NONLINEAR RANDOM RESPONSE OF HYSTERETIC SINGLE-DEGREE-OF-FREEDOM SYSTEM

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SUMMARY

The stationary random response of a single-degree-of-freedom system having "general slip" hysteresis, which lies between "bilinear" hysteresis and "pure slip" hysteresis, when subjected to the Gaussian white excitation, is presented. The approximate solution to an oscillator with the Coulomb damper is initially derived. The correspondence of this Coulomb damping characteristic to the general slip hysteresis is clarified to estimate the random response of such a hysteretic system. Important expressions thus obtained are examined by the digital simulation. Analytical solutions agree with simulation estimates with sufficient accuracy except some special cases.

INTRODUCTION

It is difficult to deal with the nonlinear response of a hysteretic oscillator through the random vibration theory in an analytical way. In the field of the earthquake engineering, however, this is the quite important problem earnestly desired to solve. This paper has been prepared as one of the trials to clear up this hard problem to some extent. To be concrete, this aims at deriving approximately the analytical expressions of the stationary random response of a single-degree-of-freedom system having the "general slip" hysteresis, when subjected to the white noise. The general slip hysteresis defined herein covers the well-known bilinear hysteresis as well as the "pure slip" hysteresis. In this sense, this is regarded as the quite general character of building structures.

APPROXIMATE SOLUTION TO THE COULOMB DAMPING SYSTEM

Consider a single-degree-of-freedom system represented by the following second order ordinary differential equation.

$$\ddot{x} + f(x, \dot{x}) = W(t) \ , \qquad (1)$$

where x is a displacement, $\cdot$ designates derivative with respect to time, t, and $f(x, \dot{x})$ represents a restoring force function. The excitation W(t) is prescribed as a Gaussian white noise having zero mean with a constant spectral density, K. Supposed that the study is limited to the random process where both the input and the output are stationary, the joint probability density function of x and $\dot{x}$, $p(x, \dot{x})$, satisfies the following Fokker-Planck equation.

$$\dot{x} \frac{\partial p}{\partial x} - \frac{\partial}{\partial \dot{x}} [p \cdot f(x, \dot{x})] - \pi K \frac{\partial^2 p}{\partial \dot{x}^2} = 0 \ . \qquad (2)$$

If the restoring force function, f, is given, in particular, by

$$f(x, \dot{x}) = c \ \mathrm{sgn}(\dot{x}) + \left(\frac{c}{2\pi K} |\dot{x}| + \frac{1}{2} \right) g(x) \ , \qquad (3)$$

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where c is a positive constant, sgn denotes a sign function and $g(x)$ is an arbitrary function of x, then the function p which satisfies Eq.(2) becomes[1)]

$$p(x, \dot{x}) = A \exp\left[-\frac{c}{\pi K} \{ |\dot{x}| + \frac{c}{2\pi K} \int_0^x g(u)du \} \right] \quad ; \quad A = \text{const.} \tag{4}$$

If the random variable $|\dot{x}|$ can be replaced by its expected value $E\left[|\dot{x}|\right]$ as its representative, the right hand side of Eq.(3) is considerably simplified. Now, since this expected value is estimated from Eq.(4) as

$$E\left[|\dot{x}|\right] = \pi K/c \ , \tag{5}$$

Eq.(3) can be reduced, in an approximate sense, into

$$f(x, \dot{x}) = c \ \mathrm{sgn}(\dot{x}) + g(x) \ . \tag{6}$$

In other words, the solution to Eq.(2) is approximately written as Eq.(4), if f is represented by Eq.(6). Moreover, if $g(x)$ is given by

$$g(x) = bx + d \ \mathrm{sgn}(x) \ , \tag{7}$$

where b and d are constants, then Eq.(6) represents a "Coulomb-set up" characteristic which can be modelled by the combination of a Coulomb damper with a set up spring[2)] as shown in Fig.1. Particularly when d=0, f becomes a "Coulomb-linear" characteristic which can be modelled by the combination of a Coulomb damper with a linear spring.

When g is given by Eq.(7), Eq.(4) becomes concretely

$$p(x, \dot{x}) = \frac{1}{2\sqrt{\pi}\, {}_c\sigma_{\dot{x}}\, {}_1\sigma_x} \cdot \frac{\exp(-\varepsilon^2)}{\mathrm{erfc}(\varepsilon)} \cdot \exp\left(-\frac{\sqrt{2}|\dot{x}|}{{}_c\sigma_{\dot{x}}} - \frac{x^2 + 2\delta|x|}{2\, {}_1\sigma_x{}^2}\right) \ , \tag{8}$$

where ${}_c\sigma_{\dot{x}} = \sqrt{2}\pi K/c$, ${}_1\sigma_x = (\pi K/c)\sqrt{2/b}$, $\delta = d/b$, $\varepsilon = \delta/({}_1\sigma_x\sqrt{2})$, and

$$\mathrm{erfc}(u) = 1 - \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt \ .$$

${}_c\sigma_{\dot{x}}$ and ${}_1\sigma_x$ stand for the root mean square (R.M.S.) value of $\dot{x}$ and x, respectively, where suffixes c and 1 denote the Coulomb-set up and Coulomb-linear system, respectively. R.M.S. value of x, ${}_c\sigma_x$, is obtained from Eq.(8) as

$${}_c\sigma_x = {}_1\sigma_x \sqrt{1 - \frac{2\varepsilon}{\sqrt{\pi}} \cdot \frac{\exp(-\varepsilon^2)}{\mathrm{erfc}(\varepsilon)} + 2\varepsilon^2} \ . \tag{9}$$

APPROXIMATE SOLUTION TO THE GENERAL SLIP HYSTERETIC SYSTEM

"General slip hysteresis" means a "stair-shape" character, as illustrated in Fig.2, which is obtained by sliding mutually two bilinear loops along the straight line which passes the origin. To prescribe this property, four parameters — the natural circular frequency in a small amplitude, ω, the ratio of a plastic stiffness to an elastic one, γ, the higher yield acceleration, α, and the ratio of the lower yield acceleration to the higher one, ν — are required. The nondimensional quantity, ν, lies between 0 and 1. The case of $\nu=0$ corresponds to "pure slip hysteresis" where the loop returns to the origin at every half cycle, while the case of $\nu=1$ to the conventional bilinear hysteresis. γ ranges $0 \leq \gamma < 1$. It will be observed from the comparison of Fig.1 with Fig.2, that this general slip hysteresis is quite similar to the Coulomb-set up characteristic with certain mutual correspondence. If this relation is detected in an analytical form, the one can be favorably estimated from the other. This will be achieved as follows.

When three parameters in the Coulomb-damping system are selected as

$$b = \gamma\omega^2, \quad c = (1+\nu)(1-\gamma)\alpha/2, \quad \text{and} \quad d = (1-\nu)(1-\gamma)\alpha/2, \tag{10}$$

then both characteristics are laid upon each other in most parts. Under these conditions, it is expected that areas surrounded by hysteresis loops are adjusted to be equal to each other approximately. This energy balance is expressed in a form of expectation as

$$E[{}_gX] = E[{}_cX] + \Delta , \tag{11}$$

where Δ is a yield displacement, X is a random variable which stands for a peak displacement and a suffix g designates the general slip characteristic.

Supposed that the probability density function of a peak, p(X), is given approximately by[1)]

$$p(X) = -\frac{d}{dX}\int_0^\infty \dot{x}p(X,\ \dot{x})d\dot{x} \Big/ \int_0^\infty \dot{x}p(0,\ \dot{x})d\dot{x} , \tag{12}$$

then substitution of Eq.(8) into Eq.(12) finally gives

$$p({}_cX) = \frac{{}_cX + \delta}{{}_1\sigma_x{}^2} \exp(- \frac{{}_cX^2 + 2\delta{}_cX}{2{}_1\sigma_x{}^2}) \tag{13}$$

$E[{}_cX]$ can be easily estimated from Eq.(13) as

$$E[{}_cX] = \sqrt{\frac{\pi}{2}}\,{}_1\sigma_x \frac{\mathrm{erfc}(\varepsilon)}{\exp(-\varepsilon^2)} . \tag{14}$$

Thus it is found from Eqs.(9) and (14) that $E[{}_cX]$ is proportional to ${}_c\sigma_x$ in a following fashion.

$$E[{}_cX] = \sqrt{\frac{\pi}{2}} \cdot \frac{\mathrm{erfc}(\varepsilon)}{\exp(-\varepsilon^2)} [1 - \frac{2\varepsilon}{\sqrt{\pi}} \cdot \frac{\exp(-\varepsilon^2)}{\mathrm{erfc}(\varepsilon)} + 2\varepsilon^2]^{-1/2} \cdot {}_c\sigma_x \tag{15}$$

Since ${}_gx$ is expected to have the property similar to ${}_cx$'s, it will be possible to use the same proportional constant as in Eq.(15) for the relation between $E[{}_gX]$ and ${}_g\sigma_x$. Therefore applying Eq.(10) to the right hand side of Eq.(9) and utilizing Eq.(11), R.M.S. displacements for the Coulomb-set up system and for the general slip system yield respectively

$${}_c\eta_x = \frac{2\pi\xi}{(1+\nu)(1-\gamma)} \cdot \sqrt{\frac{2}{\gamma}[1 - \frac{2\varepsilon}{\sqrt{\pi}} \cdot \frac{\exp(-\varepsilon^2)}{\mathrm{erfc}(\varepsilon)} + 2\varepsilon^2]} , \quad \text{and} \tag{16}$$

$${}_g\eta_x = [\frac{2\pi\xi}{(1+\nu)(1-\gamma)}\sqrt{\frac{2}{\gamma}} + \frac{\exp(-\varepsilon^2)}{\mathrm{erfc}(\varepsilon)}\sqrt{\frac{2}{\pi}}] \sqrt{1 - \frac{2\varepsilon}{\sqrt{\pi}} \cdot \frac{\exp(-\varepsilon^2)}{\mathrm{erfc}(\varepsilon)} + 2\varepsilon^2} , \tag{17}$$

where $\varepsilon = (1-\nu^2)(1-\gamma)^2/8\pi\xi\sqrt{\gamma}$. η_x identifies σ_x normalized by Δ and ξ is a nondimensional quantity regarding the input intensity as follows.

$$\eta_x \equiv \sigma_x/\Delta = \omega^2\sigma_x/\alpha , \quad \text{and} \quad \xi \equiv \omega K/\alpha^2 . \tag{18}$$

Particularly when $\nu=1$ in Eqs.(16) and (17), then

$${}_c\eta_x = \frac{\pi\xi}{1-\gamma}\sqrt{\frac{2}{\gamma}} , \quad \text{and} \tag{19}$$

$$_g\eta_x = \frac{\pi\xi}{1-\gamma}\sqrt{\frac{2}{\gamma}} + \sqrt{\frac{2}{\pi}} . \tag{20}$$

They are the expressions for the Coulomb-linear system and for the bilinear one, respectively. Eq.(20) tells that, for the bilinear system, $_g\eta_x$ takes the following minimum value when γ=1/3.

$$\min_{\gamma}\ _g\eta_x = \frac{3\pi\sqrt{6}}{2}\xi + \sqrt{\frac{2}{\pi}} \sim 11.5\xi + 0.80 \tag{21}$$

Moveover in the case of γ=0, Eqs.(16) and (17) are reduced to the following simple expressions respectively.

$$_c\eta_x = \frac{16\pi^2\sqrt{2}}{(1-\nu)(1+\nu)^2}\xi^2 \text{ , and} \tag{22}$$

$$_g\eta_x = \frac{16\pi^2\sqrt{2}}{(1-\nu)(1+\nu)^2}\xi^2 + \sqrt{2} . \tag{23}$$

Now the case that ξ is too low for the response to excurt into the plastic region enough should be excluded from the application of Eq.(17), because Eq.(11) is not considered to simply hold under such conditions.

The plastic deformation of a general slip oscillator, which is defined by the displacement which is not accompanied with the restoring force at all in the excursion on the second branch having the plastic stiffness ratio γ, is approximately equal to the displacement of an equivalent Coulomb-set up system, $_cx$, multiplied by (1-γ). Therefore the R.M.S. plastic deformation normalized by a yield displacement, η_p, is obtained from Eq.(16) as

$$\eta_p = \frac{2\pi\xi}{1+\nu}\sqrt{\frac{2}{\gamma}\left[1 - \frac{2\varepsilon}{\sqrt{\pi}} \cdot \frac{\exp(-\varepsilon^2)}{\mathrm{erfc}(\varepsilon)} + 2\varepsilon^2\right]} , \tag{24}$$

where a suffix p signifies the plastic deformation. Especially when ν=1, Eq.(24) is reduced to

$$\eta_p = \pi\xi\sqrt{2/\gamma} , \tag{25}$$

which is the expression for a bilinear oscillator. Besides, if γ=0, Eq. (24) is much simplified as

$$\eta_p = \frac{16\pi^2\sqrt{2}}{(1-\nu)(1+\nu)^2}\xi^2 . \tag{26}$$

For the evaluation of the accumulated plastic deformation, which is calculated from the accumulation of the plastic deformation with respect to time, it is necessary to estimate the velocity with which the general slip system excurses in a plastic region. Since this velocity is regarded to be equivalent to that of the Coulomb-set up system, $_c\dot{x}$, the accumulation of the absolute plastic deformation, $|x_a|$, becomes

$$|x_a| = (1-\gamma)\ t \cdot |_c\dot{x}| , \tag{27}$$

where t means duration time in the stationary state and a suffix a represents the accumulated plastic deformation. Now paying one's attention especially to the accumulated plastic deformation with positive velocity, x_a^+, its expected value will be from Eq.(27)

$$E[x_a^+] = (1-\gamma)\ t \cdot E[|_c\dot{x}|]\ /2 . \tag{28}$$

Because E[$|_c\dot{x}|$] has already given by Eq.(5), Eq.(28) is finally reduced, with the aid of Eq.(10) and the second equation of Eq.(18), to

$$\lambda_a^+ = \frac{2\pi^2\tau\xi}{1+\nu} \quad , \tag{29}$$

where λ_a^+ denotes $E[x_a^+]$ normalized by Δ and τ is the duration time normalized by the natural period of an oscillator in a small amplitude, T. Namely,

$$\lambda_a^+ \equiv E[x_a^+]/\Delta \quad , \quad \text{and } \tau \equiv t/T \quad . \tag{30}$$

Finally, the hysteretic oscillator with the viscous damping is a little studied from a viewpoint of energy balance[3]. The expected energy dissipated by the hysteretic damping is equal to $(1+\nu)\alpha E[x_a^+]$, whereas the expected energy per unit time dissipated by the viscous damping is $2h\omega\sigma_{\dot{x}}^2$. On the other hand, the expected energy per unit time supplied to the system by white noise is πK. Hence the energy balance yields

$$(1+\nu)\alpha E[x_a^+] + 2h\omega\sigma_{\dot{x}}^2 t = \pi K t. \tag{31}$$

This can be transformed into the following nondimensional form.

$$\frac{(1+\nu)\lambda_a^+}{2\pi^2\tau\xi} + \frac{2h\eta_{\dot{x}}^2}{\pi\xi} = 1, \tag{32}$$

where $\eta_{\dot{x}} \equiv \sigma_{\dot{x}}/\omega\Delta$. The first and the second term of Eq.(32) represent the ratios of the energies lost by the hysteretic and the viscous damping, respectively. If no viscosity exists, only the first term remains. This follows that all of the energy is consumed by the plastic deformation and that $(1+\nu)\lambda_a^+/2\pi^2\tau\xi=1$, which exactly agrees with Eq.(29) already derived by the different standpoint.

VERIFICATION BY THE MONTE CARLO SIMULATION

The important expressions obtained in preceding sections are verified by the digital simulation due to fifty sample functions which have properties equivalent to those of white noise. Stationary responses have been evaluated from the values averaged across the ensemble, each of which is calculated by taking the average over the stationary part that has thirty units of nondimensional time τ.

Fig.3 indicates the comparison of theoretical solutions of R.M.S. displacement with associated simulated results, when $\nu=1$ which corresponds to the bilinear hysteresis, where $\gamma=0.1$. The ordinate represents the nondimensional R.M.S. displacement, η_x, whereas the abscissa does the nondimensional input intensity, ξ. Two solid lines correspond to theoretical solutions. Two sets of points close to each line stand for the simulation estimates of the associated systems. It is found from this figure that theoretical solutions agree quite well with simulated results. Fig.4 is prepared to investigate the effect of γ on $_g\eta_x$ with the fixed ξ, where ν is still unity. It is recognized that the proposed theory traces the digital simulation with sufficient accuracy. It is surely observed that $_g\eta_x$ becomes minimum when γ is around 1/3. This valley, however, is not so sharp. Fig.5 shows the comparison similar to Fig.3 when $\nu=0$ which corresponds to the pure slip hysteresis, where $\gamma=0$. The degree of agreement is generally satisfactory, except when ξ is quite low. Fig.6 illustrates the effect of ν on $_g\eta_x$ with the fixed γ and ξ. Theoretical values agree well with simulation estimates. $_g\eta_x$ is not sensitive to ν.

The relation between η_p and γ for the bilinear case $\nu=1$, when ξ is fixed to be 0.1, is depicted in Fig.7. Simulation estimates are generally greater than theoretical ones but, with increasing γ, the difference between the both becomes less. Fig.8 is drawn to investigate the effect of ν on η_p. The both agree from the qualitative point of view, although there exists a certain amount of quantitative discrepancy. The influence of ν is not notable in this case too.

Fig.9 displays the relation between λ_a^+ and ξ for the bilinear case $\nu=1$, when $\gamma=0.1$ and $\tau=10$. Simulation results are seen to be in extremely good agreement with the theoretical solution. Fig.10 represents λ_a^+ vs. ν with the fixed γ, τ and ξ. λ_a^+ decreases in a hyperbolic way with increasing ν. The degree of agreement is quite satisfactory. It is concluded that Eq.(29) is sufficiently valid over the wide ranges of related parameters. In fact, it is quite natural that theoretical values are in considerably good agreement with simulation estimates, because the theoretical solution is, in this case, believed to be rigorous as discussed previously.

In order to understand the effect of the viscous damping, the ratios of energies lost by two different mechanisms which are expressed by Eq.(32) are depicted in Fig.11, when h=0.01 and $\gamma=0.1$, for the bilinear case $\nu=1$. The shaded region indicates the ratio of energy dissipated by the viscous damping which corresponds to the second term in the right hand side of Eq.(32). The remainder is the energy lost by the hysteresis corresponding to the first term. If h is small to this extent, most of the energy is consumed by the plastic deformation and the effect of the viscous damping can be ignored, except when ξ is quite low.

CONCLUDING REMARKS

The approximate analytical expressions of the random response for a general slip hysteretic single-degree-of-freedom system have been derived. They have been verified by the Monte Carlo simulation. As the results, together with the more detailed parametric studies[4] which are not presented here, it has been found that the theoretical solution of the R.M.S. displacement coincides with the coresponding experimental one with sufficient accuracy, except when either ξ is very low or γ is close to unity. The theoretical R.M.S. plastic deformation well predicts the associated simulation estimate, except when either ξ or γ is close to zero. Theoretical values of the expected accumulated plastic deformation agree quite well with simulated ones over the wide ranges of related parameters. In addition, it has been recognized that the foregoing expressions derived for the viscously undamped case can be approximately applied also to the oscillator with viscosity whose damping ratio is less than a few percent, unless the input intensity is very low.

As the visual representation, the relation among the nondimensional R.M.S. displacement, ${}_g\eta_x$, the nondimensional lower yield acceleration, ν, and the nondimensional plastic stiffness, γ, is spatially depicted in Fig.12, where ξ is fixed to be 0.1. The curve a corresponds to the displacement of a pure slip system whose expression is obtained by substituting $\nu=0$ in Eq. (17). The curve b means the bilinear response given by Eq.(20). The curve c is of a form of Eq.(23). Fig.13 illustrates the similar relation as for the nondimensional plastic deformation when $\xi=0.1$. The curve a is given by Eq.(24) with $\nu=0$. The curve b is expressed by Eq.(25), the plastic deformation of a bilinear system. The curve c is given by Eq.(26). Fig.14 shows the relation among, $\lambda\frac{1}{a}$, ν and γ with the fixed τ and ξ in a similar fashion. The curve a is Eq.(29) itself.

REFERENCES

1) Y. K. Lin, "Probabilistic Theory of Structural Dynamics," McGraw Hill, Inc. 1967.
2) S. H. Crandall, "Random Vibration of a Nonlinear System with a Set-up Spring," Journal of Applied Mechanics, Sep. 1962, pp. 477-482.
3) D. Karnopp, "Power Balance Method for Nonlinear Random Vibration," Journal of Applied Mechanics, Mar. 1967, pp. 212-214.
4) Y. Matsushima, "Random Response of Single-degree-of-freedom System with General Slip Hysteresis," Building Research Institute, Research paper No. 82, Jul. 1979.

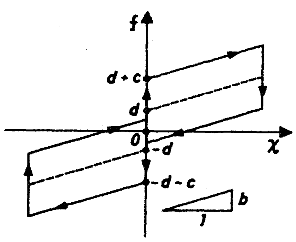

Fig.1 "Coulomb-set up" characteristic

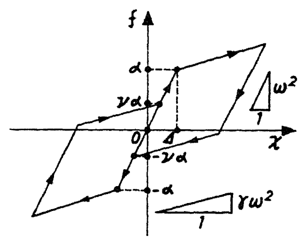

Fig.2 "General slip" hysteresis

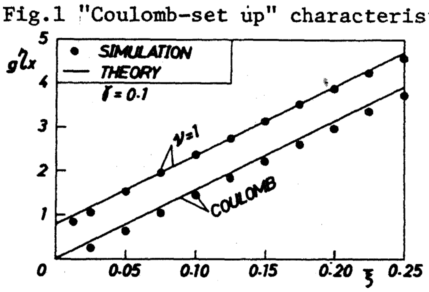

Fig.3 Displacement vs. input intensity

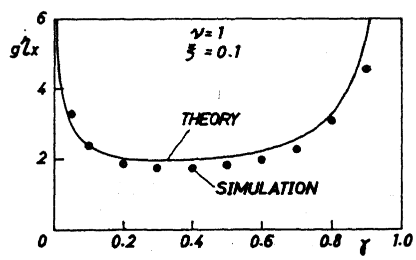

Fig.4 Displacement vs. plastic stiffness

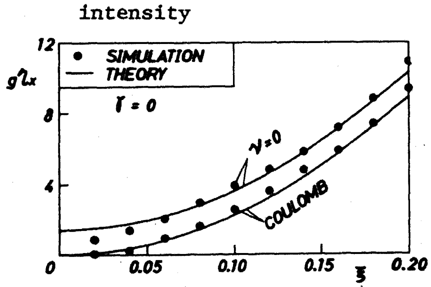

Fig.5 Displacement vs. input intensity

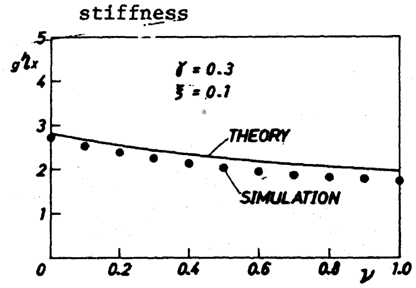

Fig.6 Displacement vs. lower yield level

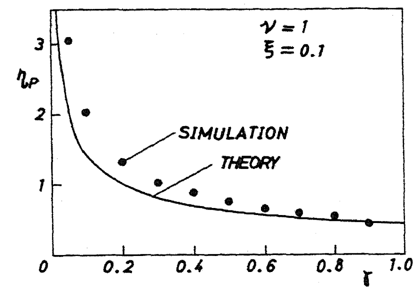

Fig.7 Plastic deformation vs. plastic stiffness

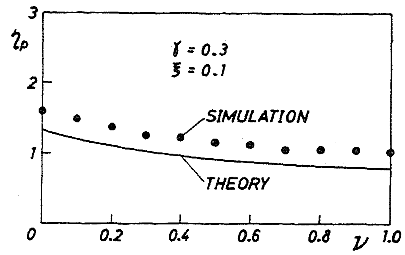

Fig.8 Plastic deformation vs. lower yield level

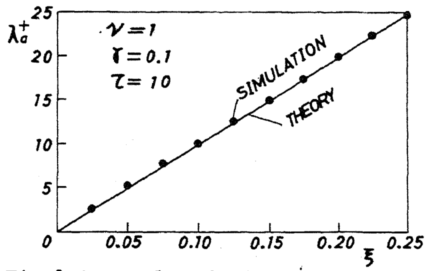

Fig.9 Accumulated plastic def. vs. input intensity

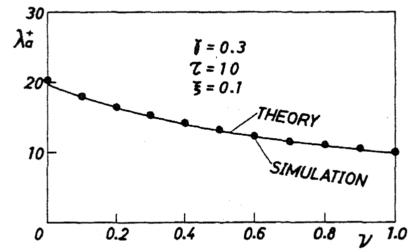

Fig.10 Accumulated plastic def. vs. lower yield level

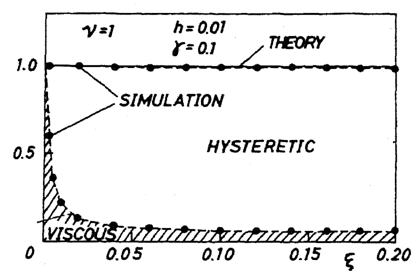

Fig.11 Ratio of dissipated energy vs. input intensity

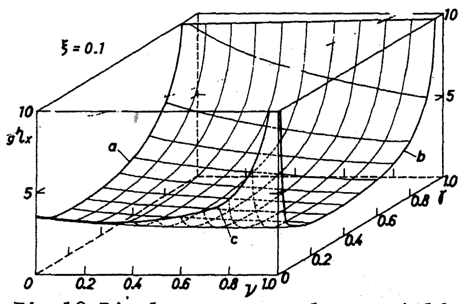

Fig.12 Displacement vs. lower yield level and plastic stiffness

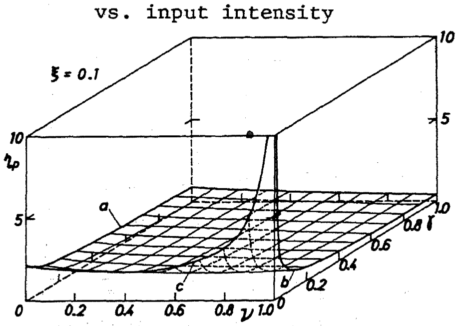

Fig.13 Plastic deformation vs. lower yield level and plastic stiffness

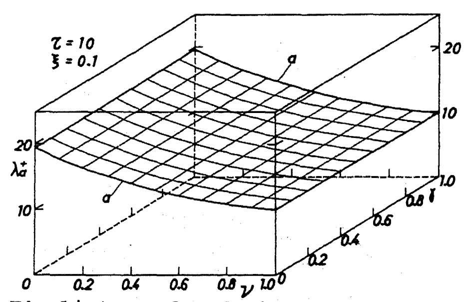

Fig.14 Accumulated plastic def. vs. lower yield level and plastic stiffness