STOCHASTIC PREDICTION OF MAXIMUM EARTHQUAKE

RESPONSE OF HYSTERETIC STRUCTURES

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SUMMARY

An approximate analytical method is presented for determining the stochastic properties of the maximum response of both linear and nonlinear hysteretic systems subjected to nonstationary random earthquake excitation. The maximum response is defined as a continuous random process in time and is described in the form of an appropriate first-order quasi-linear differential equation. It is shown that this expression of the maximum response is useful in analytical treatment of the problem. Numerical examples are given for the typical bilinear hysteretic system including the linear system. Results of the approximate analytical method are compared with results obtained by corresponding digital simulation.

INTRODUCTION

For building structures subjected to random earthquake excitations, it is one of the most important problems to evaluate the structural reliability that the building function will be performed satisfactorily with no more than superficial damage during an earthquake of moderate intensity and the structure will be able to resist a strong earthquake without extreme damage or collapse. The maximum value of displacement response during the excitation is the significant and simplest response measure representing the damage in the structure, and is closely related to the first-passage failure. Exact solutions to the maximum response problem or the first-passage problem have not yet been found and several approximate methods have been developed [1-5]. Despite the obvious importance of hysteretic nonlinearity in structural engineering, most of available methods have centered on linear systems [1-3]. Other methods have been proposed for a special class of nonlinear systems [5].

The object of this study is to present an approximate analytical method for determining the maximum displacement response of hysteretic system subjected to nonstationary random excitation. It is shown that the maximum response defined as continuous random process is described in the form of quasi-linear differential equation. It is also shown that this expression makes it possible to extend the analytical procedure recently proposed by the authors for determining the stochastic response, such as displacement and velocity, of nonlinear hysteretic systems [6]. The stochastic analysis of the maximum or minimum displacement response have been presented in Ref. 7. In this study the maximum absolute value process of the displacement response is mainly discussed. The analytical method is illustrated by examples for a linear system and bilinear hysteretic system subjected to nonstationary Gaussian white noise. The results of the proposed method are compared with those obtained by digital simulation.

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FORMULATION OF MAXIMUM RESPONSE PROCESS

The maximum absolute value $a_m(t)$ of the displacement response process $x(\tau)$ of a structure within the time interval $0 \leq \tau \leq t$ is defined by

$$a_m(t) = \max_{0 \leq \tau \leq t} |x(\tau)| \qquad (1)$$

A sample time-history of the maximum absolute displacement process is sketched in Fig. 1. The maximum absolute displacement process is continuous process in time, but the time derivative process of the maximum absolute displacement, $\dot{a}_m(t)$, is discontinuous process. In Fig. 1, it is shown that the maximum absolute displacement is coincident with the absolute displacement in the intervals A and it does not change in the intervals B. The maximum absolute displacement process can be expressed in the form of quasi-linear differntial equation:

$$\dot{a}_m = |\dot{x}|\, U(x\dot{x})\, U(|x| - a_m) \qquad (2)$$

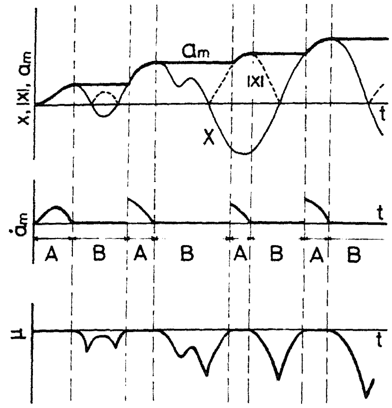

Fig. 1 Sketch for maximum absolute displacement process.

where $U(s) = 1$ for $s \geq 0$, and $= 0$ for $s < 0$. By introducing the new state variable μ defined in the range $(-\infty, 0]$ as

$$\mu = |x| - a_m \qquad (3)$$

Eq. (2) is written in terms of the variable μ as follows:

$$\dot{\mu} = \mathrm{sgn}(x)\, \dot{x} - |\dot{x}|\, U(x\dot{x})\, U(\mu) \equiv g_\mu(x, \dot{x}, \mu) \qquad (4)$$

where sgn(·) denotes the signam function. In the similar way, the maximum and minimum displacement processes are described as [7]

$$x_{max}(t) = \max_{0 \leq \tau \leq t} x(\tau) \qquad \dot{x}_{max} = \dot{x}\, U(\dot{x})\, U(x - x_{max}) \qquad (5)$$

and

$$x_{min}(t) = \min_{0 \leq \tau \leq t} x(\tau) \qquad \dot{x}_{min} = \dot{x}\, U(-\dot{x})\, U(-x + x_{min}) \qquad (6)$$

The above expressions of the maximum absolute displacement, the maximum and minimum displacement processes facilitate an acculate analytical treatment of the first passage problems appearing in structúral dynamics. To simplify the problem the structural system considered here is supposed to have a symmetrical restoring force characteristic and the stochastic excitation is also symmetrically distributed with respect to zero. In this situation, the damage associated with the excursion type failure to be expected in the structure may be closely related to the maximum absolute displacement response rather than the maximum or minimum displacement response. The maximum and minimum displacement responses are thus omitted in the following analysis.

STOCHASTIC ANALYSIS OF MAXIMUM RESPONSE

The structural system considered is the well-known bilinear hysteretic system. The nondimensional equation of motion of a single-degree-of-freedom system is given by

$$\ddot{x} + 2h\dot{x} + \Phi(x, \dot{x}) = \alpha(t)\cdot n(t) \tag{7}$$

where x denotes the nondimensional displacement with reference to the elastic limit deformation, h is the critical damping ratio. $\alpha(t)$ represents a deterministic envelope function, $n(t)$ is a Gaussian white noise with zero mean and the spectral density level S_0 , and $t=\Omega_0 T$ is nondimensional time where Ω_0 is the natural frequency of the associated linear system and T is time. The bilinear hysteretic characteristic $\Phi(x,\dot{x})$, which is normalized to have unit rigidity for the first branch and rigidity r for the second branch, is expressed as

$$\Phi(x,\dot{x}) = rx + (1-r)z \tag{8}$$

where z is the state variable defined in the range [-1, 1] and can be expressed in the following quasi-linear differential equation [6]:

$$\dot{z} = \dot{x} - g_z(\dot{x}, z) \tag{9}$$

In the above equation,

$$g_z(\dot{x}, z) = \dot{x}[U(\dot{x})U(z-1) + U(-\dot{x})U(-z-1)] \tag{10}$$

By introducing the state variable μ as an element of a Markov vector process in addition to x , $y\,(=\dot{x})$ and z , the joint probability density function $p(x,y,z,\mu;t)$ is governed by the Fokker-Planck equation

$$\mathcal{L}[p] = \frac{\partial p}{\partial t} + y\frac{\partial p}{\partial x} - y\frac{\partial}{\partial y}\{[2hy+rx+(1-r)z]p\} - \frac{\partial}{\partial z}\{[g_z(y,z)-y]p\}$$
$$-\frac{\partial}{\partial \mu}[g_\mu(x,y,\mu)p] - \frac{S(t)}{2}\frac{\partial^2 p}{\partial y^2} = 0 \tag{11}$$

where

$$S(t) = 2\pi S_0\,\alpha^2(t) \tag{12}$$

Equation (12) may be approximately solved by expanding the unknown probability density function in terms of the multi-dimensional Hermite polynomials, but it is rather complicated to take account of the higher order terms than the second. Therefore, an approximate analysis is efficiently performed by supposing a simplified form of the joint probability density function $p(x,y,z,\mu;t)$. By taking account of the non-Gaussian distribution characteristics of the state variables z and μ , which are evident from their definitions, the approximate probability density function is assumed to be the following form:

$$\bar{p}(x,y,z,\mu;t) = D_z D_\mu\, w(x,y,z,\mu;t) + D_z\,\delta(\mu)\int_0^\infty w(x,y,z,\mu';t)\,d\mu'$$
$$+ D_\mu\,\delta(z+1)\int_{-\infty}^{-1} w(x,y,z',\mu;t)\,dz' + D_\mu\,\delta(z-1)\int_1^\infty w(x,y,z',\mu;t)\,dz'$$
$$+ \delta(\mu)\,\delta(z+1)\int_{-\infty}^{-1}dz'\int_0^\infty d\mu'\,w(x,y,z',\mu';t) + \delta(\mu)\delta(z-1)\int_1^\infty dz'\int_0^\infty d\mu'\,w(x,y,z',\mu';t) \tag{13}$$

where

$$D_z = 1 - U(-z-1) - U(z-1)\,, \qquad D_\mu = 1 - U(\mu) \tag{14}$$

and $\delta(\cdot)$ is the Dirac delta function and $w(x,y,z,\mu;t)$ is the normal density function determined by the mean m_μ and the covariance matrix V . It is noted that in general, $m_\mu \neq E[\mu]$ and $V \neq K$ where $E[\mu]$ is the mean value of μ and K is the covariance matrix of the state variables x , y , z and μ . These unknown parameters m_μ and V can be determined by using the method of weighted residuals as follows:

$$\iiiint_{-\infty}^{\infty} x^{l_1} y^{l_2} z^{l_3} \mu^{l_4}\, \mathcal{L}[\bar{p}]\,dx\,dy\,dz\,d\mu = 0 \tag{15}$$

for $l_1+l_2+l_3+l_4 = 1, 2$ and $l_1, l_2, l_3, l_4 = 0, 1, 2$. Equation (15) can be reduced to a set of differential equations for moments $M(l_1, l_2, l_3, l_4)$ with respect to the state variables x, y, z and μ

$$\begin{aligned}\dot{M}(l_1,l_2,l_3,l_4) &= l_1 M(l_1-1,l_2+1,l_3,l_4) + \frac{S(t)}{2} l_2(l_2-1) M(l_1,l_2-2,l_3,l_4) \\ &\quad - l_2\{2h M(l_1,l_2,l_3,l_4) + r M(l_1+1,l_2-1,l_3,l_4) + (1-r) M(l_1,l_2-1,l_3+1,l_4)\} \\ &\quad + l_3\{M(l_1,l_2+1,l_3-1,l_4) - N_z(l_1,l_2,l_3,l_4)\} + l_4 N_\mu(l_1,l_2,l_3,l_4) \end{aligned} \tag{16}$$

where

$$M(l_1,l_2,l_3,l_4) = E[x^{l_1} y^{l_2} z^{l_3} \mu^{l_4}] \tag{17}$$

$$N_z(l_1,l_2,l_3,l_4) = E[x^{l_1} y^{l_2} z^{l_3-1} \mu^{l_4} g_z(y,z)] \tag{18}$$

$$N_\mu(l_1,l_2,l_3,l_4) = E[\{sgn(x) y - g_m(x,y,\mu)\} x^{l_1} y^{l_2} z^{l_3} \mu^{l_4-1}] \tag{19}$$

The operator $E[\cdot]$ means the ensemble average with respect to the assumed probability density function $\bar{p}(x,y,z,\mu;t)$. From Eq. (16), a set of differential equations satisfied by $E[\mu]$ and $\mathbb{K}$ are obtained as

$$\begin{aligned}
&\dot{K}_{xx} = 2K_{xy} && \dot{E}[\mu] = N_\mu(0,0,0,1) \\
&\dot{K}_{yy} = -2[2hK_{yy} + rK_{xy} + (1-r)K_{yz}] + S(t) && \dot{K}_{\mu\mu} = 2\{N_\mu(0,0,0,2) - E\mu\, N_\mu(0,0,0,1)\} \\
&\dot{K}_{zz} = 2[K_{yz} - N_z(0,0,2,0)] && \dot{K}_{x\mu} = K_{y\mu} + N_\mu(1,0,0,1) \\
&\dot{K}_{xy} = K_{yy} - 2hK_{xy} - rK_{xx} - (1-r)K_{xz} && \dot{K}_{y\mu} = -2[2hK_{y\mu} + rK_{x\mu} + (1-r)K_{z\mu}] + N_\mu(0,1,0,1) \\
&\dot{K}_{xz} = K_{yz} + K_{xy} - N_z(1,0,1,0) && \dot{K}_{z\mu} = K_{y\mu} + N_\mu(0,0,1,1) - N_z(0,0,1,1) \\
&\dot{K}_{yz} = -2[2hK_{yz} + rK_{xz} + (1-r)K_{zz}] + K_{yy} - N_z(0,1,1,0) &&
\end{aligned} \tag{20}$$

Since Eq. (20) is a implicit function with respect to m_μ and $\boldsymbol{v}$, Eq. (20) can not be directly solved. By making use of Eqs. (13), (14) and Eqs. (17) through (19), $E[\mu]$, $\dot{E}[\mu]$, $\mathbb{K}$, $\dot{\mathbb{K}}$, N_z and N_μ are expressed in terms of m_μ and $\boldsymbol{v}$. A set of differential equations for m_μ and $\boldsymbol{v}$ is then derived and the nonstationary solution under the zero initial conditions can be obtained numerically.

The mean value $E[a_m]$ and the variance σ_m^2 of the maximum absolute displacement response are given by

$$E[a_m] = \sqrt{\frac{2v_{xx}}{\pi}} - \frac{m_\mu}{2}[\theta - \frac{b}{a}] \tag{21}$$

and

$$\begin{aligned}\sigma_m^2 &= v_{xx}(1-\frac{2}{\pi}) + \frac{v_{\mu\mu}}{2}[\theta + a^2\theta(2-\theta) - b^2 - 2ab(1-\theta)] \\ &\quad + m_\mu[\theta - \frac{b}{a}]\sqrt{\frac{v_{xx}}{\pi}} - 2E[|x|\mu]\end{aligned} \tag{22}$$

where

$$a = \frac{m_\mu}{\sqrt{2v_{\mu\mu}}}, \quad b = \frac{1}{\sqrt{\pi}} e^{-a^2}, \quad \theta = erfc(a)$$

The probability density function $p(a_m;t)$ of the maximum absolute displacement response can be written as

$$p(a_m;t) = C\iiint_{-\infty}^{\infty} \bar{p}(x, y, z, |x|-a_m; t)\, dx\, dy\, dz \tag{23}$$

where C is the normalization constant. The reliability function defined as the probability that the maximum absolute displacement response within the time interval [0, t] is less than a critical level x_F is evaluated by making use of Eq. (23) as follows:

$$R(x_F;t) = \mathrm{Prob}[\,|x(\tau)| < x_F;\ 0 \leq \tau \leq t\,] = \mathrm{Prob}[a_m(t) < x_F]$$
$$= \int_0^{x_F} p(a_m;t)\, da_m \tag{24}$$

NUMERICAL RESULTS

Numerical results obtained by the proposed analytical method are presented for both linear and bilinear hysteretic systems. For purpose of verifying the results of the present approximate method, a digital simulation analysis was performed. The ensemble size for all cases was 500.

Linear System. The analytical results of a linear system to a stationary Gaussian white noise are shown in Figs. 2 to 6 for the cases h = 0.01 and 0.1. In these figures, t_o is the nondimensional natural period of the system, 2π. The mean value $E[a_m]$ and the standard deviation σ_m of the maximum absolute displacement response are shown functions of time in Fig. 2. The standard deviation σ_x of the displacement response is also plotted in Fig. 2. These values are normalized by the stationary value σ_s of σ_x where $\sigma_s = \sigma_x(\infty) = \sqrt{\pi S_o/2h}$. Fig. 3 shows the transition of the probability density function $p(a_m;t)$ with t/t_o . The reliability function is shown as a function of t in Fig. 4 for various values of the normalized critical level x_F/σ_s . The ratios $E[a_m(t)]/\sigma_x(t)$ and $\sigma_m(t)/\sigma_x(t)$ are plotted in Fig. 5. From Eqs. (21) and (22), the ratios have the asymptotic values, respectively, in the limit as t → 0

$$\lim_{t\to 0} \frac{E[a_m(t)]}{\sigma_x(t)} = \sqrt{\frac{2}{\pi}}\ , \qquad \lim_{t\to 0} \frac{\sigma_m(t)}{\sigma_x(t)} = \sqrt{1-\frac{2}{\pi}}$$

It is found that $E[a_m]/\sigma_x$ is significantly influenced by the damping ratio. Fig. 6 shows the comparison of the mean value $E[a_m]$ with the mean value $E[x_{max}]$ of the maximum displacement response. From this figure, it is shown that the difference between $E[a_m]$ and $E[x_{max}]$ increases as the damping ratio increases. The results obtained by the simulation are also plotted by dots in Figs. 2, 4 and 5 and by bar graphs in Fig. 3.

Bilinear Hysteretic System. The analytical results for a bilinear hysteretic system with h = 0.01 and r = 0.1 and 0.5 are shown in Figs. 7, 9 and 10. Also shown are the results of the simulation indicated by dots. Fig. 7 shows $E[a_m]$, σ_m and σ_x under the stationary excitation with various values of S_o . As an example of nonstationary excitation, an envelope function used here is shown in Fig. 8. In Fig. 9, $E[a_m]$, σ_m and σ_x are plotted for the case $2\pi S_o$ = 0.1. Fig. 10 shows the reliability function for various values of x_F .

Generally good agreement exists between the analytical and simulated results for the mean value and the standard deviation of the maximum absolute displacement response. The simulation estimates of reliability function are in good agreement with the analytical curves at high value range of the reliability and for the short duration of excitation. It is found that the analytical method gives satisfactory results for systems with the value of critical damping ratio in a wide range usually used. It is also shown that the present method can be used to predict the mean value and the standard deviation of the maximum absolute displacement response for the bilinear hysteretic system as well as a linear system. However, for the case of strongly nonlinear hysteretic system the method in this study is less accurate. The assumption of the approximate probability density function given by Eq. (13) becomes to be invalid because the probability density of the responses particularly concerened with μ is distorted from the assumed density function. If the more acculate analysis is required, the probability density function of the state variables must be expressed by taking account of the moments or quasi-moments of higher order than the second.

CONCLUSIONS

An analytical method to determine the maximum absolute response of both linear and bilinear hysteretic systems subjected to nonstationary Gaussian white noise has been developed. It has been shown that the maximum absolute displacement process can be expressed in the form of the first-order quasi-linear differential equation. It is emphasized that this expression of the maximum displacement response is extremely useful in analytical treatment of the problem. The nonstationary statistics of responses including the maximum displacement have been found as the solution to a set of the first-order differential equations derived from the Fokker-Planck equation. The probability density function of the maximum absolute response and the reliability function have been evaluated. A comparison of the approximate analytical results with the results obtained by digital simulation has indicated the usefulness of the proposed method in predicting the stochastic properties of the maximum displacement response of both linear and nonlinear hysteretic systems.

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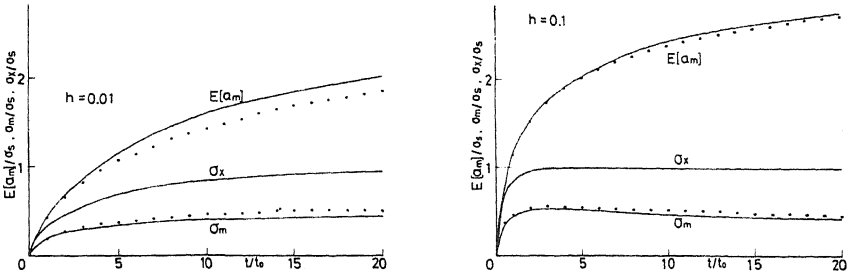
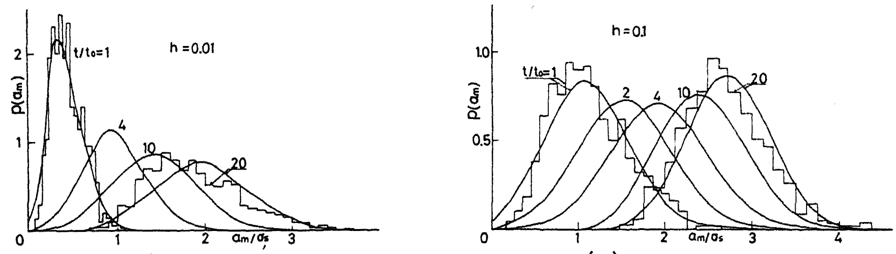


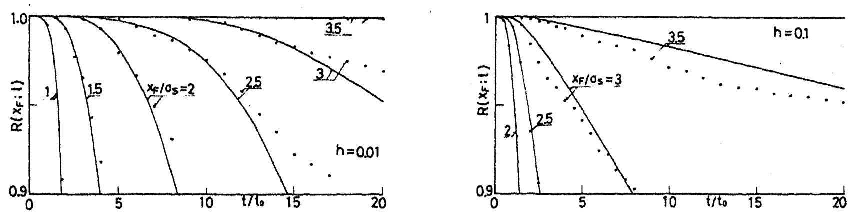
(a) h = 0.01 (b) h = 0.1

Fig. 2 Mean value and standard deviation of maximum absolute displacement response for linear system to stationary excitation.



(a) h = 0.01 (b) h = 0.1

Fig. 3 Probability density function of maximum absolute response.



(a) h = 0.01 (b) h = 0.1

Fig. 4 Reliability function for linear system to stationary excitation.

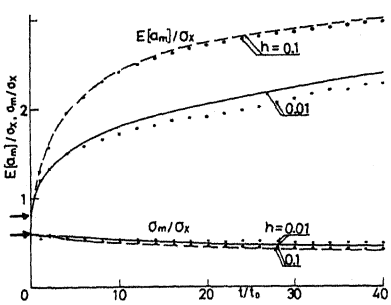


Fig. 5 Ratios $E[a_m]/\sigma_x$ and σ_m/σ_x for linear system to stationary excitation.

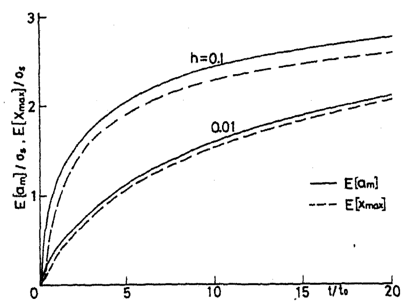
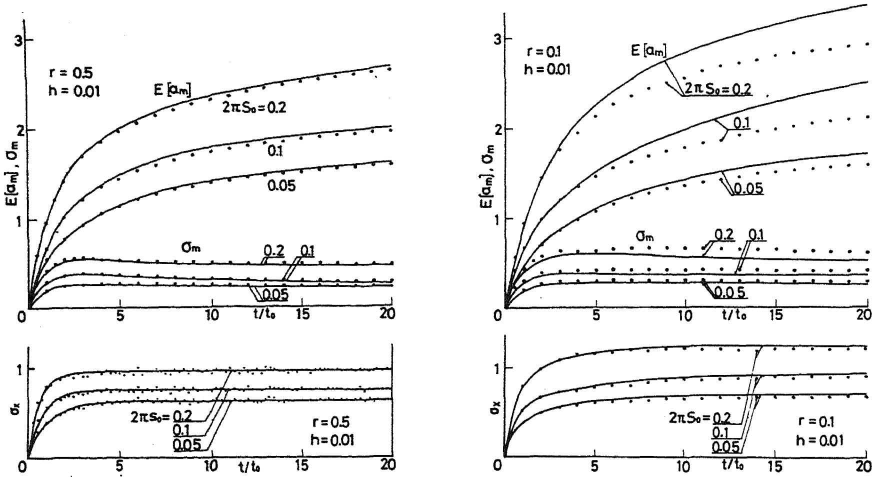


Fig. 6 Mean values $E[x_{max}]$ and $E[a_m]$.



(a) r = 0.5 (b) r = 0.1

Fig. 7 Mean value $E[a_m]$ and standard deviations σ_m and σ_x for bilinear hysteretic system to stationary excitation, h = 0.01, $2\pi S_0$ = 0.05, 0.1, 0.2.

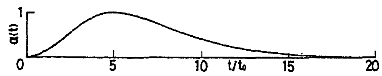
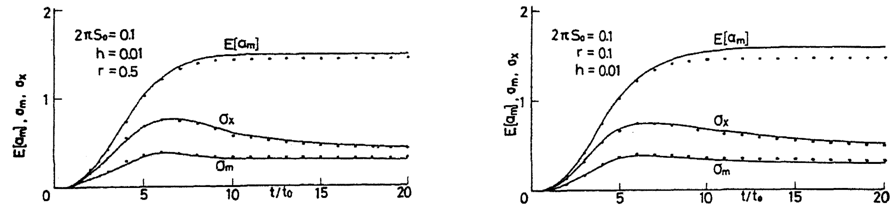
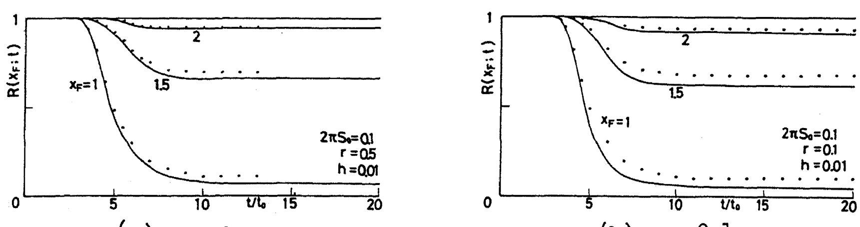


Fig. 8 Envelope function of nonstationary excitation.



(a) r = 0.5 (b) r = 0.1

Fig. 9 Mean value $E[a_m]$ and standard deviation σ_m and σ_x for bilinear hysteretic system to nonstationary excitation, h = 0.01, $2\pi S_0$ = 0.1.



(a) r = 0.5 (b) r = 0.1

Fig. 10 Reliability function for bilinear hysteretic system to nonstationary excitation, h = 0.01, $2\pi S_0$ = 0.1.