r_j}\frac{\partial \gamma_j}{\partial U_m}R^2(\omega_j) + \frac{\partial(C_{jk}+D_{jk})}{\partial r_j}\frac{\partial r_j}{\partial U_m}R^2(\omega_k) \\ &- 2(A_{jk}+B_{jk})\omega_j\frac{\partial}{\partial \omega_j}(R^2(\omega_j)/\omega_j^4) - 2(C_{jk}+D_{jk})\omega_k\frac{\partial}{\partial \omega_k}(R^2\omega_k/\omega_k^4)\Big]\end{aligned} \tag{9}$$

The expression for the derivative of R_q^2 with respect U_k is similar. Second order derivative of R_q^2 required in Eqs. 3 and 4 can also be obtained similarly.

Seismic Design Input for Secondary Systems

Seismic design inputs for secondary supported systems are commonly defined in the form of floor acceleration spectra curves. Because of structural amplification, such inputs are likely to be affected more strongly by the variability of the supporting structural parameters. To evaluate this variability again analytical and Monte Carlo approaches can be used. These are briefly described in the following sections.

Analytical Approach:

Since the analytical expression defining floor spectra has become available now, ref. 2, the use of analytical approach, which is similar to the one discussed in connection with the variability of structural response, can be made and seems attractive. The expression defining the design floor response spectrum value for floor u at an oscillator frequency, ω_o and damping β_o can be written as

$$R_u^2(\omega_o) = \sum_{j=1}^{N} \gamma_j^2\phi_j^2 \, [\bar{A}_j R^2(\omega_j) + \bar{B}_j R^2(\omega_k)]$$
$$+ 2 \sum_{j\neq k}\sum \gamma_j\gamma_k\phi_j\phi_k \, [\bar{C}_{jk}R^2(\omega_o) + \bar{D}_{jk}R^2(\omega_j) + \bar{E}_{jk}R^2(\omega_k)] \qquad (10)$$

in which $\bar{A}_j$, $\bar{B}_j$, $\bar{C}_{jk}$, $\bar{D}_{jk}$, $\bar{E}_{jk}$ are the factors which are defined in terms of frequency ratio $r_1 = \omega_j/\omega_o$ and $r_2 = \omega_k/\omega_o$ and oscillator and structural damping values of β_o and β, respectively. The expressions for these factors are given in ref. 2. (Notations are different in this reference.) With f now defined by the right hand side of Eq. 10, the mean and variances of the response spectrum value can be obtained by Eqs. 3 and 4. The approach requires the first and second order derivatives of factors A_j, B_j which can be obtained for the expression defined in ref. 2. The algebraic manipulations, though straightforward are significantly more involved. The details of these will not be provided here because of lack of space. The numerical results indicate that it is necessary to include at least the second order terms to obtain the correct mean and, more importantly, the variance of the response spectrum values. It is so because of the highly nonlinear nature of the function f in Eq. 10, especially for spectrum values near the structural frequencies. As indicated by numerical results presented later, it even seems to be necessary to include the third and maybe even the fourth-order terms in the series expansion of the function. This makes algebraic manipulation significantly more involved. However, as the analytical approach to obtain this variability has distinct advantages over the Monte Carlo approach, it is still desirable to improve the approach by inclusion of more terms. More research work in this direction is required.

Monte Carlo Approach:

To check the results obtained by the analytic approach, a modified Monte Carlo approach has been used. (A direct Monte Carlo approach in which sample values of [M] and [K] are generated to define a population of systems can become prohibitively expensive.) This approach which is essentially a combination of analytical and simulation approaches, is briefly described now.

The dynamic characteristics ω_j, ϕ_j and γ_j which are required in Eqs. 1 and 10 are functions of mass and stiffness variables U_m and U_k,

$$\omega_j = h(U_m, U_k) \quad , \quad \phi_j = g(U_m, U_k) \qquad (11)$$

These functions can be expanded in Taylor series about the mean values of variables U_m and U_k. If the variances of U_m and U_k are not large, inclusion of the first order terms may be adequate. Only the first order terms have been considered in the following development. Thus the mean values of ω_j and ϕ_j are the values obtained by eigenvalue analysis of the system with mean values of the mass and stiffness matrices. The standard deviation obtained from the linearized expansion can also be shown to be as follows:

$$\sigma_{\omega_j} = \frac{\omega_j}{2}\sqrt{\sigma_m^2 + \sigma_k^2}\,, \quad \sigma_{\phi_j} = \frac{1}{2}\,\phi_j\sigma_m\,, \quad \sigma_{\gamma_j} = \frac{1}{2}\,\gamma_j\sigma_m \tag{12}$$

Furthermore these characteristics are correlated and the coefficient of correlation between these can be shown to be

$$\rho_{\omega_j\omega_k} = 1\,, \quad \rho_{\phi_j\phi_k} = 1\,, \quad \rho_{\gamma_j\gamma_k} = 1$$
$$\rho_{\omega_j\phi_k} = \frac{\sigma_m}{\sqrt{\sigma_m^2 + \sigma_k^2}}\,, \quad \rho_{\phi_j\gamma_k} = -1 \tag{13}$$

in which $\rho_{x,y}$ represents the correlation coefficient between variables x and y. Perfect correlation between some characteristics may be noted.

With these correlations established, the linear mean square estimation theory can be used to obtain one variable if another variable is known. For the values of standard deviations and correlation coefficients in Eqs. 16 and 17, it can be shown that the value of one variable in terms of another perfectly correlated variable value can be written as follows

$$\gamma_k = \frac{\overline{\gamma}_k}{\overline{\gamma}_j}\,\gamma_j\;; \quad \omega_k = \frac{\overline{\omega}_k}{\overline{\omega}_j}\,\omega_j\,, \quad \phi_k = \frac{\overline{\phi}_k}{\overline{\phi}_j}\,\phi_j$$
$$\phi_i = -\frac{\overline{\phi}_i}{\overline{\gamma}_j}\,\gamma_j + 2\overline{\phi}_i\;; \quad \gamma_j = -\frac{\overline{\gamma}_j}{\overline{\phi}_i}\,\phi_i + 2\overline{\gamma}_j \tag{14}$$

in which a bar over a quantity represents its mean value.

Natural frequency and modeshape values are however not perfectly correlated. These values are therefore generated by generating independent values of some related variables with appropriate means and variances and transforming them back to correlated values through the modal matrix of the covariance matrix of ω_j and ϕ_j. This approach gives theoretically correct values if ω_j and ϕ_j are assumed to be jointly normal. The values of γ_j's and other ϕ_j's are then obtained using Eq. 14. A set of values of $\omega_j, \gamma_j, \phi_j$ so obtained are then used to generate a sample floor response spectra. These sample floor response spectra are then statistically analyzed to obtain the mean and variance of a floor spectrum.

It may be noted that eigenvalue analysis is not required to be carried for each set of sample values of the variables, U_m and U_k, yet the correlation between the dynamic characteristic is included in generating their values.

NUMERICAL RESULTS

Numerical results for floor spectra were obtained for the structural and earthquake input model used in ref. 2 by the analytical approach and the Monte Carlo procedure. Monte Carlo approach was used only for a few oscillator period values because of large computational cost involved. Table 1 shows the mean and standard duration of the floor response spectrum values at two selected periods. The values obtained with the 1st order and 2nd order approximations of the series expansion are shown for comparison along with the values obtained by numerical simulation.

The results though not conclusive and complete do point out the need for further improvement in the analytical approach. The results for 1st order and 2nd order approximations do not seem to differ much for the values presented in the table, though larger significant differences can be expected near structural frequencies where a response spectrum shows dominant peaks. Unfortunately, some unresolved formulational and numerical difficulties were encountered in the analytical approach for the resonance case (oscillator period equal to structural period) which also needs further investigation and research and developmental work.

Also the standard deviation obtained by either analytical or Monte Carlo approach can not be used in the calculation of the design response by the mean-plus-k-standard-deviation type of approach, as the probability distribution of the floor response in view of the structural uncertainties is a strongly skewed function near structural periods. Such skewness can be ascertained by obtaining the third order statistical moment of the response. These results must, however, be compared with the simulation approach results with a large sample size. These additional problems are under investigation at this moment.

REFERENCES

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TABLE 1

Mean and Standard Deviation of Floor Acceleration Response Spectrum Value Obtained by Analytical and Simulation Approaches

Oscillator Period	Mean Value, G-Units			Std. Deviation, G-Units		
	Analytical Approach		Simulation	Analytical Approach		Simulation
Secs	1st order	2nd order		1st order	2nd order	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
.09	1.1015	1.0710	1.1117	1.0474	1.0470	.5407
.10	.6199	.6114	.8165	.2772	.2771	.4969